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Analyticity of the fundamental solutions of hyperbolic systems*

By

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1. Introduction

Consider a hyperbolic system with analytic coefficients

$$(1.1)\qquad M[u] = \frac{\partial u}{\partial t} - \sum_{k=1}^{n} A_k(x, t)\frac{\partial u}{\partial x_k} - B(x, t)u = 0\,,$$

where the word hyperbolic means the following: $\sum A_k(x, t)\xi_k$ has, for any real $\xi \neq 0$ and all (x, t), N distinct real eigenvalues, $\lambda_1(x, t; \xi)$, $\cdots$, $\lambda_N(x, t; \xi)$. Here u are vector-valued functions with N components; A_k and B are real analytic functions of $x=(x_1, \cdots, x_n)$ and t.

In the present paper we shall show that the fundamental solutions of hyperbolic systems of partial differential equations with analytic coefficients are analytic except on the characteristic conoid. This property can also be expressed directly in terms of the solution of the equation: If at time $t=0$ the initial data of a solution u is analytic in an open set containing all points which lie on a ray issuing from some given point (x_0, t_0), then u is analytic at x_0, t_0.

J. Hadamard proved this property for second order hyperbolic equations [1] and M. Riesz also treated this problem [8]. In the case of constant coefficients, there are several papers which show

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this property. We quote among them a work of Petrowsky [7]. Recently, J. Leray published a series of works, which treat systematically general partial differential equations with analytic coefficients [3] and which elucidate this property. Now, we want to present our method, which relies assentially on a paper of P. D. Lax [2], in which he uses an asymptotic expansion of solutions and proves that, in the case of C^∞-coefficients, the fundamental solutions are C^∞ except on characteristic conoids.

We have already shown, in the case of constant coefficients, that the analyticity can be proved by using his method [5]; the present article is an extension to analytic coefficients of the work. The reasoning in both cases is essentially the same, but there are some technical complications in the case of variable analytic coefficients; for example the phase functions $l^{(i)}(x, t; \omega)$, which existed globally in the case of constant A_k, exist only for a certain time interval in the present case. For C^∞ coefficients, D. Ludwing [4] has recently overcome this difficulty by using the major principle of Huygens (Hadamard's terminology). We shall discuss this problem in another article.

Although we shall give our proof in detail later (Section 3), we give here the sketch of our proof. The fundamental solution e is defined as that solution of the equation which at $t=0$ is a δ-function, more precisely, one of its component is a δ-function and the other components vanish. We proceed by constructing explicitly *approximate* fundamental solution u_p, p an index which eventually will be taken as large, i.e. a function with the following properties:

i) u_p satisfies the differential equation approximately;

$$M[u_p] = f_p, \tag{1.2}$$

where f_p has continuous partial derivatives up to oder $p-1$,

ii) u_p satisties the initial conditions approximately, i.e.

$$u_p(x, 0) = \delta(x) + a(x)$$

where a is an analytic function of x indepent of p. The exact fundamental solution can then be written as

(1. 3) $$e = u_p + w_p + z$$

where w_p and z are defined as solutions of the following initial value problems:

(1. 4) $$\begin{cases} M[w_p] = -f_p, & w_p(x, 0) = 0. \\ M[z] = 0, & z(x, 0) = -a. \end{cases}$$

We are now ready to show the analyticity of e by finding suitable estimates for the partial derivatives of e. Let ν be a multi-index, we choose

$$p = |\nu| + \left[\frac{n}{2}\right] + 2.$$

It follows from (1. 3), by the triangle inequality that

$$|D^\nu e| \leqslant |D^\nu u_p| + |D^\nu w_p| + |D^\nu z|.$$

What we need is suitable estimates for the derivatives of u_p, w_p and z.

The explicit construction of u_p is described in Section 2; we shall derive there a suitable estimate for $D^\nu u_p$ (Proposition 1 and Theorem 1 of Section 3).

In section 3 we shall derive suitable estimates for the partial derivatives up to order p of $f_p = M[u_p]$. (Proposition 1 and its corollary). By known estimates for solutions of hyperbolic differential equations, we can then estimate the solution w_p and its partial derivatives up to order ν in terms of the estimate derived for f_p and its derivatives (to slightly higher order).

Finally, since z is the solution of an analytic initial value problem, it is itself analytic; since it is also independent of p, suitable estimates for its derivatives are immediately available.

2. Construction of approximate fundamental solution.

Hereafter we follow the notation and the definitions of [2], and also those of our previous paper [5]. Let $l^{(i)}(x, t; \omega)$ be the solution of

(2. 1) $$l_t^{(i)} = \lambda_i(x, t; l_x^{(i)})$$

with the initial value $l^{(i)}(x, 0; \omega) = x \cdot \omega$, where $\omega = (\omega_1, \cdots, \omega_n)$ real

$|\omega|=1$. Standard existence theorems guarantee that this solution exists at least if (x, t) is not far from the origin, and that this solution is an analytic function of $(x, t; \omega)$. We consider the formal expansion (cf. [2]):

$$(2.2)\quad u(x, t) \sim \sum_{i=1}^{N} \int_1^\infty \int_{\Omega_0} \exp(il^{(i)}\xi)\,\{v_0^{(i)}(x, t; \omega)+v_1^{(i)}(x, t; \omega)/\xi+\cdots +v_m^{(i)}(x, t; \omega)/\xi^m+\cdots\}\,\xi^{n-1} d\xi d\omega,$$

with the initial value $u(x, 0) = \begin{pmatrix} 0 \\ \vdots \\ \delta \\ \vdots \\ 0 \end{pmatrix}$.

We now modify this expression in the following way:

$$(2.3)\quad u_p(x, t) = \sum_{i=1}^{N} \Big\{\int_1^\infty \xi^{n-1} d\xi \int_{\Omega_0} \exp(il^{(i)}\xi)(v_0^{(i)}+v_1^{(i)}/\xi+\cdots+v_n^{(i)}/\xi^n)d\omega + \int_1^\infty \xi^{-2} d\xi \int_{\Omega_0} \exp(il^{(i)}\xi)\Big(\sum_{q=1}^{p} \frac{(il^{(i)})^{q-1}}{q!} v_{n+q}^{(i)}\Big)d\omega\Big\}.$$

Then we have, as we shall show at the end of this section,

$$(2.4)\quad M[u_p] = \sum_{i=1}^{N} \Big\{\int_1^\infty \xi^{-2} d\xi \int_{\Omega_0} \exp(il^{(i)}\xi) \frac{(il^{(i)})^{p-1}}{p!} M[v_{n+p}^{(i)}]d\omega + \int_{\Omega_0} \exp(il^{(i)})\Big(\sum_{q=2}^{p} \frac{(il^{(i)})^{q-2}}{q!} M[v_{n+q-1}^{(i)}]\Big)d\omega\Big\}.$$

We denote this right-hand side by $f_p(x, t)$.

Next we define $w_p(x, t)$ as the solution of

(1.4) $M[w_p]=-f_p(x, t)$, with initial value zero: $w_p(x, 0)=0$.

The function $u=u_p+w_p$ satisfies the equation (1.1), if i.e. satisties the stated initial condition modulo analytic data $a(x)$,

$$(2.5)\quad u(x, 0) = \begin{pmatrix} 0 \\ \vdots \\ \delta \\ \vdots \\ 0 \end{pmatrix} + a(x),$$

because, as we shall explain later, $\sum_{i=1}^{N} v_0^{(i)}(x, 0; \omega) = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$, $\sum_{i=1}^{N} v_m^{(i)}(x, 0; \omega)$

$$=0,\ m\geqslant 1,\quad a(x)=\int_0^1 \xi^{n-1}d\xi\int_{\Omega_0}\exp(ix\omega\xi)\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}d\omega\,.$$

Then, we denote by $z(x, t)$ the solution of

(1.4) $M[z]=0$, with the initial condition $z(x, 0)=-a(x)$,

we know that $z(x, t)$ is an analytic function of (x, t) (cf. [6]). We know by proposition 1 of Section 3, that $f_p(x)$ is continuously differentiable up to order $(p-1)$. Then, by Proposition of Section 3, we have estimates for the derivative of $w_p(x, t)$ up to order $\left(p-2-\left[\frac{n}{2}\right]\right)$. About those of $u_p(x, t)$, we use Proposition 1. Now we shall define the $v_m^{(i)}$, and obtain estimates for these functions.

