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A special class of spherically symmetric space-times and their imbeddings

By

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During the year 1961 a series of papers dealing with the problem of imbedding of the Schwarzschild space-time was written by one of the authors and other two, and many valuable results were obtained [3]. The process by means of which the problem was completely studied was recently applied to a treatment of the imbedding of spherically symmetric space-time (abbreviated s. s. space) [4]. The present paper is written as an addition to the paper [4].

It is generally known that the fundamental form of s. s. space, with respect to the time coordinate t and spherical ones r, θ, φ, is given by

$$eds^2 = G(t, r)dt^2 - A(t, r)dr^2 - B(t, r)(d\theta^2 + \sin^2\theta d\varphi^2).$$

If the function $B(t, r)$ is constant, such a space is called S_{11} space. On the other hand, if B is not constant, it is shown that there exists a transformation of coordinates such that B is reduced to r^2 [7, I]. The space is denoted by S_1 space. It has long been known that any s.s. space is of class at most 2 in the sense of imbedding [2].

Among S_1 spaces those of which the function $A(t, r)=1$ have some special properties. For example, those spaces are not of class 1, as proved by H. Takeno [7, III]. In the previous paper [4] we excluded a discussion of those spaces, because the general method used in the paper was not applicable to them.

The exceptional case will be treated in the present paper, and thus we consider s.s. space, the fundamental form being

$$eds^2 = G(t, r)dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{0.1}$$

A part of this form $dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$ is clearly the fundamental form of an euclidean 3-space with respect to spherical coordinates. In this point of view we shall generalize the dimensional number of (0.1) and thus obtain a notion of a *G-extension of an euclidean n-space* E^n as follows. First we consider the product manifold $R \times E^n$, where R is a field of real numbers. Then we define on this manifold a Lorentz metric

$$eds^2 = G(x^0, r)(dx^0)^2 - \sum_{i=1}^{n}(dx^i)^2, \tag{0.2}$$

where $x^0 \in R$, $r = \sqrt{\sum(x^i)^2}$, G is a positive-valued function of x^0 and r, and x^i, $i = 1, \cdots, n$, are rectangular coordinates of E^n. It is obvious that, in the case $n = 3$, the form (0.2) is equivalent to the original (0.1).

An effect of the above generalization will be seen in § 2, and it will be shown that, if $n = 2$, there exist *G-extensions of class* 1. In consequence of use of rectangular coordinates, the theory of *spaces of class* 2 will be treated uniformly in §§ 3–6. Thus we shall give an almost complete theory of the imbedding of a special class of s.s. space for which $A(t, r) = 1$.

§ 1. A *G*-extension of an euclidean space.

We consider a G-extension, whose fundamental form is given by (0.2), and denote it by G^{n+1}. We use sometimes in the following normalized coordinates (y_i), $i = 1, 2, \cdots, n$, which are defined by

$$y_i = \frac{x^i}{r} = \frac{\partial r}{\partial x^i}. \tag{1.1}$$

from which it follows immediately that

$$\sum (y_i)^2 = 1, \qquad \frac{\partial y_i}{\partial x^j} = \frac{1}{r}(\delta_{ij} - y_i y_j). \tag{1.2}$$

Making use of (y_i), Christoffel's symbols $\Gamma^{\alpha}_{\beta\gamma}$, $\alpha, \beta, \gamma = 0, 1, \cdots, n$, are

$$\Gamma^0_{00} = \frac{G_0}{2G}, \qquad \Gamma^0_{0i} = \Gamma^0_{i0} = \frac{G_r}{2G} y_i ,$$

(1. 3)*

$$\Gamma^i_{00} = \frac{G_r}{2} y_i , \qquad \textit{other} \quad \Gamma^\alpha_{\beta\gamma} = 0 ,$$

and components of the curvature tensor $R_{\alpha\beta\gamma\delta}$ are

(1. 4) $$R_{0i0j} = (P+Q)y_i y_j - Q\delta_{ij} , \qquad \textit{other} \quad R_{\alpha\beta\gamma\delta} = 0 ,$$

where we put

(1. 5) $$P = -\frac{G_{rr}}{2} + \frac{G_r^2}{4G}, \qquad Q = \frac{G_r}{2r} .$$

Furthermore, from the definitions of the Ricci tensor $R_{\alpha\beta} = g^{\gamma\delta} R_{\alpha\gamma\beta\delta}$ and the scalar curvature $R = g^{\alpha\beta} R_{\alpha\beta}$ it follows that

(1. 6) $$R_{00} = (n-1)Q - P , \quad R_{0i} = 0 , \quad R_{ij} = \frac{1}{G} R_{0i0j} ,$$

(1. 7) $$R = \frac{2}{G} [(n-1)Q - P] .$$

We shall find first a necessary and sufficient condition for G^{n+1} to be flat, that is, the curvature tensor vanishes.

From (1. 4) it follows that $R_{0i0j} y_i y_j = P = 0$. Consequently, if $n \geqq 2$, we have $Q=0$ from (1. 6). Conversely, assuming $n \geqq 2$ and $Q=0$, it is clear that $P=0$ by the definition and hence G^{n+1} is flat. In the case $n=1$, if $P=0$, it follows from (1. 4) that $R_{0101}=0$. Therefore we have

Proposition 1. *The necessary and sufficient condition for G^{n+1} to be flat is that $P=0$ for $n=1$, and $Q=0$ for $n \geqq 2$.*

Next we consider components of the conformal curvature tensor $C_{\alpha\beta\gamma\delta}$. If we put

(1. 8) $$l_{\alpha\beta} = \frac{1}{n-1} \left(R_{\alpha\beta} - \frac{R}{2n} g_{\alpha\beta} \right) ,$$

then $C_{\alpha\beta\gamma\delta}$ are expressed as

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* Throughout the paper Greek indices take the values $0, 1, \cdots, n$ and Latin $1, 2, \cdots, n$. We shall also indicate the partial derivatives of a function with respect to x^α and r by subscripts α and r respectively with commas. We shall omit comma in case there is no danger of confusion.

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - g_{\alpha\gamma}l_{\beta\delta} - g_{\beta\delta}l_{\alpha\gamma} + g_{\alpha\delta}l_{\beta\gamma} + g_{\beta\gamma}l_{\alpha\delta}\,. \tag{1.9}$$

By means of (1.6) and (1.7) we have

$$\begin{aligned} l_{00} &= \frac{1}{n}[(n-1)Q-P]\,, \qquad l_{0i} = 0\,, \\ l_{ij} &= \frac{P+Q}{(n-1)G}\left(y_iy_j - \frac{1}{n}\delta_{ij}\right), \end{aligned} \tag{1.10}$$

and hence we obtain

$$\begin{aligned} C_{0i0j} &= \frac{(n-2)(P+Q)}{n-1}\left(y_iy_j - \frac{1}{n}\delta_{ij}\right), \qquad C_{0ijk} = 0\,, \\ C_{ijkl} &= \frac{P+Q}{(n-1)G}\Big[\delta_{ik}y_jy_l + \delta_{jl}y_iy_k - \delta_{il}y_jy_k - \delta_{jk}y_iy_l \\ &\qquad - \frac{2}{n}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\Big]. \end{aligned} \tag{1.11}$$

It is well known that G^{n+1}, $n\geqq 3$, is conformally flat if and only if $C_{\alpha\beta\gamma\delta}=0$. We see easily that the condition is given by $P+Q=0$ for $n\geqq 3$ in virtue of (1.11). On the other hand, G^3 is conformally flat if and only if $l_{\alpha\beta;\gamma} - l_{\alpha\gamma;\beta} = 0$*. From (1.3) and (1.10) it follows that

$$\begin{aligned} l_{00;i} - l_{0i;0} &= \frac{1}{n}\left[(n-1)Q_r - P_r - \frac{G_r}{2G}((n-1)Q-P) + \frac{G_r}{2G}(P+Q)\right]y_i\,, \\ l_{0i;j} - l_{0j;i} &= 0\,, \\ l_{i0;j} - l_{ij;0} &= \frac{1}{(n-1)G}\left[\frac{G_0}{G}(P+Q) - (P_0+Q_0)\right]\left(y_iy_j - \frac{1}{n}\delta_{ij}\right), \\ l_{ij;k} - l_{ik;j} &= \frac{1}{n(n-1)G}\left[\frac{G_r}{G}(P+Q) - (P_r+Q_r)\right](\delta_{ij}y_k - \delta_{ik}y_j)\,. \end{aligned}$$

In the first place, equations $(l_{i0;j} - l_{ij;0})y_j = 0$ and $(l_{ij;k} - l_{ik;j})\delta^{ij} = 0$ give $P+Q = c\ G$ (c = constant). Hence we have $l_{00;i} - l_{0i;0} = (Q_r - (G_r/(2G))Q) = 0$, so that $Q_r/Q = G_r/(2G)$, and then we have easily $P+Q=0$, the same result as derived for the general case $n\geqq 3$. The converse is obvious. Thus we have

Proposition 2. *The necessary and sufficient condition for G^{n+1}, $n\geqq 2$, to be conformally flat is that $P+Q=0$.*

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* Throughout the paper semicolons denote covariant derivatives.

