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Some results on Hausdorff $\mathfrak{m}$-adic modules and $\mathfrak{m}$-adic differentials

Dedicated to Prof. Akizuki for commemoration of his 60-th birthday

By

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Introduction. This paper consists of two parts. In Part I, we study some properties of Hausdorff $\mathfrak{m}$-adic modules and we introduce the notions of $\mathfrak{m}$-adic free modules (see, Definition 1). We see in §2 that $\mathfrak{m}$-adic free modules have very similar properties as free modules, especially when coefficients rings are local rings. In §4, we see that $\mathfrak{m}$-adic free modules are nothing but free modules, so far as only semi-finite modules (see, Definition 2) over local rings are considered (Theorem 2). Making use of the results in Part I, we develope in Part II the theory of $\mathfrak{m}$-adic differentials which has been introduced in [9]. The main results are as follows.

(1) We see in §2 that the notion of the $\mathfrak{m}$-adic differentials coincides with the notion of usual differentials, so far as we treat only localities defined over fields.

(2) In §3, we seek conditions for a prime ideal in an $\mathfrak{m}$-adic ring to be unramified over its subring, in terms of $\mathfrak{m}$-adic differentials. This is an analogous subject to that in Nakai [8], §5 or Kunz [4], §3.

(3) Regular local rings are characterized in our languages (§4 and §7). These are generalizations of a number of results in [9], §6 and §7. These are also generalizations of Satz 1 in Kunz [5] and Kunz's Satz 1 itself is proved by a different way. This shows that the theory of $\mathfrak{m}$-adic differentials is useful even in order to study the usual theory of differentials, like the fact that the method

of completions is useful in order to study the theory of local rings.

(4) §5 and §6 are written as preliminary parts of §7. §5 includes some precise statements of a structure theorem of complete local rings (Proposition 1 and Proposition 2).

(5) In §8 we see that our notion of $\mathfrak{m}$-adic differentials is dual to Nagata's derivations defined in [6].

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Part I

1. General properties of Hausdorff $\mathfrak{m}$-adic modules. All rings in this paper are commutative rings with unit elements. Let R be a ring and let $\mathfrak{m}$ be its ideal. When we call R an $\mathfrak{m}$-adic ring, we always assume that $\mathfrak{m}$ has a finite basis and $\bigcap_n \mathfrak{m}^n=(0)$. When we say that an R-module M is an $\mathfrak{m}$-adic module, we mean that R is an $\mathfrak{m}$-adic ring and we consider $\mathfrak{m}$-adic topology in M (i.e., $\{\mathfrak{m}^n M: n=1, 2, \cdots\}$ is defined to be a system of neighbourhoods of (0)). But we do not always mean that M is a Hausdorff space, that is, we do not always assume that $\bigcap_n \mathfrak{m}^n M=(0)$. If M is a Hausdorff $\mathfrak{m}$-adic R-module, if R^* is the $\mathfrak{m}$-adic completion of R and if M^* is the $\mathfrak{m}$-adic completion of M, then, as in the theory of finite modules, the following are true:

(1) M is a Hausdorff $\mathfrak{m}^*$-adic R-module, where $\mathfrak{m}^*=\mathfrak{m}R^*$,

(2) $\mathfrak{m}^{*n}M^*\cap M=\mathfrak{m}^n M$,

(3) $M^*/\mathfrak{m}^{*n}M^*=M/\mathfrak{m}^n M$,

(4) if $M=\mathfrak{m}M$, then $M=(0)$.

Remark. Let R be an $\mathfrak{m}$-adic ring and let R' be an over-ring of R which is an $\mathfrak{m}R'$-adic ring. Let M be a Hausdorff R-module and let M^* be the $\mathfrak{m}$-adic completion of M. It is well known that if R is an $\mathfrak{m}$-adic Zariski ring and R' and M are finite R-modules, then it holds that $R'\otimes_R M^*$ is the $\mathfrak{m}R'$-adic completion of $R'\otimes_R M$. Generally, the following assertion is true. "$R'\otimes_R M/\bigcap_n \mathfrak{m}^n(R'\otimes_R M)$ is a dense subspace of $R'\otimes_R M^*/\bigcap_n \mathfrak{m}^n(R'\otimes_R M^*)$. If R' is a finite R-module and if it is an $\mathfrak{m}R'$-adic ring, the latter is the $\mathfrak{m}R'$-adic completion of the former."

The following lemma will be used later.

Lemma 1. *Let M be a Hausdorff $\mathfrak{m}$-adic R-module and let N be its dense submodule. We denote by N^* the $\mathfrak{m}$-adic completion of N. Assume that there exists an R-homomorphism ψ of M into N^* such that the restriction of ψ on N is the natural injection of N into N^*. Then ψ is injective and N^* is the $\mathfrak{m}$-adic completion of $\psi(M)$.*

Proof. Let n be an arbitrary non-negative integer and let a be an element of $\mathfrak{m}^n M \cap N$. Then $\psi(a) \in \mathfrak{m}^n N^* \cap N = \mathfrak{m}^n N$. This shows that $\mathfrak{m}^n M \cap N \subset \mathfrak{m}^n N$, hence $\mathfrak{m}^n M \cap N = \mathfrak{m}^n N$. Hence the $\mathfrak{m}$-adic R-module N is a subspace of the $\mathfrak{m}$-adic R-module M. From this our assertions are proved, if we notice that ψ is a continuous map.