Now we return to (2.2). Apply the operator M to the right-hand side and set the coefficient of $1/\xi^{m-1}$ equal to 0. We obtain

$$(2.6)\qquad i(l_t^{(i)}-A\cdot l_x^{(i)})v_m^{(i)}+M[v_{m-1}^{(i)}]=0\,,\qquad (i=1, 2, \cdots, N)\,,$$

where $v_{-1}^{(i)}=0$; the factor i in the first term is $\sqrt{-1}$.
By the way, this operation M, more precisely, the differentiation under the integral sign is justified as follows: For $p\geqslant -1$, we interpret the expression $\int_1^\infty\int_{\Omega_0}\exp(il\xi)\xi^p f(x, t\,;\omega)d\xi d\omega$ as a distribution depending on the parameter t. This possibility comes from the fact $l_x\neq 0$. (See the beginning of the proof of Fundamental Lemma of Section 4). Therefore the above distribution is defined as $\lim_{M\to\infty}\int_1^M \xi^p d\xi\int_{\Omega_0}\exp(il\xi)f(x, t\,;\omega)d\omega$, where the convergence is in the topology of distributions.
Remarking that $l^{(i)}(x, 0\,;\omega)=x\cdot\omega$, we take the initial data

$$(2.7)\qquad \sum_{i=1}^N v_0^{(i)}(x, 0\,;\omega)=\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix},$$

$$(2.8)\qquad \sum_{i=1}^N v_m^{(i)}(x, 0\,;\omega)=0\qquad \text{for}\quad m\geqslant 1\,.$$

To express (2.6) more explicitly, we use the eigen-vectors $R_i(x, t\,;\omega)$ $(i=1, \cdots, N)$ of $A(x, t)\cdot\omega$; R_i corresponds to the eigenvalue $\lambda_i(x, t\,;\omega)$; we assume that R_i are of unit length. We define

$R_j^{(i)}(x, t; \omega)=R_j(x, t; l_x^{(i)}/|l_x^{(i)}|)$; thus $R_j^{(i)}$ $(i, j=1, \cdots, N)$ are analytic functions of $(x, t; \omega)$. Now we put

$$(2.9)\qquad v_0^{(i)} = \omega_i(x, t; \omega) R_i^{(i)}(x, t; \omega);$$
$$v_m^{(i)} = \sum_{j=1}^{N} \sigma_m^{ij}(x, t; \omega) R_j^{(i)}(x, t; \omega) \qquad \text{for} \quad m \geqslant 1.$$

Moreover we use the left eigenvectors $L_i(x, t; \omega)$ corresponding to $\lambda_i(x, t; \omega)$: $(\lambda_i I - {}^t A \cdot \omega) L_i(x, t; \omega)=0$. We normalize L_i in such a way that $\langle L_i, R_j \rangle = \delta_i^j$, where $\langle\ ,\ \rangle$ means usual scalar product; δ_i^j is the Kronecker symbol. Analogously to $R_j^{(i)}$, we define $L_j^{(i)}(x, t; \omega)=L_j(x, t; l_x^{(i)}/|l_x^{(i)}|)$. Then $\langle L_j^{(i)}, R_k^{(i)} \rangle = \delta_k^j$ and $L_j^{(i)}$ are analytic functions of $(x, t; \omega)$. Similarly, we define $\lambda_j^{(i)}(x, t; \omega)= \lambda_j(x, t; l_x^{(i)}/|l_x^{(i)}|)$.

Now, we assume that $v_0^{(i)}, v_1^{(i)}, \cdots, v_{m-1}^{(i)}$ are determined, and want to show how to determine $v_m^{(i)}$. Take the scalar product of (2.6) with $L_j^{(i)}$. Taking account of (2.1), we have

$$(2.10)\quad |l_x^{(i)}|(\lambda_i^{(i)}-\lambda_j^{(i)})\sigma_m^{ij} - i\langle L_j^{(i)}, M[v_{m-1}^{(i)}]\rangle = 0, \qquad \text{where } i=\sqrt{-1}.$$

In order that this system of equations have a solution it is necessary to assume that we have determined $v_{m-1}^{(i)}$ in such a way that

$$(2.11)\qquad \langle L_i^{(i)}, M[v_{m-1}^{(i)}]\rangle = 0.$$

We make this assumption. Then we have

$$(2.12)\quad \sigma_m^{ij}(x, t; \omega) = i\langle L_j^{(i)}, M[v_{m-1}^{(i)}]\rangle / |l_x^{(i)}|(\lambda_i^{(i)}-\lambda_j^{(i)}), \qquad \text{for } i \neq j;$$

more explicitly, denoting the denominator by $s_{ij}(x, t; \omega)$,

$$(2.13)\quad \sigma_m^{ij}(x, t; \omega) = s_{ij}(x, t; \omega)\{(\sigma_{m-1}^{ij})_t + \sum_{k=1}^{N} L_k^{(ij)}[\sigma_{m-1}^{ik}]\}, \quad \text{for } i \neq j,$$

where $L_k^{(ij)}$ is a first order differential operator of the form

$$(2.14)\qquad L_k^{(ij)} = \sum_{\nu=1}^{n} p_{k\nu}^{(ij)}(x, t; \omega) \cdot \frac{\partial}{\partial x_\nu} + q_k^{(ij)}(x, t; \omega),$$

with analytic coefficients. Now we determine σ_m^{ii} in such a way that (2.11) is satisfied for $v_m^{(i)}$, namely

$$(2.15)\qquad \langle L_i^{(i)}, M[\sigma_m^{ii} R_i^{(i)}]\rangle = -\sum_{j; j\neq i} \langle L_i^{(i)}, M[\sigma_m^{ij} R_j^{(i)}]\rangle.$$

Here the left-hand side is a differential operator acting on σ_m^{ii}:

(2.16) $$L_i[\sigma_m^{ii}] = \left(\frac{\partial}{\partial t} - \sum_\nu a_\nu^{(i)} \frac{\partial}{\partial x_\nu} - b^{(i)}\right)[\sigma_m^{ii}],$$

with $a_\nu^{(i)} = \langle L_i^{(i)}, A_\nu R_i^{(i)} \rangle$. We remark that the direction of differentiation is that of the bicharacteristic of M (cf. [2], p. 630), therefore this direction is real. Of course $a_\nu^{(i)}(x, t; \omega)$ and $b^{(i)}(x, t; \omega)$ are real analytic functions.
Therefore, (2.16) is written

(2.17) $$L_i[\sigma_m^{ii}] = -\sum_{k;\, k\neq i} L_k^{(ii)}[\sigma_m^{ik}],$$

where $L_k^{(ii)}$ are of the form (2.14).
The initial condition of σ_m^{ii} is determined by (2.8), namely

(2.18) $$\sigma_m^{ii}(x, 0; \omega) = -\sum_{j;\, j\neq i} \sigma_m^{ji}(x, 0; \omega).$$

Finally, $v_0^{(i)} = \sigma_0^i R_i^{(i)}(x, t; \omega)$ are determined by

(2.19) $L_i[\sigma_0^i] = 0$, with the initial condition

(2.20) $\sigma_0^i(x, 0; \omega) = \sigma^i(x; \omega)$, where

(2.21) $$\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \sum_{i=1}^{N} \sigma^i(x; \omega) R_i(x, 0; \omega).$$

Now let us show how (2.4) is obtained. Take an integrand of (2.3):

(2.22) $$\frac{\exp(il\xi)}{q!}(il)^{q-1} v_{n+q}/\xi^2, \qquad q \geqslant 2,$$

where we omitted the index i. Apply M to this integrand; omitting the factor $\exp(il\xi)$ we then have

(2.23) $$\frac{i(l_t - A\cdot l_x)}{q!}(il)^{q-1} v_{n+q}/\xi + \frac{i(l_t - A\cdot l_x)}{q!}(q-1)(il)^{q-2} v_{n+q}/\xi^2 + \frac{(il)^{q-1}}{q!} M[v_{n+q}]/\xi^2.$$

If (2.6) is taken into account, this can be written as follows:

(2.24) $$-\frac{(il)^{q-1}}{q!} M[v_{n+q-1}]/\xi - \frac{(q-1)(il)^{q-2}}{q!} M[v_{n+q-1}]/\xi^2 + \frac{(il)^{q-1}}{q!} M[v_{n+q}]/\xi^2.$$

Now we integrate the first term by parts:

$$\text{(2.25)}\quad \int_1^{\infty} \exp(il\xi)(il)^{q-1}\xi^{-1}d\xi = [(il\xi)(il)^{q-2}\xi^{-1}]_1^{\infty}$$

$$+\int_1^{\infty} \exp(il\xi)(il)^{q-2}\xi^{-2}d\xi = -\exp(il)(il)^{q-2}+\int_1^{\infty}\exp(il\xi)(il)^{q-2}\xi^{-2}d\xi .$$

This is merely formal expression. For the rigorous justification, see the beginning of the proof of the Fundamental Lemma of Section 4. Then (2.24) becomes

$$\text{(2.26)}\quad -\frac{(il)^{q-2}}{(q-1)!}M[v_{n+q-1}]/\xi^2+\frac{(il)^{q-1}}{q!}M[v_{n+q}]/\xi^2$$

$$+\frac{(il)^{q-2}}{q!}M[v_{n+q-1}]/\xi^2 .$$

here the last term expresses the integrated part.
From this, we can see easily (2.4).

Finally, we state again our

Main Theorem. *The fundamental solutions* $e_k(x, t)$, $t\geqslant 0$:

$$\text{(2.27)}\quad M[e_k]=0\,,\quad \textit{with the initial data}\ \ e_k(x,0)=\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}k\,,$$

$$(k=1, 2, \cdots, N)\,,$$

are analytic functions of (x, t) *except on the characteristic conoids issuing from the origin.*

Remark. As we have remarked earlier, our proof is, in general, valid only in a neighborhood of the origin.

3. Proof of Main Theorem.

At first we state Theorem 1, Proposition 1 and 2. Using these, we shall prove Main Theorem.

From Position 5 of Section 5, we have

Theorem 1. *There exists a complex neighborhood* V_c *of* Ω_0 *such that all* $v_m^{(f)}(x, t\,;\omega)$ *can be extended analytically there and*

$$\sup_{\omega\in V_c} |D_x^\nu v_m^{(i)}(x, t; \omega)| \leqslant \frac{(m+|\nu|)!}{\rho^{|\nu|}} K^m A,$$

$$\sup_{\omega\in V_c} |D_x^\nu \cdot M[v_m^{(i)}]| \leqslant \frac{(m+|\nu|+1)!}{\rho^{|\nu|+1}} K^m A, \qquad \text{for} \quad 0\leqslant t\leqslant t_0 \,(\leqslant 1), \quad \begin{bmatrix} m = 0, 1, 2, \cdots \\ i = 1, \cdots, N \end{bmatrix},$$

where K, A, and ρ are constants.