In particular, we are interested in the case where G^{n+1} is an Einstein space from the physical point of view. We have, however, the following theorem.

Theorem 1. *If G^{n+1}, $n\geqq 2$, is an Einstein space, then G is necessarily flat.*
In fact, if G^{n+1} is an Einstein space, that is $R_{\alpha\beta}=(R/(n+1))g_{\alpha\beta}$, it follows from (1.6) and (1.7) that

$$(n-1)Q-P = \frac{2}{n+1}[(n-1)Q-P],$$

$$(P+Q)y_iy_j-Q\delta_{ij} = -\frac{2}{n+1}[(n-1)Q-P]\delta_{ij}.$$

The first equation gives $(n-1)Q-P=0$ and then we have from the second that $R_{0i0j}=0$.

§ 2. A G-extension G^{n+1} of class one.

Since G^4 is a S_1 space such that $A(t, r)=1$, it is known that G^4 is not of class 1, in consequence of the theorem proved by Takeno [7, III]. However, we generalized the dimensional number as defined in the introduction, and hence we may expect the existence of a G^{n+1} of class 1. In this section such a G^{n+1} will be considered.

A space G^{n+1} is of class 1, that is, G^{n+1} is looked upon as a hypersurface of a pseudo-euclidean E^{n+2}, if and only if there exist $e=\pm 1$ and $b_{\alpha\beta}(=b_{\beta\alpha})$, satisfying the Gauss equation

$$(G_1) \qquad eR_{\alpha\beta\gamma\delta} = b_{\alpha\gamma}b_{\beta\delta}-b_{\alpha\delta}b_{\beta\gamma},$$

and the Codazzi equation

$$(C_1) \qquad b_{\alpha\beta;\gamma}-b_{\alpha\gamma;\beta} = 0.$$

The sign e in (G_1) is an indicator of the normal of G^{n+1}, and $b_{\alpha\beta}$ are components of the second fundamental tensor of G^{n+1}. By means of (1.4), (G_1) is expressed

$$(2.1) \qquad eR_{0i0j} = b_{00}b_{ij}-b_{0i}b_{0j},$$

$$(2.2) \qquad b_{0j}b_{ik}-b_{0k}b_{ij} = 0,$$

$$(2.3) \qquad b_{ik}b_{jl}-b_{il}b_{jk} = 0.$$

From (2.3) it follows immediately that the rank of the matrix (b_{ij}) is at most 1.

We consider first the case where $b_{ij}=0$, $i, j=1, 2, \cdots, n$. Making use of (1.4), the equation (2.1) is written concretely

$$Q\delta_{ij} = (P+Q)y_i y_j + eb_{0i}b_{0j}\,. \tag{2.4}$$

The matrix, whose elements are right-hand members of (2.4), is of rank at most 2. Since we should assume that G^{n+1} under consideration be not flat, the quantity Q does not vanish by Proposition 1. Therefore we know from (2.4) that $n=2$ is necessary in this case.

Contracting (2.4) by y_j we have

$$Py_i + eb_{0i}\cdot b_{0j}y_j = 0\,.$$

If $b_{0j}y_j \neq 0$, there exists a factor λ such that $b_{0i}=\lambda y_i$. Substitution of this into (2.4) gives a contradiction $n=1$. Therefore we have

$$b_{0j}y_j = 0\,, \quad P=0\,. \tag{2.5}$$

Thus (2.4) is a system of three equations

$$Q(\delta_{ij}-y_i y_j) = eb_{0i}b_{0j}\,, \quad i, j=1, 2\,,$$

from which we obtain easily

$$b_{0i} = \lambda\varepsilon_{ij}y_j\,, \quad \lambda=\pm\sqrt{eQ}\,, \quad i, j=1, 2\,, \tag{2.6}$$

where $e\pm1$ must be taken as $eQ>0$, and we used skew-symmetric quantities ε_{ij} such that $\varepsilon_{12}=-\varepsilon_{21}=1$, $\varepsilon_{11}=\varepsilon_{22}=0$.

Next we consider the case where the rank of $(b_{ij})=1$. Then there exist b_i, $i=1, 2, \cdots, n$, such that

$$b_{ij} = \eta b_i b_j\,, \quad \eta = \pm 1\,. \tag{2.7}$$

From (2.2) we see also an existence of b_0, by means of which b_{0j} are expressed in the form

$$b_{0j} = \eta b_0 b_j\,. \tag{2.8}$$

We substitute in (2.1) from (2.7) and (2.8), and then obtain

$$Q\delta_{ij} = (P+Q)y_i y_j - e(\eta b_{00}-b_0^2)b_i b_j\,. \tag{2.9}$$

The process which was used to obtain $n=2$ from (2.4) applied equally well to (2.9), and then we have $n=2$ as well. Further contraction of (2.9) by y_j gives

$$Py_i - e(\eta b_{00} - b_0^2)b_i \cdot b_j y_j = 0 .$$

If $(\eta b_{00} - b_0^2)b_j y_j \neq 0$, there exists λ such that $b_i = \lambda y_i$, and substitution in (2.9) leads to a contradiction. But, if $\eta b_{00} - b_0^2 = 0$, we have again a contradiction from (2.9). Hence we obtain

$$b_j y_j = 0 , \quad P = 0 , \tag{2.10}$$

similar to (2.5). Finally we solve (2.9) for b_i and obtain

$$b_i = \mu \varepsilon_{ij} y_j , \quad \mu = \pm\sqrt{-eQ/(\eta b_{00} - b_0^2)} , \quad i, j = 1, 2 . \tag{2.11}$$

Summarizing the results of the above two cases we obtain

Proposition 3. *If non-flat G^{n+1}, $n \geqq 2$, is of class one, it is necessary that $n=2$ and the quantity $P=0$. Then we have only two systems of solutions $e=\pm 1$ and $b_{\alpha\beta}$ of the Gauss equation (G_1) as follows.*

Case 1. b_{00} *is arbitrary,* $b_{ij} = 0$,

$$b_{0j} = \lambda \varepsilon_{jk} y_k , \qquad i, j, k = 1, 2 ,$$

$$\lambda = \pm\sqrt{eQ} , \quad eQ > 0 .$$

Case 2. b_{00} *is arbitrary,* $b_{ij} = \eta b_i b_j$,

$$b_{0j} = \eta b_0 b_j , \quad \eta = \pm 1 , \qquad i, j, k = 1, 2 ,$$

$$b_j = \mu \varepsilon_{jk} y_k , \quad \mu = \pm\sqrt{-eQ/(\eta b_{00} - b_0^2)} .$$

The following theorem is a consequence of the above proposition and Proposition 2.

Theorem 2. *If G^{n+1}, $n \geqq 2$, is conformally flat, and of class one, then $n=2$ and G^{n+1} is flat.*

In this place we observe that, in both of the above two cases, the matrix $(b_{\alpha\beta})$ is of rank less than 3. It is generally known that, if the rank is more than 3, the Codazzi equation (C_1) are automatically satisfied as a consequence of (G_1) [1, p. 281], [6]. That is, however, not the case for our problem, and hence we must treat furthermore the equation (C_1).

Now, (C_1) is, by means of (1.3), expressible in the form

$$b_{00,i}-b_{0i,0}-b_{00}\frac{G_r}{2G}y_i+b_{0i}\frac{G_0}{2G}+b_{ij}y_j\frac{G_r}{2}=0\,, \tag{2.12}$$

$$b_{0i,j}-b_{0j,i}-b_{0i}\frac{G_r}{2G}y_j+b_{0j}\frac{G_r}{2G}y_i=0\,, \tag{2.13}$$

$$b_{0i,j}-b_{ij,0}+b_{0j}\frac{G_r}{2G}y_i=0\,, \tag{2.14}$$

$$b_{ij,k}-b_{ik,j}=0\,. \tag{2.15}$$

Case 1. We shall first treat the case 1 in the proposition 3. Since $b_{ij}=0$, (2.15) is trivial. We see easily from (2.5) and (1.2) that

$$b_{0j,0}y_j=0\,,\quad b_{0j,i}y_j=-\frac{1}{r}b_{0i}\,.$$

Accordingly, applying contraction of (2.14) by y_i, we obtain

$$\frac{G_r}{2G}=\frac{1}{r}\,. \tag{2.16}$$

It is easily verified that the condition $P=0$ as shown in the above proposition is obtained as direct result of (2.16).

We now return to consideration of (2.14) itself which is written

$$b_{0i,j}+b_{0j}\frac{1}{r}y_i=0\,.$$

Substituting from (2.6), the above equation can be rewritten

$$\lambda_j\varepsilon_{ik}y_k+\frac{\lambda}{r}(\varepsilon_{ij}-\varepsilon_{ik}y_ky_j+\varepsilon_{jk}y_ky_i)=0\,. \tag{2.17}$$

It is easily seen that identities

$$\varepsilon_{ij}-\varepsilon_{ik}y_ky_j+\varepsilon_{jk}y_ky_i=0 \tag{2.18}$$

hold, and hence (2.17) gives $\lambda_j=0$ only. On the other hand, we have from (2.16)

$$G=r^2g(x^0)\,,\quad Q=g(x^0)\,, \tag{2.19}$$

where g is a function of x^0 alone. Therefore $\lambda_j=0$ are automatically satisfied.

Equation (2. 13) is clearly obtained from (2. 14). Finally we are concerned with (2. 12), which, in virtue of (2. 6) and (2. 19), is rewritten as follows.

$$b_{00,i}-\frac{1}{r}b_{00}y_i = 0\,,$$

and hence we have $b_{00}=rb(x^0)$, where b is an arbitrary function of x^0. Consequently, in the case 1, we obtain e and $b_{\alpha\beta}$ satisfying (C_1) as follows.

$$\begin{aligned}&b_{00} = rb(x^0)\,, \quad b_{ij} = 0\,,\\ &b_{0j} = \lambda\varepsilon_{jk}y_k\,, \qquad\qquad\qquad (2.20)\\ &\lambda = \pm\sqrt{eg(x^0)}\,, \quad eg(x^0)>0\,, \quad \textit{where} \quad G = r^2g(x^0)\,.\end{aligned}$$

Case 2. If follows from (2. 10) that

$$b_{j,0}y_j = 0\,, \quad b_{j,i}y_j = -\frac{1}{r}b_i\,.$$

With the aid of (2. 11), equation (2. 14) is written

$$\begin{aligned}&(\mu b_{0,j}+\mu_j b_0)\varepsilon_{ik}y_k+\frac{\mu}{r}b_0(\varepsilon_{ij}-\varepsilon_{ik}y_ky_j)\\ &-2\mu\mu_0\varepsilon_{ik}y_k\varepsilon_{jl}y_l+\mu b_0\frac{G_r}{2G}y_i\varepsilon_{jk}y_k = 0\,.\end{aligned} \tag{2.21}$$

In applying contraction by y_i, we have

$$\mu b_0\varepsilon_{ij}y_i\left(\frac{G_r}{2G}-\frac{1}{r}\right) = 0\,.$$

These equations lead us to classify this case into

Case 2-1: $b_0=0$. In this case (2. 21) gives $\mu_0=0$.

Case 2-2: $b_0\neq 0$, and $G_r/(2G)=r^{-1}$. In this case, making use of (2. 18), we have from (2. 21)

$$\mu b_{0,j}+\mu_j b_0 = 2\mu\mu_0\varepsilon_{jk}y_k\,. \tag{2.22}$$

Next we consider (2. 15), which, making use of (2. 18), is expressed in the form

$$2(\mu_k\varepsilon_{jh}y_h-\mu_j\varepsilon_{kh}y_h)\varepsilon_{il}y_l+\frac{\mu}{r}(\varepsilon_{ik}\varepsilon_{jh}-\varepsilon_{ij}\varepsilon_{kh})y_h = 0\,.$$

Since $n=2$, this is equivalent to the single equation

$$(2.23)\qquad 2\mu_i y_i + \frac{\mu}{r} = 0\,.$$

It is clear that (2.13) is automatically satisfied from (2.14).

Finally we shall deal with (2.12). In the case 2-1, the equations are of the form

$$b_{00,i} - b_{00}\frac{G_r}{2G} y_i = 0\,,$$

and hence we obtain $b_{00} = (G)^{1/2} b(x^0)$, where b is an arbitrary function of x^0. From the definition of μ we have

$$\mu^2 = -\frac{eQ}{\eta\sqrt{G}\,b(x^0)}\,.$$

Because of $\mu_0 = 0$ as above obtained, Q is expressed in the form

$$(2.24)\qquad Q = \sqrt{G}\,b(x^0)h(x^i)\,,$$

where $h(x^i)$ is independent of x^0. Now (2.23) is written $h_i x^i = -h$, from which it follows that $h(x^1, x^2)$ is homogeneous function of degree -1. By means of $P=0$ and (2.24), we have $rh(x^i) = 1/b(x^0) = c$ (=constant). Consequently we have, in the case 2-1

$$(2.25)\qquad \begin{aligned} b_{ij} &= -\frac{e\eta c}{r}\varepsilon_{ik} y_k \varepsilon_{jl} y_l\,, \\ b_{0j} &= 0\,, \quad b_{00} = \frac{\sqrt{G}}{c}\,, \qquad \textit{where} \quad G_r = 2\sqrt{G}\,. \end{aligned}$$

We now turn to a consideration of (2.12) in the case 2-2. Equation (2.12) is written

$$(2.26)\qquad b_{00,i} - b_{00}\frac{1}{r} y_i = \eta\varepsilon_{ij} y_j \left(b_{0,0}\mu + b_0\mu_0 - \frac{G_0}{2G} b_0\mu\right).$$

Since the condition (2.19) has been imposed as well, we obtain from the definition of μ

$$(2.27)\qquad b_{00} = \eta\left(b_0^2 - \frac{eg(x^0)}{\mu^2}\right).$$

In order to find solutions of (2.22), (2.23), (2.26) and (2.27) concretely, we now suppose that *both of* $b_0(x^0, x^1, x^2)$ *and* $\mu(x^0, x^1, x^2)$ *depend upon* x^0 *and* r *only.* Then (2.23) is written in the form $2\mu_r+(\mu/r)=0$, and hence we have $\mu=f(x^0)/\sqrt{r}$, where f is a function of x^0. Next (2.22) is expressible as

$$(2.22') \qquad (\mu b_0)_r y_j = 2\mu\mu_0 \varepsilon_{jk} y_k \, .$$

Contraction of these by y_j gives $\mu b_0=h(x^0)$, where h is a function of x^0. Hence (2.22′) gives $\mu_0=0$ only, from which it follows that the above function f is necessarily constant. If we denote by c the constant f, then we obtain $\mu=c/\sqrt{r}$, $b_0=\sqrt{r}\,h(x^0)/c$. In virtue of (2.27) we have $b_{00}=\eta r(h^2-eg)/c^2$. If we substitute the expression of b_{00} into (2.26), and integrate the resulting equations, then we have $h=\bar{c}\sqrt{\zeta g}$, where $\zeta=\pm 1$, $\zeta g>0$ and $\bar{c}$ is an another constant. Finally we obtain

$$\begin{aligned} & b_{00} = \frac{\eta r g(x^0)}{c^2}(\zeta\bar{c}^2-e)\,, \qquad b_{0i} = \eta\bar{c}\sqrt{\zeta g(x^0)}\,\varepsilon_{ij}y_j\,, \\ (2.28) \qquad & b_{ij} = \frac{\eta c^2}{r}\varepsilon_{ik}y_k\varepsilon_{jl}y_l\,, \qquad \eta,\ \zeta,\ e = \pm 1\,, \qquad \zeta g>0\,, \\ & c(\neq 0)\,,\ \bar{c} = \textit{constants}, \quad \textit{where } G = r^2 g(x^0)\,, \end{aligned}$$

in the case 2-2, under the restriction that both of b_0 and μ are function of x^0 and r.