Remark 1. We derived Proposition 3, 4, and 5, by assuming ω real. However, these propositions are also true even if we extend these functions $\sigma_m^{i,j}(x, t; \omega)$ analytically in a small complex neighborhood of the real unit sphere Ω_0. This fact follows from the analyticity of all functions which appear in (2.13), (2.14), and (2.16), and from the fact that the direction of differentiation of (2.16) is real if ω is real.

Remark 2. In above statement, we have mentioned nothing about the domain of x. This domain, together with t_0, depends on that of the phase functions $l^{(i)}(x, t; \omega)$. About t_0 we assumed $t_0\leqslant 1$. This limitation does not diminish the generality, because by a linear transformation in t, we can always bring any fixed point to $t_0\leqslant 1$.

We state the following two key propositions, whose proofs shall be given later (section 4).

Proposition 1. *Consider the mapping*: $(il)^p\psi(x, t; \omega)\to\varphi(x, t)$ *defined by*

$$\varphi(x, t) = \int_1^\infty \xi^{-2} d\xi \int_{\Omega_0} \exp(il\xi)(il)^p \psi(x, t; \omega) d\omega .$$

Let us fix a compact set U in (x, t)-space, which does not meet the characteristic conoid with vertex at the origin. Assume

$$(3.1) \qquad \sup_{(x,t)\in U,\, \omega\in V_c} |D_x^\nu \psi(x, t; \omega)| \leqslant \frac{(q+|\nu|)!}{\rho^{|\nu|}} A, \qquad \text{for} \quad \nu\geqslant 0 .$$

Then we have

$$(3.2) \qquad \sup_{(x,t)\in U} |D_x^\nu \varphi(x, t)| \leqslant C_0 C^p \frac{(q+|\nu|)!}{\rho^{|\nu|}} A, \qquad \text{for} \quad \nu\geqslant 0,$$

provided ρ is small: $\rho\leqslant\rho_0$. *ρ_0 depends on U and V_c. Moreover,* *for* $|\nu|\leqslant p$, $\varphi(x, t)$ *is a continuous function, and* (2.2) *is true for any compact set* U (*i.e. without above condition*); C *and* C_0 *are constants depending only on* U *and* V_c, *they are independent of* q *and* A.

According to Proposition 1, consider now the following simpler transformation:

$$\varphi(x, t) = \int_{\Omega_0} \exp(il)(il)^p \psi(x, t; \omega) d\omega. \tag{3.3}$$

As we see it in the proof of the above proposition, we have the same property as above, namely.

Corollary 1. *Let* U *be a compact set* (*without the condition in Prop.* 1), *Assume* (3.1), *then we have* (3.2). $\varphi(x, t)$ *is an analytic function in* x.

As we see in (2.3), we should also consider the distribution of the form:

$$\varphi(x, t) = \int_1^\infty \xi^r d\xi \int_{\Omega_0} \exp(il\xi) \psi(x, t; \omega) d\omega, \tag{3.4}$$

where r is an integer positive 0 or negative. Let U be a compact set satisfying the condition of Proposition 1; assume $\psi(x, t; \omega)$ satisfies (3.1), then we have

Corollary 2. $\varphi(x, t)$ *is an analytic function in* x *for* $x \in U$.

In order to estimate the derivatives of w_p in (1.4), we use the

Proposition 2.[1] *Consider the mapping* $f(x, t) \to v(x, t)$ *defined by* $M[v] = f(x, t)$, $v(x, 0) = 0$. *Let* U *be a compact set in the space* $t = t_0$. *We assume*

(3.5)
$$\sup |D_x^\nu f(x, t)| \leqslant \frac{(q+|\nu|)!}{\rho^{|\nu|}} A, \quad 0 \leqslant t \leqslant t_0, \quad \textit{for} \quad |\nu| \leqslant k + \left[\frac{n}{2}\right] + 1,$$

then there exist constants C_1 *and* ρ_0 *such that*

$$\sup_{x \in U} |D_x^\nu v(x, t)| \leqslant C_1 \frac{(q+|\nu|)!}{\rho_0^{|\nu|}} A \qquad \textit{for} \quad |\nu| \leqslant k, \tag{3.6}$$

1) Since the proof is given in [6], we don't reproduce it here.

C_1 and ρ_0 are independent of f. More precisely, they depend on U, and are independent of q, k, and A.

Remark. Here we assumed ρ fixed. What is needed for the proof of Main Theorem is that C_1 and ρ_0 can be taken independently of q and k. Here we need not assume that f is analytic in x. The estimate (3.6) claims only for $|\nu|\leqslant k$, assuming (3.5) for $|\nu|\leqslant k+\left[\frac{n}{2}\right]+1$. The symbol sup of the left-hand side of (3.6) is taken in the domain defined as follows: we choose first a fixed retrograde convex cone C such that the domain of dependence with respect to any point (x, t) is contained in $(x, t)+C$, (for instance, we take as C the cone $|\xi/\tau|\leqslant\max_i(\sup_{x,t;\omega}|\lambda_i(x, t;\omega)|)$, $\tau<0$. Then the domain swept by $(x, t_0)+C$ when x runs a neighborhood of U replies to our demand.

Now we prove Main Theorem. Take a point (x_0, t_0) which does not belong to the characteristic conoid. We want to prove the distribution $u(x, t_0)=u_p(x, t_0)+w_p(x, t_0)$, u_p and w_p being difined by (2.3) and (1.4), is analytic in x at $x=x_0$. We choose a small neighborhood U of x_0 in the space $t=t_0$ in such a way that it has a positive distance from the characteristic conoid.

Decompose (2.3) in two parts: $u_p=u^0+\tilde{u}_p$, where

$$u^0(x, t) = \sum_{i=1}^{N}\int_1^\infty \xi^{n-1}d\xi\int_{\Omega_0}\exp(il^{(i)}\xi)(\sum_{j=0}^{N}v_j^{(i)}/\xi^j)d\omega\,. \tag{3.7}$$

By the Corollary 2 or Proposition 1, we see that $u^0(x, t_0)$ is analytic in x for $x\in U$.

Take an arbitrary ν. According to this ν, we fix p by

$$p = |\nu|+\left[\frac{n}{2}\right]+2\,, \tag{3.8}$$

and consider u_p and w_p corresponding to this p. By Proposition 1 and Theorem 1, we have

$$\sup_{x\in U}|D_x^\nu\tilde{u}_p(x, t_0)|\leqslant C_0NA\sum_{q=1}^{p}C^q\frac{(n+q+|\nu|)!}{q!}\left(\frac{1}{\rho}\right)^{|\nu|}K^{n+q}\,.$$

Since we can assume $C, K\geqslant 2$, we have

(3.9) $$\sup_{x\in U}|D_x^\nu \tilde{u}_p(x, t_0)| \leqslant (2C_0NA)\frac{(n+p+|\nu|)!}{p!}\left(\frac{1}{\rho}\right)^{|\nu|}C^pK^{n+p}.$$

In order to estimate $D_x^\nu w_p(x, t_0)$, at first we estimate $f_p(x, t)$. By proposition 1 and its corollary 1, we have for $0\leqslant t\leqslant t_0$,

$$\sup|D_x^\mu f_p(x, t)| \leqslant (C_0NA)C^{p-1}\frac{(|\mu|+1+n+p)!}{p!}\left(\frac{1}{\rho}\right)^{|\mu|+1}K^{n+p}$$
$$+ (C_0NA)\sum_{q=2}^{p} C^{q-2}\frac{(|\mu|+1+n+q-1)!}{q!}\left(\frac{1}{\rho}\right)^{|\mu|+1}K^{n+q-1}$$
$$\leqslant (3C_0NA)C^{p-1}\frac{(|\mu|+1+n+p)!}{p!}\left(\frac{1}{\rho}\right)^{|\mu|+1}K^{n+p}$$

for $|\mu|\leqslant p-1=|\nu|+\left[\frac{n}{2}\right]+1$. Then, by Proposition 2, we have

(3.10)
$$\sup_{x\in U}|D_x^\nu w_p(x, t_0)| \leqslant C_1(3C_0NA)C^{p-1}\frac{(|\nu|+1+n+p)!}{p!}\left(\frac{1}{\rho}\right)^{|\nu|+1}K^{n+p}.$$

Since $\rho_0<\rho$, combining this estimate with (3.9), we have

(3.11) $$\sup_{x\in U}|D_x^\nu\{\tilde{u}_p(x, t_0)+w_p(x, t_0)\}| \leqslant (CK)^p\frac{(n+p+1+|\nu|)!}{p!}\left(\frac{1}{\rho}\right)^{|\nu|}B,$$

where $B=(2C_0NA+3C_0NAC_1)/\rho_0$.

We remark here $\frac{(n+p+1+|\nu|)!}{p!}\leqslant(n+1)!\ 2^{n+p+1+|\nu|}|\nu|!$.

Therefore taking account of (3.8) and of the analyticity of $u^0(x, t_0)$, we see easily that $\sup_{x\in U}|D_x^\nu u(x, t_0)|$ is majorized by the form $c\frac{|\nu|!}{\rho'^{|\nu|}}$, for $\nu\geqslant 0$. This shows $u(x, t_0)$ is analytic in x on $x\in U$.

The solution $e_k(x, t_0)=u(x, t_0)+z(x, t_0)$ is thus analytic in x on U, because $z(x, t)$ is an analytic function. The proof is thus complete, since the analyticity in (x, t) of $e_k(x, t)$ follows from Cauchy-Kowalewski Theorem.