Thus we obtained three systems of solutions (2.20), (2.25), and (2.28). We observe that (2.20) and (2.28) were obtained under the condition that the function G is expressed as $r^2g(x^0)$. On the other hand (2.25) was found when G satisfies the equation $G_r=2\sqrt{G}$, from which $G=(r+g(x^0))^2$, where g depends on x^0 alone. Furthermore it is observed that there remain some freedoms of determinations of e and $b_{\alpha\beta}$, and hence G^3 as a hypersurface of E^4 is not rigid. Summarizing those results we have established

Theorem 3. *A G-extension* G^{n+1}, $n\geqq 2$, *is of class* 1 *if and only if* $n=2$ *and the function* $G(x^0, r)$ *is of the form*

$$G = r^2 g(x^0) \qquad or \qquad G = (r+g(x^0))^2 \, .$$

Then G^3 *of class* 1 *is not rigid as a hypersurface of a pseudo-euclidean* 4*-space.*

§ 3. The Gauss equations of G^{n+1} of class 2, I.

In this and the following sections we shall be concerned with a space G^{n+1}, $n \geqq 2$, of class 2. Since G^{n+1} is looked upon as a subspace of a pseudo-euclidean $(n+3)$-space, we take two mutually orthogonal unit vectors normal to G^{n+1}. Then we denote by e and $e'(=\pm 1)$ indicators of normals and by $b_{\alpha\beta}$ and $c_{\alpha\beta}$ the second fundamental tensors with respect to these normals. These quantities should satisfy the Gauss and Codazzi equations.

In this and the next sections we shall deal with the Gauss equation

$$(G_2) \qquad R_{\alpha\beta\gamma\delta} = e(b_{\alpha\gamma}b_{\beta\delta} - b_{\alpha\delta}b_{\beta\gamma}) + e'(c_{\alpha\gamma}c_{\beta\delta} - c_{\alpha\delta}c_{\beta\gamma}) .$$

We observe that this equation is algebraic in character. It is clear that a system of solutions e, e', $b_{\alpha\beta}$ and $c_{\alpha\beta}$ of (G_2) will be uniquely determined, because two mutually orthogonal unit normals may be chosen arbitrarily, and hence a system of solutions will be transformed to an another one according as normals are changed [5].

In this section we shall assume that there exists a system of two normals such that

$$(3.1) \qquad b_{00} \neq 0 , \quad c_{00} = 0$$

hold, while we shall consider, in the next section, the case where there does not exist such a system of normals. For details on a transformation of the second fundamental tensors corresponding to a transformation of normals, we refer to a forthcoming book [5].

Now, when (3.1) is satisfied, (G_2) is of the form

$$(3.2) \qquad R_{0i0j} = e(b_{00}b_{ij} - b_{0i}b_{0j}) - e'c_{0i}c_{0j} ,$$

$$(3.3) \qquad e(b_{0j}b_{ik} - b_{0k}b_{ij}) + e'(c_{0j}c_{ik} - c_{0k}c_{ij}) = 0 ,$$

$$(3.4) \qquad e(b_{ik}b_{jl} - b_{il}b_{jk}) + e'(c_{ik}c_{jl} - c_{il}c_{jk}) = 0 .$$

The first of them is rewritten as

$$(3.2') \qquad b_{ij} = \frac{e}{b_{00}}(R_{0i0j} + eb_{0i}b_{0j} + e'c_{0i}c_{0j}) .$$

We substitute in (3.3) from (3.2'), and obtain

$$(3.3') \quad \frac{1}{b_{00}}(b_{0j}R_{0i0k}-b_{0k}R_{0i0j})+\frac{e'}{b_{00}}c_{0i}(b_{0j}c_{0k}-b_{0k}c_{0j})+e'(c_{0j}c_{ik}-c_{0k}c_{ij})=0\,.$$

Multiplying (3.3') by c_{0l} and summing three equations obtained from it by cyclic permutation of indices j, k and l, we have

$$(3.5) \quad R_{0i0j}(b_{0k}c_{0l}-b_{0l}c_{0k})+R_{0i0k}(b_{0l}c_{0j}-b_{0j}c_{0l})+R_{0i0l}(b_{0j}c_{0k}-b_{0k}c_{0j}) = 0\,.$$

It follows from (1.4) that the det. $R_{0i0j}=P(-Q)^{n-1}$. Since we are concerned with non-flat G^{n+1}, the quantity Q does not vanish by Proposition 1. It will be convenient that det. R_{0i0j} does not vanish, and hence we will assume $P\neq 0$ for a while. Then there exists an inverse matrix (S^{ij}) of the (R_{0i0j}), and we see easily

$$S^{ij} = \left(\frac{1}{P}+\frac{1}{Q}\right)y_i y_j - \frac{1}{Q}\delta_{ij}\,.$$

In applying contraction of (3.5) by S^{il}, we obtain

$$(3.6) \qquad (n-2)(b_{0j}c_{0k}-b_{0k}c_{0j}) = 0\,.$$

Therefore, if we suppose

$$(3.7) \qquad P\neq 0\,, \quad \text{and} \quad n\geqq 3\,,$$

then we have $b_{0j}c_{0k}-b_{0k}c_{0j}=0$. From this we see that there are only two possible cases as follows.

I. $b_{0j}=0$, $j=1, 2, \cdots, n$.

II. At least one of b_{0j} does not vanish.

In the latter case there exists a factor ρ such that

$$(3.8) \qquad c_{0j} = \rho b_{0j}\,, \quad j = 1, 2, \cdots, n\,.$$

Case I. Equations (3.2') and (3.3') are of the simple forms

$$(3.2'') \qquad b_{ij} = \frac{e}{b_{00}}(R_{0i0j}+e'c_{0i}c_{0j})\,,$$

$$(3.3'') \qquad c_{0j}c_{ik}-c_{0k}c_{ij} = 0\,,$$

respectively. We shall show that c_{0j} are all equal to zero. For, otherwise (3.3'') gives immediately

$$(3.9) \qquad c_{ij} = \lambda c_{0i}c_{0j}\,,$$

where λ is a function. Substituting in (3.4) from (3.2″) and (3.9) we obtain

$$
\begin{aligned}
(3.10)\qquad & R_{0i0k}R_{0j0l}-R_{0i0l}R_{0j0k}+e'(R_{0i0k}c_{0j}c_{0l}+R_{0j0l}c_{0i}c_{0k} \\
& \qquad -R_{0i0l}c_{0j}c_{0k}-R_{0j0k}c_{0i}c_{0l})=0\,,
\end{aligned}
$$

and contraction by S^{jl} gives

$$
(3.11)\qquad (n-1)R_{0i0k}+e'R_{0i0k}\cdot c_{0j}c_{0l}S^{jl}+e'(n-2)c_{0i}c_{0k}=0\,.
$$

Furthermore, contracting by S^{ik} we obtain

$$
n(n-1)+2e'(n-1)c_{0j}c_{0l}S^{jl}=0\,.
$$

It follows from this that $c_{0j}c_{0l}S^{jl}=-n/(2e')$, and hence we have $R_{0i0j}=-2e'c_{0i}c_{0j}$ from (3.11), where we made use of (3.7). It implies that the det. $R_{0i0j}=0$, contrary to our hypothesis.