4. Proof of Proposition 1.

We rely on the following fundamental lemma:

Fundamental lemma. *Consider the function $g(x)$, more precisely, distribution defined by*

(4.1) $g(x)=\int_{1}^{\infty}\xi^{q}d\xi\int_{\Omega_0}\exp(il\xi)(il)^{p}f(x;\omega)d\omega,\qquad p\geqslant 0,\ q\geqslant -2.$

Denote

(4.2) $$L=\max(2,\sup_{x\in U,\,\omega\in V_c}|l(x;\omega)|),$$

where U satisfies the condition of proposition 1.
Then there exist constants C_0 and $\delta(<1)$ such that

(4.3) $$\sup_{x\in U}|g(x)|\leqslant\frac{(q+2)!}{\delta^{q+2}}L^{p}M\{f;V_c\}C_0,$$

where $M\{f;V_c\}=\sup_{x\in U,\,\omega\in V_c}|f(x;\omega)|$,
and δ and C_0 depend only on U and V_c; they do not depend on p, q and f. Moreover, if $p\geqslant q+2$, we need not to assume on U the condition that U does not meet the characteristic conoid with vertex at the origin.

We shall give the proof of this lemma at the end, and we admit this lemma as proved.

Now we want to estimate $D^{\nu}g(x)$, where

(4.4) $$g(x)=\int_{1}^{\infty}\xi^{-2}d\xi\int_{\Omega_0}\exp(il\xi)(il)^{p}f(x;\omega)d\omega.$$

We assnme

(4.5) $$\sup_{x\in U,\,\omega\in V_c}|D_x^{\nu}f(x;\omega)|\leqslant\frac{(r+|\nu|)!}{\rho^{|\nu|}}A.$$

Since $l(x;\omega)$ is an analytic function of x and ω, we assume

(4.6) $$\sup_{x\in U,\,\omega\in V_c}|D_x^{\nu}l(x;\omega)|\leqslant\frac{|\nu|!}{\rho'^{|\nu|-1}}L,\qquad\text{for}\quad|\nu|\geqslant 1.$$

For the simplicity of estimates, we assume $L\geqslant 1$.

If necessary, by taking ρ' smaller we can assume that

(4.7) $\rho'\leqslant\frac{\delta}{2L}$, δ being defined in Fundamental Lemma.

Moreover, we assume (by taking ρ smaller in (4.5)) that

(4.8) $$\rho\leqslant\rho'/8.$$

Under this condition, we want to show that

(4.9) $$\sup_{x\in U}|D^{\nu}g(x)|\leqslant\frac{(r+|\nu|)!}{\rho^{|\nu|}}3C_0L^pA.$$

Now we operate D_x^ν to (4.4). We shall denote hereafter D^ν instead of D_x^ν. Then, under the sign of integration, we have

(4.10) $$\sum_\mu C_\mu^\nu D^\mu\{\exp(il\xi)(il)^p\}D^{\nu-\mu}f.$$

Now again by Leibniz,

$$D^\mu\{\exp(il\xi)(il)^p\}=\sum_\lambda C_\lambda^\mu D^\lambda\exp(il\xi)D^{\mu-\lambda}\{(il)^p\}.$$

Here, we have

$$D^\nu\exp(il\xi)=\sum_{k=1}^{|\nu|}\exp(il\xi)\frac{(i\xi)^k}{k!}l_{\nu,k};$$

$$D^\nu\{(il)^p\}=\sum_{k=1}^{\min(p,|\nu|)}C_k^p i^k(il)^{p-k}l_{\nu,k},$$

where

$$l_{\nu,k}=\nu!\sum_{\rho_1+\cdots+\rho_k=\nu}\left(\frac{D^{\rho_1}l}{\rho_1!}\right)\left(\frac{D^{\rho_2}l}{\rho_2!}\right)\cdots\left(\frac{D^{\rho_k}l}{\rho_k!}\right),$$

where the summation is taken over all partitions of ν into k positive ($|\rho_i|\geqslant 1$) vectors.

Now, we want to estimate $l_{\nu,k}$. We shall write simply $|l_{\nu,k}|\leqslant$, instead of $\sup_{x\in U,\,\omega\in V_c}|l_{\nu,k}|\leqslant$. At first,

(4.11) $$|l_{\nu,k}|\leqslant\frac{L^k}{\rho'^{|\nu|-k}}\nu!\sum_{\rho_1+\cdots+\rho_k=\nu}1.$$

This last summation can be majorized by the coefficient of x^ν in $\left(\frac{1}{1-(x_1+\cdots+x_n)}\right)^k$, namely

$$\frac{1}{\nu!}k(k+1)(k+2)\cdots(k+|\nu|-1)=\frac{1}{\nu!}C_{|\nu|}^{k+|\nu|-1}|\nu|!.$$

Since, $C_{|\nu|}^{k+|\nu|-1}\leqslant C_{|\nu|}^{2|\nu|-1}<2^{2|\nu|-1}$, we have

(4.12) $$|l_{\nu,k}|\leqslant\frac{1}{2}\cdot|\nu|!\,(L\rho')^k\left(\frac{4}{\rho'}\right)^{|\nu|}.$$

Now, we apply Fundamental Lemma to the integral

$$\iint\exp(il\xi)\frac{(i\xi)^k}{k!}l_{\lambda,k}(il)^{p-k'}l_{\mu-\lambda,k'}D^{\nu-\mu}f\,\xi^{-2}d\xi\,d\omega.$$

This function can be majorized by

$$C_0\left\{\left(\frac{L\rho'}{\delta}\right)^k\frac{1}{2}\left(\frac{4}{\rho'}\right)^{|\lambda|}|\lambda|!\right\}\left\{L^{p-k'}(L\rho')^{k'}\frac{1}{2}\left(\frac{4}{\rho'}\right)^{|\mu-\lambda|}|\mu-\lambda|!\right\}\frac{(r+|\nu-\mu|)!}{\rho^{|\nu-\mu|}}A.$$

Since $\left(\frac{L\rho'}{\delta}\right)^k\leqslant\left(\frac{1}{2}\right)^k$ and fortiori $(L\rho')^{k'}\leqslant\left(\frac{1}{2}\right)^{k'}$, we see the following: In order to majorize $D^\nu g(x)$, we expand (4.10) by Leibniz, and we majorize there

$$l_{\nu,k} \text{ by } \left(\frac{1}{2}\right)^{k+1}|\nu|!\left(\frac{4}{\rho'}\right)^{|\nu|}; \; il \text{ by } L; \; D^{\nu-\mu}f \text{ by } \frac{(r+|\nu-\mu|)!}{\rho^{|\nu-\mu|}}A.$$

Therefore, we can marjorize $D^\nu g(x)$ by

$$C_0\sum_\mu C_\mu^\nu\left\{\sum_\lambda C_\lambda^\mu\left(|\lambda|!\left(\frac{4}{\rho'}\right)^{|\lambda|}\sum_{k=1}^{|\lambda|}\left(\frac{1}{2}\right)^{k+1}\right)\right.$$
$$\left.\times\left(|\mu-\lambda|!\left(\frac{4}{\rho'}\right)^{|\mu-\lambda|}\sum_{k'=1}^{p}C_{k'}^p\left(\frac{1}{2}\right)^{k'+1}L^{p-k'}\right)\right\}\frac{(r+|\nu-\mu|)!}{\rho^{|\nu-\mu|}}A.$$

Since the first factor $\leqslant\left(\frac{4}{\rho'}\right)^{|\lambda|}|\lambda|!\frac{1}{2}$, and the second factor $\leqslant\left(\frac{4}{\rho'}\right)^{|\mu-\lambda|}|\mu-\lambda|!\frac{1}{2}L^p$, the term between $\{\cdots\}$ is majorized by

$$\left(\frac{1}{2}\right)^2L^p\left(\frac{4}{\rho'}\right)^{|\mu|}\sum C_\lambda^\mu|\lambda|!\,|\mu-\lambda|!. \tag{4.13}$$

The summation is majorized by $\sum_{s=0}^{|\mu|}C_s^{|\mu|}s!\,(|\mu|-s)!$. Now by (4.8) $\frac{4}{\rho'}\leqslant\frac{1}{2\rho}$. Since $\left(\frac{1}{2\rho}\right)^{|\mu|}=\left(\frac{3}{4}\right)^{|\mu|}\left(\frac{2}{3\rho}\right)^{|\mu|}\leqslant\left(\frac{2}{3\rho}\right)^{|\mu|}\left(\frac{3}{4}\right)^{|\lambda|}$ for $|\lambda|\leqslant|\mu|$, (4.13) is marjorized by

$$|\mu|!\left(\frac{2}{3\rho}\right)^{|\mu|}L^p\frac{1}{4}\sum_{s=0}^{\infty}\left(\frac{3}{4}\right)^s=|\mu|!\left(\frac{2}{3\rho}\right)^{|\mu|}L^p.$$

Finally, we see that $D^\nu g(x)$ is marjorized by

$$C_0L^p\frac{(r+|\nu|)!}{\rho^{|\nu|}}A\sum_{s=0}^{\infty}\left(\frac{2}{3}\right)^s=3C_0L^p\frac{(r+|\nu|)!}{\rho^{|\nu|}}A.$$

This completes the proof (4.9) and therefore that of Proposition 1.

Proof of the Fundamental Lemma.

The proof is carried out in the same way as in [5]. Namely, we follow the way of P. D. Lax ([2]). However, we need a more precise argument. We adapted an argument of L. Schwartz to our case (cf. [9], Exposé 6).