Therefore we have all of $c_{0j}=0$ in this case. Then (3.3″) holds good and (3.2″) gives $b_{ij}=eR_{0i0j}/b_{00}$. Hence, because of our assumption the det. b_{ij} does not vanish also. Now (3.4) and the uniqueness theorem [6, 1, p. 200] lead us to $c_{ij}=\rho b_{ij}$, $\rho=\pm 1$. Substituting in (3.4) we see that $e'=-e$. Thus we have arrived at the conclusion for the case I as follows.

$$
(3.12)\qquad
\begin{aligned}
& b_{00}\neq 0\,, \quad c_{00}=0\,, \quad b_{0j}=c_{0j}=0\,, \\
& b_{ij}=\frac{e}{b_{00}}R_{0i0j}\,, \quad c_{ij}=\rho b_{ij}\,, \\
& e'=-e\,, \quad \rho=\pm 1\,, \quad i,\, j=1,\,2,\,\cdots,\,n\,.
\end{aligned}
$$

Case II. We substitute in (3.2′) from (3.8), and obtain

$$
(3.13)\qquad b_{ij}=\frac{e}{b_{00}}(R_{0i0j}+(e+\rho^2e')b_{0i}b_{0j})\,.
$$

On the other hand, we substitute in (3.3′) and obtain

$$
b_{0j}\left(\frac{1}{b_{00}}R_{0i0k}+e'\rho c_{ik}\right)-b_{0k}\left(\frac{1}{b_{00}}R_{0i0j}+e'\rho c_{ij}\right)=0\,,
$$

from which it follows that there exists a factor λ such that

$$
\frac{1}{b_{00}}R_{0i0j}+e'\rho c_{ij}=\lambda b_{0i}b_{0j}\,.
$$

Since an assumption $\rho=0$ leads immediately to a contradiction det. $R_{0i0j}=0$, the above equation are written in the form

$$c_{ij} = \frac{e'\lambda}{\rho} b_{0i}b_{0j} - \frac{e'}{\rho b_{00}} R_{0i0j} . \tag{3.14}$$

We substitute for b_{ij} and c_{ij} in (3.4) expressions of the forms (3.13) and (3.14) respectively, and the resulting equations are of the form

$$\begin{aligned} \kappa(R_{0i0k}R_{0j0l}-R_{0i0l}R_{0j0k})+\sigma(R_{0i0k}b_{0j}b_{0l} \\ +R_{0j0l}b_{0i}b_{0k}-R_{0i0l}b_{0j}b_{0k}-R_{0j0k}b_{0i}b_{0l}) = 0 , \end{aligned} \tag{3.15}$$

where putting

$$\kappa = \frac{e}{b_{00}^2}+\frac{e'}{\rho^2 b_{00}^2} , \qquad \sigma = \frac{e(e+\rho^2 e')}{b_{00}^2} - \frac{e'\lambda}{\rho^2 b_{00}} .$$

The process by means of which we obtained the det. $R_{0i0j}=0$ from (3.10) is applied to (3.15) as well, and it is easily seen that $\kappa=\sigma=0$. Hence we have $\lambda=0$, $\rho=\pm 1$ and $e'=-e$. Consequently we have arrived at the conclusion in the case II as follows.

$$\begin{aligned} & b_{00} \neq 0 , \quad c_{00} = 0 , \quad c_{0j} = \rho b_{0j} , \quad \rho = \pm 1 , \\ & b_{ij} = \frac{e}{b_{00}} R_{0i0j} , \quad c_{ij} = \rho b_{ij} , \quad e' = -e , \end{aligned} \tag{3.17}$$

$i, j=1, 2, \cdots, n$, *where at least one of* b_{0j} *does not vanish.*

Thus we obtain only two system of solutions (3.12) and (3.17), under the hypothesis (3.7). We can, however, see that those are still solutions, even if $n=2$ or $P=0$. Furthermore, if we take $b_{0j}=0$, $j=1, 2, \cdots, n$, in (3.17), then we have (3.12). Therefore we conclude that

Proposition 4. *The Gauss equation* (G_2) *for a space* G^{n+1}, $n\geqq 2$, *of class 2 have a system of solutions*

$$\begin{aligned} & c_{0j} = \rho b_{0j} , \quad c_{ij}=\rho b_{ij} , \quad \rho = \pm 1 , \\ & b_{ij} = \frac{e}{b_{00}} R_{0i0j} , \quad e' = -e , \quad i, j = 1, 2, \cdots, n , \end{aligned} \tag{3.18}$$

provided that $b_{00}\neq 0$ *and* $c_{00}=0$.

§ 4. The Gauss equations of G^{n+1} of class 2, II.

In the preceding section we dealt with a general case where there exists a system of two normals such that $b_{00} \neq 0$ and $c_{00}=0$. This is, however, not always the case [5]. That is, we have first a case where both of b_{00} and c_{00} vanish, and secondly a case where $e'=-e$ and $b_{00}^2=c_{00}^2 \neq 0$. But we shall show that the first does not take place, provided (3.7). In fact, in this case (3.2) gives

$$R_{0i0j} = -eb_{0i}b_{0j}-e'c_{0i}c_{0j} \,,$$

from which it follows that the rank of (R_{0i0j}) is less than 3, contrary to (3.7). In this section, we shall continue to suppose (3.7) for a while.

Now we shall consider in the following the second case, namely

$$e' = -e\,, \quad c_{00} = \rho b_{00} \neq 0\,, \quad \rho = \pm 1\,. \tag{4.1}$$

Then equation (G_2) can be written

$$eR_{0i0j} = (b_{ij}-\rho c_{ij})b_{00}-b_{0i}b_{0j}+c_{0i}c_{0j}\,, \tag{4.2}$$

$$b_{0j}b_{ik}-b_{0k}b_{ij} = c_{0j}c_{ik}-c_{0k}c_{ij}\,, \tag{4.3}$$

$$b_{ik}b_{jl}-b_{il}b_{jk} = c_{ik}c_{jl}-c_{il}c_{jk}\,. \tag{4.4}$$

We have first from (4.2)

$$b_{ij} = \rho c_{ij}+\frac{1}{b_{00}}(eR_{0i0j}+b_{0i}b_{0j}-c_{0i}c_{0j})\,. \tag{4.2$'$}$$

Substitution of this into (4.3) gives

$$\begin{aligned} &c_{ik}(\rho b_{0j}-c_{0j})-c_{ij}(\rho b_{0k}-c_{0k}) \\ &\quad +\frac{e}{b_{00}}(b_{0j}R_{0i0k}-b_{0k}R_{0i0j})-\frac{1}{b_{00}}c_{0i}(b_{0j}c_{0k}-b_{0k}c_{0j}) = 0\,. \end{aligned} \tag{4.3$'$}$$

We multiply (4.3′) by b_{0l} and sum three equations obtained from it by cyclic permutation of indices j, k, l. Then we have

$$c_{ij}(b_{0k}c_{0l}-b_{0l}c_{0k})+c_{ik}(b_{0l}c_{0j}-b_{0j}c_{0l})+c_{il}(b_{0j}c_{0k}-b_{0k}c_{0j}) = 0\,. \tag{4.5}$$

Next multiplying (4.3′) by c_{0l} and using an entirely similar way, we have in virtue of (4.5)

$$R_{0i0j}(b_{0k}c_{0l}-b_{0l}c_{0k})+R_{0i0k}(b_{0l}c_{0j}-b_{0j}c_{0l})+R_{0i0l}(b_{0j}c_{0k}-b_{0k}c_{0j}) = 0\,. \tag{4.6}$$

By means of our hypothesis (3.7), we can make use of contraction of (4.6) by S^{il}, and it follows that $b_{0j}c_{0k}-b_{0k}c_{0j}=0$ from (4.6), Similarly to the classification in the last section, we have the following two cases.

I. $b_{0j}=0$, $j=1, 2, \cdots, n$.

II. At least one of b_{0j} does not vanish.

In the latter case, there exists a factor λ such that

$$c_{0j} = \lambda b_{0j}\,, \quad j = 1, 2, \cdots, n\,. \tag{4.7}$$

Case I. Equations (4.2′) and (4.3′) are written

$$b_{ij} = \rho c_{ij} + \frac{1}{b_{00}}(eR_{0i0j} - c_{0i}c_{0j})\,, \tag{4.8}$$

$$c_{0j}c_{ik} - c_{0k}c_{ij} = 0 \tag{4.9}$$

respectively. The process which was used in the last section to obtain $c_{0j}=0$ from (3.2″), (3.9) and (3.4) is applied to this case as well, and we see then $c_{0j}=0$, $j=1, 2, \cdots, n$, from (4.9). Thus (4.8) is rewritten

$$b_{ij} = \rho c_{ij} + \frac{e}{b_{00}} R_{0i0j}\,. \tag{4.8′}$$

We put (4.8′) into (4.4), and obtain

$$\frac{1}{b_{00}}(R_{0i0k}R_{0j0l} - R_{0i0l}R_{0j0k}) + e\rho(R_{0i0k}c_{jl} + R_{0j0l}c_{ik} - R_{0i0l}c_{jk} - R_{0j0k}c_{il}) = 0\,.$$