At first, we remark that the integral

$$(4.1)\qquad g(x) = \int_1^\infty \xi^q d\xi \int_{\Omega_0} \exp(il\xi)(il)^p f(x\,;\,\omega)d\omega$$

has no more meaning as function when $q \geqslant -1$. We understand $g(x)$ in the following way: $g(x)$ is the distribution defined by, $(\varepsilon > 0)$,

$$(4.14)\qquad \lim_{\varepsilon\to 0}\int_0^\infty \xi^q d\xi \int_{\Omega_0} \exp\{(il-\varepsilon)\xi\} h(x\,;\,\omega)d\omega\,, \qquad h(x\,;\,\omega) = (il)^p f\,,$$

where the convergence is taken in the topology of distribution (i.e. $\mathfrak{D}'$). We can also define $g(x)$ as follows: For $\varphi(x)\in\mathfrak{D}$.

$$\langle g(x),\ \varphi(x)\rangle = \int_1^\infty \xi^q d\xi \int_{\Omega_0} d\omega \int \exp(il\xi)\, h(x\,;\,\omega)\varphi(x)\,dx\,.$$

We see that $g(x)$ is a continuous linear form on $\mathfrak{D}$. This fact relies on the condition $l_x \neq 0$. Namely, by taking l as one of local coordinates, we can show that the integral $\int_{\Omega_0} d\omega \int \exp(il\xi)\, h(x\,;\,\omega)\varphi(x)\,dx$ is majorized by $C_m \xi^{-m}$, where C_m is a constant majorized by $C(m, K)\sum_{|\nu|\leqslant m} \|D^\nu \varphi\|_{L^1}$ for $\varphi\in\mathfrak{D}_K$ (K is a compact), and here we can take m as large as we like.

Now we return to our purpose. We remark that in (4.14) we can take $(il-\varepsilon)^p$ instead of $(il)^p$. Then, if $p\geqslant 1$, the integration by parts gives

$$\int_1^\infty \xi^q \exp\{(il-\varepsilon)\xi\}(il-\varepsilon)^p d\xi$$
$$= \exp(il-\varepsilon)(il-\varepsilon)^{p-1} - q\int_1^\infty \exp\{(il-\varepsilon)\xi\}(il-\varepsilon)^{p-1}\xi^{q-1}d\xi\,,$$

here we assume $q\geqslant 1$. We denote this relation by

$$\int_1^\infty \xi^q \exp(il\xi)(il)^p d\xi \approx \exp(il)(il)^{q-1} - q\int_1^\infty \exp(il\xi)(il)^{p-1}\xi^{q-1}d\xi\,.$$

Therefore, for $p > q \geqslant 0$, the integration by parts yields

$$(4.15)\quad \int_1^\infty \exp(il\xi)(il)^p \xi^q d\xi \approx \exp(il)\sum_{j=1}^{q+1}(-1)^j q(q-1)\cdots(q-j+2)(il)^{p-j},$$

where $0!=1$, $(-1)!=1$. Therefore we have

(4.16) $$\int_1^{\infty} \xi^q d\xi \int_{\Omega_0} \exp(il\xi)(il)^p f(x;\omega)\,d\omega$$
$$=\begin{cases} \displaystyle\sum_{j=1}^{q+1}(-1)^j q(q-1)\cdots(q-j+2)\int_{\Omega_0}\exp(il)(il)^{p-j}f(x;\omega)\,d\omega\,, & \text{for } p>q\geqslant 0\,, \\ \displaystyle\sum_{j=1}^{p}(\text{the same term as above})+q(q-1)\cdots(q-p+1)\int_1^{\infty}\xi^{q-p}d\xi \\ \qquad\times\displaystyle\int_{\Omega_0}\exp(il\xi)f(x;\omega)\,d\omega\,, & \text{for } q\geqslant p\geqslant 0\,. \end{cases}$$

We see here that in the first case (i.e. $p>q\geqslant 0$), the integral is a continuous function.

Now consider

(4.17) $$h(x)=\int_1^{\infty}\xi^m d\xi\int_{\Omega_0}\exp(il\xi)f(x;\omega)\,d\omega\,,\qquad m=q-p\,.$$

We want to prove that there exist $\delta\,(<1)$ and C_0 such that

(4.18) $$\sup_{x\in U}|h(x)|\leqslant\frac{(m+2)!}{\delta^{m+2}}M\{f;\,V_c\}C_0'\,,\qquad\text{for}\quad m\geqslant -1\,,$$

where δ and C_0' depend only on U and V_c, they don't depend on m and f.

At first we assume that $l(x;\omega)$ does not vanish on the sphere Ω_0, namely

(4.19) $$|l(x;\omega)|\geqslant\delta'>0\qquad\text{for}\quad x\in U\text{ and }\omega\in\Omega_0\,.$$

We assume here $\delta'<1$. We remark that $h(x)$ is defined by

$$h(x)=\lim_{\varepsilon\to 0}\int_1^{\infty}\xi^m d\xi\int\exp\{(il-\varepsilon)\xi\}f(x;\omega)\,d\omega\,,$$

where the convergence is taken in the topology of $\mathcal{D}'$. Then, by integration by parts, we have

(4.20)
$$\int_1^{\infty}\exp\{(il-\varepsilon)\xi\}\xi^m d\xi=\exp(il-\varepsilon)\sum_{i=1}^{m+1}\frac{m(m-1)\cdots(m-i+2)}{(il-\varepsilon)^i}(-1)^i\,.$$

Hence

$$\sup_{x\in U}|h(x)|\leqslant\frac{(m+1)!}{\delta^{m+1}}\sup_{x\in U,\omega\in\Omega_0}|f(x,\omega)|\,.$$

If we consider the case $m=-1,\,-2$, we have

(4.21) $|h(x)| \leqslant 2\frac{(m+2)!}{\delta^{m+2}} \sup |f(x;\omega)|\,|\Omega_0|, \qquad \text{for} \quad m \geqslant -2.$

Consider now the general situation. We denote $V_x = \{\omega\,;\, l(x\,;\,\omega) = 0\}$. The hypothesis that U (we can assume hereafter U small set) has a positive distance from the characteristic conoid implies that, on V_x, $l_\omega \neq 0$ (see [2] p. 645). Then, we can take l as one of the local coordinates in the neighborhood of V_x on Ω_0. We introduce also analytic local coordinates $(u_1, \cdots, u_{n-2})$ of V_x, then $(u_1, \cdots, u_{n-2}, l) = (u, l)$ forms analytic local coordinates on Ω_0 (see our previous paper, [5]). We can cover the V_x by such a finite number of local coordinates, and define a partition of unity subordinate to this covering: $\sum_{i=1}^{p} \alpha_i(u) \equiv 1$. Then, take a function $\beta(l)$ of small support, which is 1 in a small neighborhood of $l=0$, and $0 \leqslant \beta(l) \leqslant 1$. Then $h(x) = h_1(x) + \sum_{j=1}^{p} h_2^{(j)}(x)$, where

(4.22)
$$\begin{cases} h_1(x) = \int_1^\infty \xi^m d\xi \int_{\Omega_0} \exp(il\xi)[1-\beta(l)] f d\omega, \\ h_2^{(j)}(x) = \int_1^\infty \xi^m d\xi \int_{\Omega_0} \exp(il\xi)\alpha_j(u)\beta(l) f d\omega. \end{cases}$$

About $h_1(x)$ there is nothing to say, because this case is essentially the same one as $l \neq 0$ on Ω_0. Now we consider $h_2^{(j)}$. $d\omega = J_j(u, l)\,du dl$, where J_j is an analytic function.

$$f d\omega = \tilde{f}(u, l) J_j(u, l)\,du dl \equiv f_j(u, l)\,du dl.$$

Here we have

(4.23) $$|D_l^k f_i(u, l)| \leqslant \frac{k!}{\rho^k} M\{f; V_c\} J, \qquad \begin{matrix} i = 1, \cdots, p. \\ k = 0, 1, 2, \cdots. \end{matrix}$$

where ρ does not depend on f; it depends on U and V_c.

$$h_2^{(j)}(x) = \int_1^\infty \xi^m d\xi \int \alpha_j(u)\,du \int \exp(il\xi)\beta(l) f_j(u, l)\,dl.$$

By integration by parts, the last integral is equal to

$$\frac{1}{(-i)^{m+2}} \int \exp(il\xi)\, D_l^{m+2}[\beta(l) f_j(u, l)]\,dl$$
$$= \frac{1}{(-i)^{m+2}} \int \exp(il\xi) \{\beta(l) D_l^{m+2} f_j + \sum_{s \neq 0} C_s^{m+2} D_l^s \beta D_l^{m+2-s} f_j\}\,dl.$$

Now look at the integral :

$$\int \exp(il\xi) D_l^s \beta D_l^{m+2-s} f_j \, dl .$$

Integration by parts gives,

$$= (-1)^{s-1} \int \beta'(l) D_l^{s-1} \{\exp(il\xi) D_l^{m+2-s} f_i\} \, dl .$$

Since

$$D_l^{s-1}\{\exp(il\xi) D_l^{m+2-s} f_j\} = \exp(il\xi) \sum_t C_t^{s-1} (i\xi)^t D_l^{m+1-t} f_j ,$$

taking into account of the fact that $|l| \geqslant \delta'$, where $\beta'(l) \neq 0$, and applying (4. 21), we see that

$$\int_1^\infty \xi^{-2} d\xi \int \alpha_j(u) \, du \int \exp(il\xi) D_l^s \beta D_l^{m+2-s} f_j \, dl$$

is majorized by

$$2 \sum_t C_t^{s-1} \frac{t!}{\delta'^t} \frac{(m+1-t)!}{\rho^{m+1-t}} M\{f;\, V_c\} S_j J , \tag{4. 24}$$

where $S_j = \int \alpha_j(u) |\beta'(l)| \, du dl$.

Denote

$$\delta = \frac{1}{2} \min(\delta', \rho) , \tag{4. 25}$$

then, (4. 24) is majorized by

$$2 \frac{(m+1)!}{(2\delta)^{m+1}} (s-1) M\{f;\, V_c\} S_j J .$$

Finally, $h_2^{(j)}(x)$ is majorized by

$$2 \frac{(m+2)!}{(2\delta)^{m+2}} M\{f;\, V_c\} \left[S^0 + \frac{2\delta}{m+2} S_j J \sum_{s \neq 0} C_s^m (s-1) \right] ,$$

where $S^0 = \int_{\beta \neq 0} d\omega$. The quantity between [ ] is majorized by $2^{m+2}(S^0 + JS_j) = 2^{m+2} S'_j$.

Therefore

$$|h_2^{(j)}(x)| \leqslant 2 \frac{(m+2)!}{\delta^{m+2}} M\{f;\, V_c\} S'_j , \qquad j = 1, 2, \cdots, p .$$

Since

$|h_1(x)|\leqslant 2\frac{(m+2)!}{\delta^{m+2}}\sup|f(x,\omega)|\,|\Omega_0|$, we have finally

$$|h(x)|\leqslant\frac{(m+2)!}{\delta^{m+2}}M\{f;\,V_c\}\,C_0\,. \tag{4.26}$$

where $C_0=2|\Omega_0|+2\sum_{j=1}^{p}S'_j$. This is nothing but (4.18).

Now we want to prove (4.3). From (4.16) we get

i) for $p>q\geqslant 0$, $|g(x)|\leqslant L^p\ q!\,|\Omega_0|\sup_{\omega\in\Omega_0}|f(x\,;\omega)|$;

ii) for $q\geqslant p\geqslant 0$,

$$\sup_{x\in U}|g(x)|\leqslant L^p q!\,|\Omega_0|\sup_{x,\omega}|f(x\,;\omega)|+\frac{(q+2)!}{\delta^{q+2}}\cdot M\{f;\,V_c\}C'_0\,.$$

Therefore, if we take

$$C_0\geqslant|\Omega_0|+C'_0\,, \tag{4.27}$$

we get (4.3). Now we look at the case $q=-1,-2$. For $q=-2$, (4.3) is evidently true. For $q=-1$, we have

$$\int_1^\infty\exp(il\xi)(il)^p\xi^{-1}d\xi\approx\begin{cases}-(il)^{p-1}\exp(il)+\int_1^\infty\exp(il\xi)(il)^{p-1}\xi^{-2}d\xi\,, & \text{for } p\geqslant 1\,;\\ \int_1^\infty\exp(il\xi)\xi^{-1}d\xi\,, & \text{for } p=0\,.\end{cases}$$

Therefore, for $p\geqslant 1$,

$$|g(x)|\leqslant 2L^{p-1}|\Omega_0|\sup|f(x\,;\omega)|\leqslant L^p|\Omega_0|\sup|f(x\,;\omega)|\,;$$

for $p=0$, by (4.18) we have

$$\sup_{x\in U}|g(x)|\leqslant\frac{1}{\delta}M\{f;\,V_c\}C'_0\,.$$

This completes the proof of Fundamental Lemma.

5. Proof of Theorem 1.*)

We start from the

*) The part (p. 347–352) is the same as that of our previous paper (i.e. p. 275–281): Solutions nulles et solutions non analytiques, J. Math. Kyoto Univ. 1-2 (1962). To make this article self-contained, we reproduce it.

Lemma 1. *Let $a(x)$ and $b(x)$ be two analytic functions, we assume*

$$|D^{\nu}a(x)|\leqslant\frac{(r+|\nu|)!}{(k\rho)^{|\nu|}}A, \qquad k>1,$$
$$|D^{\nu}b(x)|\leqslant\frac{(s+|\nu|)!}{\rho^{|\nu|}}B,$$

where r and s are non negative integers. Then we have the following estimate:

$$|D^{\nu}(ab)(x)|\leqslant\frac{(r+s+|\nu|)!}{\rho^{|\nu|}}(k/k-1)AB/C_r^{r+s}.$$

Proof:

$D^{\nu}(ab)=\sum_{\mu}C_{\mu}^{\nu}D^{\mu}a\cdot D^{\nu-\mu}b$. Since $\sum_{|\mu|=p}C_{\mu}^{\nu}\leqslant C_{p}^{|\nu|}$, we have

$$|D^{\nu}(ab)|\leqslant\frac{AB}{\rho^{|\nu|}}\sum_{p=0}^{|\nu|}C_{p}^{|\nu|}(r+p)!\,(s+|\nu|-p)!\,(1/k)^{p}.$$

Now $C_{p}^{|\nu|}(r+p)!\,(s+|\nu|-p)!=C_{p}^{|\nu|}\dfrac{(r+s+|\nu|)!}{C_{r+p}^{r+s+|\nu|}}\leqslant(r+s+|\nu|)!/C_r^{r+s}$,

because $\qquad C_{r+p}^{r+s+|\nu|}/C_{p}^{|\nu|}\geqslant C_r^{r+s}$.

Hence,

$$|D^{\nu}(ab)|\leqslant\frac{AB}{C_r^{r+s}}\frac{(r+s+|\nu|)!}{\rho^{|\nu|}}\sum_{p=0}^{|\nu|}(1/k)^{p},$$

and the last factor $\leqslant k/k-1$.

Consider the equation $L[u]=f(x,t)$, where

$$L=\frac{\partial}{\partial t}+\sum_{i=1}^{n}a_i(x,t)\frac{\partial}{\partial x_i}+b(x,t),$$

where $a_i(x,t)$ are functions with real values. We assume: a_i and b are real analytic functions, namely:

$$(5.1)\qquad |D_{x,t}^{\nu}a_i(x,t)|\leqslant\frac{(|\nu|-1)!}{(3\rho)^{|\nu|-1}}\gamma, \qquad |\nu|\geqslant1,\ |a_i(x,t)|\leqslant\gamma_0,$$
$$|D_{x,t}^{\nu}b(x,t)|\leqslant\frac{|\nu|!}{(3\rho)^{|\nu|}}\gamma, \qquad \nu\geqslant0.$$

Under this condition, we want to estimate the solution

$$(5.2)\qquad L[u]=f, \quad \text{with inital data } 0: u(x,0)=0.$$

Lemma 2. *Assume*

(5.3) $$|D_x^\nu f(x,t)| \leqslant \frac{(r+|\nu|)!}{\rho^{|\nu|}} \exp(\gamma t) K(t)^{r+|\nu|} A, \quad \text{where} \quad r \geqslant 1.$$

Then

(5.4) $$|D_x^\nu u(x,t)| \leqslant 2\frac{(r+|\nu|-1)!}{\rho^{|\nu|}} \exp(\gamma t) K(t)^{r+|\nu|} A/\gamma n,$$

where $K(t) = \exp(\gamma n t)(1+\gamma n t)$.

Proof. At first,

$$|u(x,t)| \leqslant r! \exp(\gamma t) A \int_0^t K(s)^r ds$$
$$\leqslant r! \exp(\gamma t)(1+\gamma n t)^r \frac{\exp(r\gamma n t)}{r(\gamma n)},$$

this proves (5.4) for $\nu = 0$.

Now we want to prove (5.4) as follows: Denote $u_m(t) = \max_{|\nu|=m} \sup_x |D_x^\nu u(x,t)|$.

Then, we shall have, taking the differentiation of (5.2), and taking account of (5.1),[2]

$$\left(\frac{d}{dt} - \gamma - mn\gamma\right) u_m(t) \leqslant f_m(t) + \sum_{p=1}^{m} u_{m-p}(t)(C_{p+1}^m n + C_p^m)\frac{p!}{(3\rho)^p}\gamma.$$

Assume here that for $u_{m-p}(t)$, $p \geqslant 1$, the inequality (5.4) is true. Hence, in the above inequality, the term $u_{m-p}(t) \cdots$, can be majorized by

$$(1/3)^p\left(C_{p+1}^m + \frac{1}{n} C_p^m\right)\frac{(r+m-1-p)!\,p!}{\rho^m} 2A \exp(\gamma t) K(t)^{r+m-p}.$$

---

2) The argument is the same as given in [6], even simpler. Let us reproduce it. We operate $D_i\left(=\frac{\partial}{\partial x_i}\right)$ to (5.2). Then

$$L[D_i u] + \sum_{k=1}^{n} (D_i a_k)(D_k u) + (D_i b)(u) = D_i f.$$

In general we have

$$L[D_{i_1}\cdots D_{i_m} u] + \sum_{k=1}^{n}\sum_{p=1}^{m} (D_{i_p} a_k)(D_{i_1}\cdots \hat{D}_{i_p}\cdots D_{i_m} D_k u) + \sum_k \sum_{p,q} (D_{i_p} D_{i_q} a_k)(D_{i_1}\cdots \hat{D}_{i_p}\cdots \hat{D}_{i_q}$$
$$\cdots D_{i_m} D_k u) + \cdots + \sum_k (D_{i_1}\cdots D_{i_m} a_k)(D_k u) +$$
$$+ \sum_{p=1}^{m} (D_{i_p} b)(D_{i_1}\cdots \hat{D}_{i_p}\cdots D_{i_m} u) + \sum_{p,q} (D_{i_p} D_{i_q} b)(D_{i_1}\cdots \hat{D}_{i_p}\cdots \hat{D}_{i_q}\cdots D_{i_m} u) + \cdots$$
$$+ (D_{i_1}\cdots D_{i_m} b)(u) = D_{i_1}\cdots D_{i_m} f.$$

Taking account of (5.1), we have the desired differential inequality.