Contraction of this by S^{jl} and furthermore by S^{ik} leads us to the following two equations.

$$\frac{1}{b_{00}}(n-1)R_{0i0k} + e\rho R_{0i0k}\cdot c_{jl}S^{jl} + e\rho(n-2)c_{ik} = 0\,,$$

$$\frac{1}{b_{00}}n(n-1) + e\rho 2(n-1)c_{jl}S^{jl} = 0\,.$$

From the above three systems of equations and (4.8′) we obtain

$$\begin{aligned} &c_{00} = \rho b_{00} \neq 0\,, \quad \rho = \pm 1\,, \\ &b_{0j} = c_{0j} = 0\,, \quad e_{ij} = -\rho b_{ij}\,, \\ &b_{ij} = \frac{e}{2b_{00}} R_{0i0j}\,, \quad e' = -e\,, \quad i, j = 1, 2, \cdots, n\,. \end{aligned} \tag{4.10}$$

This is the conclusion in the case I,

Case II. The equation (4. 2′) is then written

$$b_{ij} = \rho c_{ij} + \frac{1}{b_{00}}(eR_{0i0j} + (1-\lambda^2)b_{0i}b_{0j}), \tag{4.11}$$

where (4. 7) was used. On the other hand, substituting from (4. 7) in (4. 3′), we deduce equations, which mean that there exists a factor μ such that

$$\frac{e}{b_{00}}R_{0i0j} + (\rho-\lambda)c_{ij} = \mu\, b_{0i}b_{0j}.$$

The assumption $\rho-\lambda(=\sigma)=0$ leads us to a contradiction, namely $\det. R_{0i0j}=0$. Thus σ does not vanish, and hence the above equation gives

$$c_{ij} = \frac{\mu}{\sigma}b_{0i}b_{0j} - \frac{e}{\sigma b_{00}}R_{0i0j}. \tag{4.12}$$

Substituting in (4. 11), we obtain

$$b_{ij} = \left(\frac{\rho\mu}{\sigma} + \frac{1-\lambda^2}{b_{00}}\right)b_{0i}b_{0j} - \frac{e\lambda}{\sigma b_{00}}R_{0i0j}. \tag{4.13}$$

Substitution of these expressions (4. 12) and (4. 13) into (4. 4) gives

$$\begin{aligned}&\kappa(R_{0i0k}R_{0j0l} - R_{0i0l}R_{0j0k}) \\ &\quad + \tau(R_{0i0k}b_{0j}b_{0l} + R_{0j0l}b_{0i}b_{0k} - R_{0i0l}b_{0j}b_{0k} - R_{0j0k}b_{0i}b_{0l}) = 0,\end{aligned} \tag{4.14}$$

where coefficients κ and τ are

$$\kappa = \frac{\lambda^2-1}{\sigma^2 b_{00}^2}, \qquad \tau = \frac{e\mu(1-\lambda\rho)}{\sigma^2 b_{00}} - \frac{\lambda(1-\lambda^2)e}{\sigma b_{00}^2}.$$

Equation (4. 14) has the similar form with (3. 15), and hence $\kappa=\tau=0$ is obtained as well, from which it follows that $\lambda=\pm 1$, $\mu(1-\lambda\rho)=0$. The supposition $\mu\neq 0$ gives, however, $\lambda=\rho$, that is $\sigma=0$. Therefore we have $\mu=0$. Since both ρ and λ are equal to ± 1 and $\rho-\lambda=\sigma\neq 0$, we have $\lambda=-\rho$, $\sigma=2\rho$. Thus (4. 12) and (4. 13) give the final equations:

$$\begin{aligned}&c_{00} = \rho b_{00} \neq 0, \qquad \rho = \pm 1, \\ &c_{0j} = -\rho b_{0j}, \qquad c_{ij} = -\rho b_{ij}, \\ &b_{ij} = \frac{e}{2b_{00}}R_{0i0j}, \qquad e' = -e, \qquad i, j = 1, 2, \cdots, n, \\ &\textit{at least one of } b_{0j} \textit{ does not vanish.}\end{aligned} \tag{4.15}$$

We should notice the fact that, if all of b_{0j} are put to be zero, from (4.15) we have (4.10). Further, we see that those are still solutions, even if our assumption (3.7) is removed. Therefore we establish

Proposition 5. *The equation* (G_2) *has a system of solutions*

$$c_{0j} = -\rho b_{0j}, \quad c_{ij} = -\rho b_{ij},$$
$$b_{ij} = \frac{e}{2b_{00}} R_{0i0j}, \quad e' = -e, \quad i, j = 1, 2, \cdots, n, \tag{4.16}$$

provided that $c_{00} = \rho b_{00} \neq 0$, $\rho = \pm 1$.

The following proposition is a consequence of the above two propositions.

Proposition 6. *Let* G^{n+1} *be such that* $n \geqq 3$, *and the quantity* $P \neq 0$. *Then Gauss equation* (G_2) *has only two systems of solutions* e, e', $b_{\alpha\beta}$, $c_{\alpha\beta}$, *those given by* (3.18) *and* (4.16).

On the other hand, if $n=2$ or $P=0$, we may think that there exists a lot of solutions of different type. However it seems us to be complicated to discuss such a special case completely.

§ 5. Codazzi and Ricci equations of G^{n+1} of class 2, I.

Quantities e, e', $b_{\alpha\beta}$, and $c_{\alpha\beta}$ satisfying the Gauss equation (G_2) were found in the preceding sections. We know that a space G^{n+1} under consideration is of class two, if and only if there exist e, e', $b_{\alpha\beta}$, $c_{\alpha\beta}$ and further ν_α satisfying (G_2) and the following two systems of equations. The first is Codazzi equation

$$b_{\alpha\beta;\gamma} - b_{\alpha\gamma;\beta} = -e'(c_{\alpha\beta}\nu_\gamma - c_{\alpha\gamma}\nu_\beta),$$
$$c_{\alpha\beta;\gamma} - c_{\alpha\gamma;\beta} = e(b_{\alpha\beta}\nu_\gamma - b_{\alpha\gamma}\nu_\beta), \tag{C_2}$$

and the second is Ricci equation

$$\nu_{\alpha;\beta} - \nu_{\beta;\alpha} + g^{\gamma\delta}(b_{\gamma\alpha}c_{\delta\beta} - b_{\gamma\beta}c_{\delta\alpha}) = 0. \tag{R_2}$$

This and the following sections are devoted to the study of (C_2) and (R_2) for solutions (3.18) and (4.16) respectively.

We shall be concerned here with (3.18). By means of (1.3), equation (C_2) is written

$$b_{00,i}-b_{0i,0}-b_{00}\frac{G_r}{2G}y_i+b_{0i}\frac{G_0}{2G}+b_{ij}\frac{G_r}{2}y_j=-e\rho\nu_0 b_{0i}\,, \tag{5.1}$$

$$b_{0i,j}-b_{0j,i}-b_{0i}\frac{G_r}{2G}y_j+b_{0j}\frac{G_r}{2G}y_i=e\rho(\nu_j b_{0i}-\nu_i b_{0j})\,, \tag{5.2}$$

$$b_{i0,j}-b_{ij,0}+b_{0j}\frac{G_r}{2G}y_i=e\rho(\nu_j b_{i0}-\nu_0 b_{ij})\,, \tag{5.3}$$

$$b_{ij,k}-b_{ik,j}=e\rho(\nu_k b_{ij}-\nu_j b_{ik})\,, \tag{5.4}$$

$$-b_{0i,0}+b_{0i}\frac{G_0}{2G}+b_{ij}\frac{G_r}{2}y_j=e\rho(\nu_i b_{00}-\nu_0 b_{0i})\,. \tag{5.5}$$

It is easily seen that the other equations among (C_2) are automatically satisfied by the above and $c_{0j}=\rho b_{0j}$, $c_{ij}=\rho b_{ij}$. Furthermore (R_2) is written

$$\nu_{0,i}-\nu_{i,0}+\frac{\rho}{G}b_{00}b_{0i}=0\,, \tag{5.6}$$

$$\nu_{i,j}-\nu_{j,i}=0\,. \tag{5.7}$$

Now, in the first place, from (5.1) and (5.5) equation

$$e\rho\nu_i=\frac{G_r}{2G}y_i-\frac{b_{00,i}}{b_{00}} \tag{5.8}$$

is obtained. Then we define ν_i by (5.8), and hence (5.1) is a consequence of (5.5). If we substitute (5.8) and $b_{ij}=eR_{0i0j}/b_{00}$ in (5.4), the resulting equations are satisfied, as will be easily seen from equation

$$R_{0i0j,k}-R_{0i0k,j}=\frac{G_r^2}{4rG}(\delta_{ik}y_j-\delta_{ij}y_k)\,,$$

which follows from (1.4) immediately. Next, equation (5.7) is evidently satisfied in virtue of (5.8).