Since $(r+m-1-p)!\,p!=(r+m-1)!/C_p^{r+m-1}$, and $C_{p+1}^m\leqslant mC_p^m$, $C_p^{r+m-1}\geqslant C_p^m$, we have

$$\left(C_{p+1}^m+\frac{1}{n}C_p^m\right)(r+m-1-p)!\,p!\leqslant(r+m-1)!\left(m+\frac{1}{n}\right)$$
$$\leqslant(r+m-1)!\,(m+1)\,.$$

Hence

$$\sum_{p=1}^m u_{m-p}\cdots\leqslant\frac{(r+m-1)!}{\rho^m}2A\exp(\gamma t)K(t)^{r+m-1}(m+1)\sum_{p=1}^m\left(\frac{1}{3}\right)^p.$$

Since $f_m(t)\leqslant\frac{(r+m)!}{\rho^m}\exp(\gamma t)K(t)^{r+m}A$, the integration gives,

$$u_m(t)\leqslant\frac{(r+m-1)!}{\rho^m}\exp(\gamma t)K(t)^{r+m}/\gamma n+\frac{(r+m-1)!}{\rho^m}\exp(\gamma t)K(t)^{r+m}A$$
$$\times(m+1)/(r+m)(\gamma n)\,.$$

Since $r\geqslant1$, $(m+1)/(r+m)\leqslant1$, the above inequality shows that (5.4) is true for $|\nu|\leqslant m$.

Lemma 3. *Under the same condition as lemma* 2, *consider the solution u of $L[u]=0$. We assume on the intial value*

$$|D_x^\nu u(x,0)|\leqslant\frac{(r+|\nu|)!}{\rho^{|\nu|}}A\,,\quad r\geqslant0,\ \textit{then we have}$$

$$|D_x^\nu u(x,t)|\leqslant2\frac{(r+|\nu|)!}{\rho^{|\nu|}}\exp(\gamma t)K(t)^{|\nu|}A\,.$$

Since the proof can be carried out in the same way as the previous, we omit the proof.

Proposition 3. *Consider the solution $u(x,t)$ of $L[u]=f(x,t)$ with the condition $u(x,0)=0$. We assume*

$$(5.5)\quad |D_x^\nu D_t^q f(x,t)|\leqslant\frac{(r+q+|\nu|)!}{\rho^{q+|\nu|}}\exp(\gamma t)K(t)^{r+q+|\nu|}(\gamma n)^qA\,,\quad r\geqslant1\,,$$

then

$$(5.6)\quad |D_x^\nu D_t^q u(x,t)|\leqslant2\frac{(r-1+q+|\nu|)!}{\rho^{q+|\nu|}}\exp(\gamma t)K(t)^{r+q+|\nu|}(\gamma n)^qA\,,$$

where γ and ρ satisfy, in addition to (5.1), *the following condition*:

$$(5.7)\qquad \gamma\geqslant\min(6\gamma_0,\ 27)\,;\quad \rho\leqslant1/18\,.$$

Proof. For $q=0$, (5.6) is nothing but Lemma 2. We are going to prove this lemma by induction on q. Let us assume that (5.6) is true for $q=0, 1, \cdots, p-1$, and we want to prove for $q=p$. $D_t^p u(x, t)$ is the solution of

$$(5.8)\qquad L[D_t^p u] = D_t^p f - \sum_{s=0}^{p-1} C_s^p \{\sum_{i=1}^{n} (D_t^{p-s} a_i)(D_t^s D_i u) + (D_t^{p-s} b)(D_t^s u)\} .$$

We put this second hand, $=D_t^p f+\varphi(x, t)$, and divide $D_t^p u$ in three functions:

$$D_t^p u = u_0+u_1+u_2 ,$$

where

1) $u_0(x, t)$ is defined by

$$L[u_0] = 0 , \quad u_0(x, 0) = D_t^p u(x, 0) ,$$

2) u_1, u_2 are defined by

$$\left.\begin{aligned} L[u_1] &= D_t^p f \\ L[u_2] &= \varphi \end{aligned}\right\} \text{ with zero initial condition.}$$

At first, consider $u_1(x, t)$. By Lemma 2, we have

$$(5.9)\qquad |D_x^\nu u_1(x, t)| \leqslant \frac{(r+p-1+|\nu|)!}{\rho^{p+|\nu|}} \exp(\gamma t) K(t)^{r+p+|\nu|} (\gamma n)^p A\left(\frac{2}{\gamma n}\right) .$$

Now consider $u_2(x, t)$. We want to estimate $D_x^\nu \varphi(x, t)$. Take the term $(D_t^{p-s} a_i)(D_t^s D_i u)$ in (5.8). The derivatives D_x^ν of this function is majorized, using Lemma 1, by

$$\frac{(r-1+p+|\nu|)!}{\rho^{p+|\nu|}} \left(\frac{1}{3}\right)^{p-s-1} 2\gamma / C_{p+s}^{r+p-1} \times \exp(\gamma t) K(t)^{r+s+|\nu|} (\gamma n)^s A .$$

For the simplicity, we write this fact by,

$$(D_t^{p-s} a_i)(D_t^s D_i u) \underset{\nu}{\rightarrow} (r-1+p+|\nu|)! \left(\frac{1}{3}\right)^{p-s-1} 2\gamma (\gamma n)^s / C_{r+s}^{r+p-1} .$$

In the same way, we have

$$(D_t^{p-s} b)(D_t^s u) \underset{\nu}{\rightarrow} (r-1+p+|\nu|)! \left(\frac{1}{3}\right)^{p-s} 2\gamma (\gamma n)^s / C_{p-s}^{r+p-1} .$$

Since $C_s^p / C_{r+s}^{r+p-1} \leqslant C_s^p / C_s^{p-1} \leqslant p$, and $C_{p-s}^{r+p-1} \geqslant C_{p-s}^p = C_s^p$, we see that $D_x^\nu \varphi(x, t)$ is majorized by

$$\frac{(r-1+p+|\nu|)!}{\rho^{p+|\nu|}} \exp(\gamma t) K(t)^{r-1+p+|\nu|} 2\left(\frac{3}{2}p+\frac{1}{2n}\right)(\gamma n)(\gamma n)^{p-1}A.$$

Then by Lemma 2, we have

(5.10)

$$|D_x^\nu u_2(x, t)| \leqslant \frac{(r-1+p+|\nu|)!}{\rho^{p+|\nu|}} \exp(\gamma t) K(t)^{r+p+|\nu|}(\gamma n)^{p-1} A \left\{ \frac{4\left(\frac{3}{2p}+\frac{1}{2n}\right)}{r+p-1+|\nu|} \right\},$$

where the last factor is majorized by $4\left(\frac{3}{2}+\frac{1}{2pn}\right) \leqslant 4.2=8.$

Finally consider $u_0(x, t)$. Now we want to estimate $D_x^\nu u_0(x, t)$ by means of Lemma 3. For this purpose, we are going to estimate

$$D_x^\nu u_0(x, 0) \equiv D_x^\nu D_t^p u(x, 0).$$

(5.11)

$$D_t^p u(x, 0) = D_t^{p-1} f(x, 0) - \sum_{s=0}^{p-2} C_t^{p-1} \left\{ \sum_{i=1}^{n} (D_t^{p-1-s} a_i)(D_t^s D_i u) + (D_t^{p-1-s})(D_t^s u) \right\}$$
$$- \left(\sum_{i=1}^{n} a_i (D_t^{p-1} D_i u) + b(D_t^{p-1} u) \right).$$

We put this second member $= D_t^{p-1} f + v_1(x) + v_2(x)$. At first, we remark that the estimate of $D_x^\nu v_1(x)$ was obtained previously, namely

$$(5.12) \qquad |D^\nu v_1(x)| \leqslant \frac{(r-2+p+|\nu|)!}{\rho^{p-1+|\nu|}} 2\left\{\frac{3}{2}(p-1)+\frac{1}{n}\right\}(\gamma n)^{p-1}A.$$

Or, since $2\left\{\frac{3}{2}(p-1)+\frac{1}{n}\right\} \leqslant 3p, \quad (r+p-1+|\nu|) \geqslant p,$

$$(5.13) \qquad |D^\nu v_1(x)| \leqslant \frac{(r+p-1+|\nu|)!}{\rho^{p-1+|\nu|}} (\gamma n)^{p-1} A \cdot 3.$$

$$(5.14) \qquad |D^\nu v_2(x)| \leqslant \frac{(r+p-1+|\nu|)!}{\rho^{p+|\nu|}} (\gamma n)^p \left(2\frac{\gamma_0}{\gamma}+6\rho\right).$$

In fact,

$$D^\nu(a_i \cdot D_t^{p-1} D_i u) = a_i(D_x^\nu D_t^{p-1} D_i u) + \sum_{\mu \neq 0} C_\mu^\nu (D^\mu a_i)(D_x^{\nu-\mu} D_t^{p-1} D_i u).$$