If we substitute from (5.8) in (5.2), the resulting equation is written in the form

$$\left(\frac{b_{00}b_{0i}}{G}\right)_{,j}=\left(\frac{b_{00}b_{0j}}{G}\right)_{,i}\,,$$

from which it follows that there must exist a function λ such that

$$b_{0i}=\frac{G}{b_{00}}\lambda_i\,. \tag{5.9}$$

Substituting from (5.8) and (5.9) in (5.6), we have

$$e\rho\nu_{0,i} = \left(\frac{G_0}{2G} - \frac{b_{00,0}}{b_{00}}\right)_{,i} - e\lambda_i\,.$$

Integration of this gives

$$e\rho\nu_0 = \frac{G_0}{2G} - \frac{b_{00,0}}{b_{00}} - e\lambda + f(x^0)\,, \tag{5.10}$$

where $f(x^0)$ is a function of x^0 only.

We have already (5.8), (5.9) and (5.10), and we are now in a position to treat remaining equations (5.3) and (5.5). Inserting these results in (5.3) we have first

$$\lambda_{ij} + \frac{G_r}{2G}(\lambda_i y_j + \lambda_j y_i) - \frac{e}{G} R_{0i0j,0} + \frac{e}{G}\left(\frac{G_0}{2G} - e\lambda + f\right) R_{0i0j} = 0\,. \tag{5.3$'$}$$

It seems us to be difficult to find a general λ satisfying (5.3′), and in order to obtain a concrete form of λ, we will assume, henceforth, that λ *is a function of* x^0 *and* r. Then it is easily seen that

$$\lambda_i = \lambda_r y_i\,, \qquad \lambda_{ij} = \left(\lambda_{rr} - \frac{\lambda_r}{r}\right) y_i y_j + \frac{\lambda_r}{r}\delta_{ij}\,,$$

Making use of these expressions, (5.3′) is then written

$$\begin{aligned}&\Big[\lambda_{rr} - \frac{\lambda_r}{r} + \frac{G_r}{G}\lambda_r - \frac{e}{G}(P_0+Q_0) + \frac{e}{G}(P+Q)\\ &\quad\times\left(\frac{G_0}{2G} - e\lambda + f\right)\Big] y_i y_j + \left[\frac{\lambda_r}{r} + \frac{e}{G}Q_0 - \left(\frac{G_0}{2G} - e\lambda + f\right)\frac{eQ}{G}\right]\delta_{ij} = 0\,.\end{aligned} \tag{5.11}$$

from which it follows evidently that

$$\lambda_r + \frac{rQ}{G}\lambda + \frac{er}{G}\left(Q_0 - \frac{QG_0}{2G} - Qf\right) = 0\,, \tag{5.12}$$

$$\lambda_{rr} + \left(\frac{G_r}{G} - \frac{1}{r}\right)\lambda_r - \frac{e}{G}(P_0+Q_0) + \frac{e}{G}(P+Q)\left(\frac{G_0}{2G} - e\lambda + f\right) = 0\,. \tag{5.13}$$

The first (5.12) gives immediately $\left(\sqrt{G}\lambda + \frac{eG_0}{2\sqrt{G}} - ef\sqrt{G}\right)_{,r} = 0$, and hence we obtain

$$\lambda = -\frac{eG_0}{2G} + ef(x^0) + \frac{1}{\sqrt{G}}g(x^0)\,, \tag{5.14}$$

where $g(x^0)$ is an arbitrary function of x^0. It is easily verified that (5.13) follows from (5.14). Substituting from (5.14) in (5.9), we obtain expressions of b_{0j} as follows.

$$b_{0j} = \frac{e}{b_{00}}\left(-\frac{G_{0r}}{2}+\frac{G_0 G_r}{2G}-\frac{eG_r}{2\sqrt{G}}\, g(x^0)\right) y_j . \tag{5.15}$$

Finally we consider (5.5). These are now written

$$\left(\frac{b_{00}^2}{2G}\right)_{,\,j} = \left[\lambda_{0r}+(e\lambda-f)\lambda_r+\left(\frac{eG_r^2}{8G}\right)_{,\,r}\right] y_j .$$

The fact that $b_{00}^2/(2G)$ is a function of x^0 and r is easily verified from the above, and hence we have by integration

$$b_{00}^2 = 2G\left(\lambda_0+\frac{e}{2}\lambda^2-\lambda f+\frac{eG_r^2}{8G}+h(x^0)\right),$$

where $h(x^0)$ is as well an arbitrary function of x^0. Substitution of (5.14) into the above equation gives

$$\begin{aligned} b_{00}^2 &= e\left(-G_{00}+\frac{5G_0^2}{4G}+\frac{G_r^2}{4}\right)+2eGf_0+2\sqrt{G}\,g_0 \\ &\quad -eGf^2+eg^2-\frac{G_0}{\sqrt{G}}\,g+2Gh . \end{aligned} \tag{5.16}$$

Therefore we have arrived at the end. That is, (5.16) gives b_{00}, (5.15) b_{0j}, (5.8) ν_j, and (5.10) ν_0. The final (5.10) is

$$e\rho\nu_0 = \frac{G_0}{G}-\frac{b_{00,0}}{b_{00}}-\frac{e}{\sqrt{G}}\,g , \tag{5.17}$$

where we put (5.14) into (5.10).

Three functions f, g and h of one variable x^0 are taken arbitrarily, and hence if we take those equal to zero, then we have the simplest expressions as follows.

$$\begin{aligned} b_{00}^2 &= e\left(-G_{00}+\frac{5G_0^2}{4G}+\frac{G_r^2}{4}\right), \\ b_{00}b_{0j} &= e\left(-\frac{G_{0j}}{2}+\frac{G_0G_j}{2G}\right), \\ b_{00}b_{ij} &= e\left(-\frac{G_{ij}}{2}+\frac{G_iG_j}{4G}\right), \\ e\rho\nu_0 &= \frac{G_0}{G}-\frac{b_{00,0}}{b_{00}}, \qquad e\rho\nu_j = \frac{G_j}{2G}-\frac{b_{00,j}}{b_{00}} . \end{aligned} \tag{5.18}$$

We can state this result as follows.

Theorem 4. *Let a space G^{n+1}, $n\geqq 2$, be non-flat and the quantity $-G_{00}+\frac{5G_0^2}{4G}+\frac{G_r^2}{4}\neq 0$. Then there exists an imbedding of G^{n+1} in a pseudo-euclidean $(n+3)$-space such that $b_{00}\neq 0$ and $c_{00}=0$. Equations* (3.18) *and* (5.18) *give such an imbedding.*

§ 6. Codazzi and Ricci equations of G^{n+1} of class 2, II.