The first term of the second hand is majorized by

$$2\gamma_0 \frac{(r+p-1+|\nu|)!}{\rho^{p+|\nu|}} (\gamma n)^{p-1} A,$$

the second term is majorized by

$$2\gamma\left(\frac{3}{2}\right)\frac{(r+p-2+|\nu|)!}{\rho^{p-1+|\nu|}}(\gamma n)^{p-1}A\,,$$

and $D^{\nu}\{b(D_t^{p-1}u)\}$ is majorized by

$$2\gamma\left(\frac{3}{2}\right)\frac{(r+p-1+|\nu|)!}{\rho^{p-1+|\nu|}}(\gamma n)^{p-1}A\,.$$

Since

$$|D_x^{\nu}D_t^{p-1}f(x,0)|\leqslant\frac{(r+p-1+|\nu|)!}{\rho^{p+|\nu|}}(\gamma n)^{p-1}A\,,\quad\text{we have}$$

$$|D_x^{\nu}u_0(x,0)|\leqslant\frac{(r+p-1+|\nu|)!}{\rho^{p+|\nu|}}(\gamma n)^{p}\left(\frac{4}{\gamma n}+2\frac{\gamma_0}{\gamma}+6\rho\right)A\,.$$

Hence, by using Lemma 3, we have

(5.15)
$$|D_x^{\nu}u_0(x,t)|\leqslant\frac{(r+p-1+|\nu|)!}{\rho^{p+|\nu|}}(\gamma n)^{p}\exp(\gamma t)K(t)^{|\nu|}2\left(\frac{4}{\gamma n}+2\frac{\gamma_0}{\gamma}+6\rho\right)A\,.$$

Adding (5.9), (5.10), and (5.15) we have finally

(5.16)
$$|D_x^{\nu}D_t^{p}u(x,t)|\leqslant\frac{(r+p-1+|\nu|)!}{\rho^{p+|\nu|}}\exp(\gamma t)K(t)^{r+p+|\nu|}(\gamma n)^{p}2\left(\frac{9}{\gamma n}+2\frac{\gamma_0}{\gamma}+6\rho\right)A\,.$$

By the condition on γ and ρ, mentioned in the statement of this lemma, the last factor is less than 2, which proves (5.6) for $q=p$. Our proof is thus complete.

Proposition 4. *Consider the solution $u(x,t)$ of $L[u]=0$. Concerning the initial value $u(x,0)$, we assume*

(5.17) $\quad |D_x^{\nu}u(x,0)|\leqslant\frac{(r+|\nu|)!}{\rho^{|\nu|}}A\,,\quad r\geqslant 0,$ *then*

(5.18) $\quad |D_x^{\nu}D_t^{q}u(x,t)|\leqslant 2\frac{(r+q+|\nu|)!}{\rho^{q+|\nu|}}\exp(\gamma t)K(t)^{r+q+|\nu|}(\gamma n)^{q}A\,,$

where γ and ρ are assumed to satisfy (5.7) *of Proposition* 3.

Proof. For $q=0$, (5.18) is nothing but Lemma 3. For $q\geqslant 1$, the proof is almost same as the previous Lemma, so we omit it.

Now we turn to the estimate of σ_m^{ij}. Remark that σ_m^{ij} are defined as follows: For $i \neq j$,

$$\sigma_m^{ij}(x, t; \omega) = s_{ij}(x, t; \omega)\{(\sigma_{m-1}^{ij})_t + \sum_{k=1}^{N} L_k^{(ij)}[\sigma_{m-1}^{ik}]\}, \tag{5.19}$$

where $L_k^{(ij)}$ is the first order differential operator of the form:

$$L_k^{(ij)}[u] = \sum_{\nu=1}^{n} p_{k\nu}^{(ij)}(x, t; \omega)\frac{\partial u}{\partial x_\nu} + q_k^{(ij)}(x, t; \omega)u,$$

where all the coefficients, together with s_{ij}, are analytic functions of (x, t, ω).

After (5.19), we define finally σ_m^{ii} as the solution of

$$L_i[\sigma] = -\sum_{k;\, k\neq i} \mathrm{L}_k^{(ii)} L[\sigma_m^{ik}], \tag{5.20}$$

with the initial condition:

$$\sigma_m^{ii}(x, 0; \omega) = -\sum_{k;\, k\neq i} \sigma_m^{ki}(x, 0; \omega). \tag{5.21}$$

$\sigma_0^{ij}(x, t; \omega)$ are defined as follows: $\sigma_0^{ij}=0$, for $i \neq j$. σ_0^{ii} are defined as the solution of (5.20), namely,

$$L_i[\sigma_0^{ii}] = 0, \quad \text{with the initial value} \tag{5.22}$$

$\sigma_0^{ii}(x, 0; \omega) = \sigma^i(x; \omega)$, where

$$\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \sum_{i=1}^{N} \sigma^i(x, \omega) R_i(x, 0; \omega).$$

Then we have the following

Proposition 5. *We assume*

$$|D_x^\nu \sigma_i(x; \omega)| \leqslant \frac{|\nu|!}{\rho^{|\nu|}} A, \quad \textit{then}$$

(5.23)

$$|D_x^\nu D_t^p \sigma_m^{ij}(x, t; \omega)| \leqslant \frac{(m+p+|\nu|)!}{\rho^{m+p+|\nu|}} \exp(\gamma t) K(t)^{m+p+|\nu|} (\gamma n)^{m+p} (2A) C_0^m,$$

for $0 \leqslant t \leqslant 1$, *where* C_0 *is a large constant conveniently chosen.*

Proof. For $m=0$, (5. 22) and the assumption, yields, by Proposition 4,

$$|D_x^{\nu} D_t^{p} \sigma_0^{ij}| \leqslant 2 \frac{(p+|\nu|)!}{\rho^{p+|\nu|}} \exp(\gamma t) K(t)^{p+|\nu|} (\gamma n)^p A .$$

We assume now that (5. 23) is true for σ_0^{ij}, σ_1^{ij}, $\cdots$, σ_{m-1}^{ij}, and prove (5. 23) for σ_m^{ij}. We can assume that all the coefficients of $L_m^{(ij)}$, together with s_{ij} have the radii of convergence greater than 3ρ. Then, from (5. 19), by using Lemma 1, we have for $i \neq j$,

(5. 24)

$$|D_x^{\nu} D_t^{p} \sigma_{m+1}^{ij}| \leqslant M \frac{(m+1+p+|\nu|)!}{\rho^{m+1+p+|\nu|}} \exp(\gamma t) K(t)^{m+1+p+|\nu|} (\gamma n)^{m+1+p} A C_0^m ,$$

where M is determined by $L_k^{(i)}$, s_{ij}.

Next, putting

$f_{m+1}^{(i)} = -\sum_{k \neq i} L_k^{(i)} [\sigma_{m+1}^{ik}]$, we divide σ_{m+1}^{ii} in two functions:

$\sigma_{m+1}^{ii} = \delta_{m+1}^{(i)} + \delta_{m+1}^{\prime(i)}$, where

$L_i^{(i)}[\delta_{m+1}^{(i)}] = f_{m+1}^{(i)}$, with zero initital value: $\delta_{m+1}^{(i)}(x, 0; \omega) = 0$;

$L_i^{(i)}[\delta_{m+1}^{\prime(i)}] = 0$, with the given initial data given by (5. 21): $\sigma_{m+1}^{ii}(x, 0; \omega) = -\sum_{k \neq i} \sigma_{m+1}^{ki}(x, 0; \omega)$.

By the hypothesis, we have

$$|D_x^{\nu} D_t^{p} f_{m+1}^{(i)}| \leqslant M^2 \frac{(m+2+p+|\nu|)!}{\rho^{m+2+p+|\nu|}} \exp(\gamma t) K(t)^{m+2+p+|\nu|} (\gamma n)^{m+1+p} A C_0^m ,$$

By using Proposition 3, we have

(5. 25)

$$|D_x^{\nu} D_t^{p} \delta_{m+1}^{(i)}| \leqslant 2M^2 \frac{(m+1+p+|\nu|)!}{\rho^{m+2+p+|\nu|}} \exp(\gamma t) K(t)^{m+2+p+|\nu|} (\gamma n)^{m+1+p} A C_0^m .$$

On the other hand

$$|D_x^{\nu} \delta_{m+1}^{\prime(i)}(x, 0; \omega)| \leqslant M(N-1) \frac{(m+1+|\nu|)!}{\rho^{m+1+|\nu|}} (\gamma n)^{m+1} A C_0^m .$$

By using Proposition 4, we have

(5. 26)

$$|D_x^{\nu} D_t^{p} \delta_{m+1}^{\prime(i)}| \leqslant 2M(N-1) \frac{(m+1+p+|\nu|)!}{\rho^{m+1+p+|\nu|}} \exp(\gamma t) K(t)^{m+1+p+|\nu|} (\gamma n)^{m+1} A C_0^m .$$

Adding (5.25) and (5.26), we have

$$(5.27) \qquad |D_x^\nu D_t^p \sigma_{m+1}^{ii}| \leqslant \frac{(m+1+p+|\nu|)!}{\rho^{m+1+p+|\nu|}} \times \exp(\gamma t) K(t)^{m+1+p+|\nu|} (\gamma n)^{m+1+p} 2AC_0^m \left\{ \frac{M^2}{\rho} K(1) + M(N-1) \right\},$$

for $0 \leqslant t \leqslant 1$. Therefore, if we choose C_0 in such a way that

$$(5.28) \qquad C_0 \geqslant \frac{M^2}{\rho} K(1) + M(N-1),$$

(5.27) shows that (5.23) is true for σ_{m+1}^{ii}, which completes our proof.

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