We now turn to a consideration of equations (C_2) and (R_2) for solutions (4.16) of (G_2). In this case, equation (C_2) is

$$(6.1)\quad b_{00,j}-b_{0j,0}-b_{00}\frac{G_r}{2G}y_j+b_{0j}\frac{G_0}{2G}+b_{jk}\frac{G_r}{2}y_k = e\rho(\nu_j b_{00}+\nu_0 b_{0j}),$$

$$(6.2)\quad b_{0i,j}-b_{0j,i}-b_{0i}\frac{G_r}{2G}y_j+b_{0j}\frac{G_r}{2G}y_i = -e\rho(\nu_j b_{0i}-\nu_i b_{0j}),$$

$$(6.3)\quad b_{i0,j}-b_{ij,0}+b_{0j}\frac{G_r}{2G}y_i = -e\rho(\nu_j b_{i0}-\nu_0 b_{ij}),$$

$$(6.4)\quad b_{ij,k}-b_{ik,j} = -e\rho(\nu_k b_{ij}-\nu_j b_{ik}),$$

$$(6.5)\quad b_{00,j}+b_{0j,0}-b_{00}\frac{G_r}{2G}y_j-b_{0j}\frac{G_0}{2G}-b_{jk}\frac{G_r}{2}y_k = e\rho(\nu_j b_{00}-\nu_0 b_{0j}),$$

and the orther equations among (C_2) are satisfied as a consequence of the above equations and $c_{0j}=-\rho b_{0j}$, $c_{ij}=-\rho b_{ij}$. Further, the Ricci equation (R_2) is of the form

$$(6.6)\quad \nu_{0,j}-\nu_{j,0}-\frac{2\rho}{G}b_{00}b_{0j} = 0,$$

$$(6.7)\quad \nu_{i,j}-\nu_{j,i} = 0.$$

We can first derive from (6.1) and (6.5) the following two equations.

$$(6.8)\quad \frac{b_{00,j}}{b_{00}}-\frac{G_r}{2G}y_j = e\rho\nu_j,$$

$$(6.9)\quad b_{0j,0}-b_{0j}\frac{G_0}{2G}-b_{jk}\frac{G_r}{2}y_k = -e\rho\nu_0 b_{0j}.$$

We notice that ν_j of (6.8) have the similar forms as one of (5.8). It is easily verified that (6.4) holds good as a consequence of (6.8) and $b_{ij}=eR_{0i0j}/(2b_{00})$. Similarly (6.7) is obtained in virtue of (6.8). Next, in similar manner as in the last section, (6.2) gives

$$b_{0j} = \frac{G}{b_{00}} \lambda_j \, . \tag{6.10}$$

Now, substituting from (6.8) and (6.10) in (6.6), the equation

$$e\rho\nu_0 = \frac{b_{00,0}}{b_{00}} - \frac{G_0}{2G} + 2e\lambda + f(x^0) \tag{6.11}$$

is obtained, where $f(x^0)$ is an arbitrary function. Then (6.3) is rewritten in the form

$$\begin{aligned} \lambda_{ij} + \frac{G_r}{2G}(\lambda_i y_j + \lambda_j y_i) - \frac{e}{2G} R_{0i0j,0} \\ + \frac{e}{2G}\left(\frac{G_0}{2G} - 2e\lambda - f\right) R_{0i0j} = 0 \, . \end{aligned} \tag{6.12}$$

By the same reason as in the last section, we assume that λ *is a function of* x^0 *and* r. Then we obtain from (6.12)

$$\lambda_r + \frac{rQ}{G}\lambda + \frac{er}{2G}\left(Q_0 - \frac{QG_0}{2G} + Qf\right) = 0 \, , \tag{6.13}$$

$$\lambda_{rr} + \left(\frac{G_r}{G} - \frac{1}{r}\right)\lambda_r - \frac{e(P_0+Q_0)}{2G} + \frac{e(P+Q)}{2G}\left(\frac{G_0}{2G} - 2e\lambda - f\right) = 0 \, . \tag{6.14}$$

The first (6.13) gives by integration

$$\lambda = -\frac{eG_0}{4G} - \frac{e}{2} f(x^0) + \frac{1}{\sqrt{G}} g(x^0) \, , \tag{6.15}$$

where $g(y^0)$ is an arbitrary function. Further, it is easify verified that the second is a consequence of (6.15).

The preceding process is formaly analogous to the one used in the last section. We have already seen there that (5.5) gave the quantity b_{00}. On the other hand, (6.9) is not so. In fact we obtain from (6.9)

$$\lambda_0 + e\lambda^2 + f\lambda + \frac{eG_r^2}{16G} = h(x^0) \, ,$$

where $h(x^0)$ is another arbitrary function. Substitution of (6.15) gives then

$$\frac{e}{4G}\left(-G_{00} + \frac{5G_0^2}{4G} + \frac{G_r^2}{4}\right) - \frac{e}{2} f_0 + \frac{1}{\sqrt{G}} g_0 - \frac{e}{4} f^2 + \frac{e}{G} g^2 - \frac{g}{G\sqrt{G}} = h \, . \tag{6.16}$$

Therefore we know here that f, g and h are *not arbitrary*. It is concluded that there must exist three functions f, g and h of only one variable x^0 such that (6.16) holds, in order that (4.16) satisfy (C_2) and (R_2), provided that λ is a function of x^0 and r. Thus (6.16) is thought of as a restriction for the function G.

However it may be, we have now quantities e, e', $b_{\alpha\beta}$, $c_{\alpha\beta}$, and ν_α of the type (4.16) satisfying (C_2) and (R_2), as follows.

$$
(6.17)\qquad
\begin{aligned}
b_{00}b_{ij} &= \frac{e}{2}\left(-\frac{G_{ij}}{2}+\frac{G_iG_j}{4G}\right),\\
b_{00}b_{0j} &= \frac{e}{2}\left(-\frac{G_{0j}}{2}+\frac{G_0G_j}{2G}\right)-\frac{G_j}{2\sqrt{G}}\,g(x^0),\\
e\rho\nu_0 &= \frac{b_{00,0}}{b_{00}}-\frac{G_0}{G}+\frac{2e}{\sqrt{G}}\,g(x^0),\\
e\rho\nu_j &= \frac{b_{00,j}}{b_{00}}-\frac{G_j}{2G},
\end{aligned}
$$

where $b_{00}(\neq 0)$ remains still arbitrarily. Thus we have

Theorem 5. *Consider a non-flat G^{n+1}, $n\geqq 2$, such that there exist three functions f, g and h of only one variable x^0 satisfying* (6.16). *The G^{n+1} can be imbedded in a pseudo-euclidean $(n+3)$-space such that $e'=-e$, $c_{00}=\rho b_{00}\neq 0$, $\rho=\pm 1$. Such an imbedding is given by* (4.16) *and* (6.17).

We saw that Theorem 4 was not applicable to those spaces for which the quantity

$$
H = -G_{00}+\frac{5G_0^2}{4G}+\frac{G_r^2}{4}
$$

vanishes. On the other hand, Theorem 5 is fortunately applied to those exceptional case. In fact, if we take $f=g=h=0$, then (6.16) is redued to $H=0$. Then (6.17) is as follows.

$$
(6.18)\qquad
\begin{aligned}
b_{00}b_{ij} &= \frac{e}{2}\left(-\frac{G_{ij}}{2}+\frac{G_iG_j}{4G}\right),\\
b_{00}b_{0j} &= \frac{e}{2}\left(-\frac{G_{0j}}{2}+\frac{G_0G_j}{2G}\right),\\
e\rho\nu_0 &= \frac{b_{00,0}}{b_{00}}-\frac{G_0}{G},\qquad e\rho\nu_j = \frac{b_{00,j}}{b_{00}}-\frac{G_j}{2G}.
\end{aligned}
$$

Thus we have the following, a supplement to Theorem 4.

Theorem 6. *Let* G^{n+1}, $n\geqq 2$, *be such that the quantity* $H=0$. *Then* G^{n+1} *is imbedded in a pseudo-euclidean* $(n+3)$*-space, especially to satisfy* $e'=-e$, $c_{00}=\rho b_{00}\neq 0$, $\rho=\pm 1$. *Such an imbedding is given by* (4.16) *and* (6.18).

We consider the imbedding vector $\boldsymbol{z}$ and denote by $\boldsymbol{m}$ and $\boldsymbol{n}$ orthogonal unit normals to G^{n+1}. Then we have the Gauss formula

$$\boldsymbol{z}_{\alpha;\beta} = eb_{\alpha\beta}\boldsymbol{m}+e'c_{\alpha\beta}\boldsymbol{n}\,.$$

In the case of Theorem 6, these are written

$$\boldsymbol{z}_{0;0} = eb_{00}(\boldsymbol{m}-\rho\boldsymbol{n})\,,$$
$$\boldsymbol{z}_{0;j} = eb_{0j}(\boldsymbol{m}+\rho\boldsymbol{n})\,, \qquad \boldsymbol{z}_{i;j} = eb_{ij}(\boldsymbol{m}+\rho\boldsymbol{n})\,.$$

It will be easily verified that normals $\boldsymbol{m}-\rho\boldsymbol{n}$ and $\boldsymbol{m}+\rho\boldsymbol{n}$ are not orthogonal to each other, but both are null vectors.

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