2. $\mathfrak{m}$-adic free modules. Let M be an $\mathfrak{m}$-adic Hausdorff R-module.

Definition 1. *We call M an $\mathfrak{m}$-adic free R-module, when M contains a set of elements $\{a_\iota\}_{\iota \in I}$ such that:*

(1) *the submodule N generated by $\{a_\iota \mid \iota \in I\}$ is dense in M,*

(2) *the a_ι are linearly independent mod. $\mathfrak{m}^n M$ over $R/\mathfrak{m}^n$ for every positive integer n. In this case, we call $\{a_\iota\}_{\iota \in I}$ an $\mathfrak{m}$-adic free base of M.*

It follows immediately that a free module over an $\mathfrak{m}$-adic ring R is $\mathfrak{m}$-adic free. When M is an $\mathfrak{m}$-adic Hausdorff R-module, if the a_ι are elements of M and if we write $\Sigma \alpha_\iota a_\iota$ ($\alpha_\iota \in R$), we shall always understand that $\alpha_\iota \in \mathfrak{m}^n$ except for a finite number of ι for each n, hence $\Sigma \alpha_\iota a_\iota$ has meaning (at least in the completion of M).

Let M be an $\mathfrak{m}$-adic free R-module and $\{a_\iota\}_{\iota \in I}$ be its $\mathfrak{m}$-adic free base. If $\Sigma \alpha_\iota a_\iota = 0$, then $\alpha_\iota = 0$ for all ι's. If we denote by R^* and M^* the $\mathfrak{m}$-adic completions of R and M respectively and if $\mathfrak{m}^* = \mathfrak{m}R^*$, then by (3) in § 1 we see that M^* is an $\mathfrak{m}^*$-adic free R^*-module with an $\mathfrak{m}^*$-adic free base $\{a_\iota\}_{\iota \in I}$. Hence it follows also immediately that the completion of a free module over an $\mathfrak{m}$-adic ring is $\mathfrak{m}$-adic free.

Proposition 1. *Let R be a complete $\mathfrak{m}$-adic ring and let M be a complete Hausdorff $\mathfrak{m}$-adic R-module. Then M is $\mathfrak{m}$-adic free, if and only if there exists a set of elements $\{a_\iota\}_{\iota\in I}$ of M such that every element of M is expressed uniquely in the form $\Sigma\alpha_\iota a_\iota$ $(\alpha_\iota\in R)$. In this case $\{a_\iota\}_{\iota\in I}$ is an $\mathfrak{m}$-adic free base of M.*

Proof. Let M be $\mathfrak{m}$-adic free and let $\{a_\iota\}_{\iota\in I}$ be its $\mathfrak{m}$-adic free base. Let a be an element of M, then $a=\lim_{n\to\infty}(\Sigma\alpha_\iota^{(n)}a_\iota)$, where $\alpha_\iota^{(n)}\in R$ and $\alpha_\iota^{(n)}=0$ except for a finite number of ι for each n. We may assume that $\Sigma\alpha_\iota^{(n+1)}a_\iota-\Sigma\alpha_\iota^{(n)}a_\iota\in\mathfrak{m}^nM$, hence, by definition, $\alpha_\iota^{(n+1)}-\alpha_\iota^{(n)}\in\mathfrak{m}^n$ for every n. Then for every ι there exists $\alpha_\iota=\lim_{n\to\infty}\alpha_\iota^{(n)}$, because R is complete. It is easy to see that $\alpha_\iota\in\mathfrak{m}^n$ except for a finite number of ι, for each n. Hence $\Sigma\alpha_\iota a_\iota=a'$ exists in M. It is also easily seen that $a=a'$. Conversely, assume the existence of $\{a_\iota\}_{\iota\in I}$ of the said property. Then it follows directly from our assumptions that $N=\Sigma_\iota Ra_\iota$ is dense in M. We shall show that the a_ι are linearly independent mod. $\mathfrak{m}^nM$ over $R/\mathfrak{m}^n$ for every n. Assume that $\sum_{i=1}^{m}\alpha_{\iota(i)}a_{\iota(i)}\in\mathfrak{m}^nM$ where $\alpha_{\iota(i)}\in R$ and $\iota(i)\in I$. It is easy to see that every element in $\mathfrak{m}^nM$ is expressed in the form $\Sigma\beta_\iota a_\iota$ with $\beta_\iota\in\mathfrak{m}^n$. From this we see that $\alpha_{\iota(i)}\in\mathfrak{m}^n$.

Proposition 2. *Let M be a Hausdorff $\mathfrak{m}$-adic R-module and let R^* and M^* be the $\mathfrak{m}$-adic completion of R and M respectively. We put $\mathfrak{m}^*=\mathfrak{m}R^*$. Then the following three conditions are equivalent to each other.*

(a) *M is $\mathfrak{m}$-adic free,*

(b) *M^* is $\mathfrak{m}^*$-adic free with its $\mathfrak{m}^*$-adic free base in M,*

(c) *M^* is an $\mathfrak{m}$-adic completion of a free R-module contained in M.*

The proof is easy. Hence we omit it. In the corollary to Theorem 1, we shall see that if R is a local ring, then this proposition can be stated in a simpler way.

Lemma 2. *Let M be an $\mathfrak{m}$-adic free R-module and let $\{a_\iota\}_{\iota\in I}$ be its $\mathfrak{m}$-adic free base. Let n be a natural number. Then $M/\mathfrak{m}^nM$ is a free $(R/\mathfrak{m}^n)$-module and the classes of the elements $a_\iota(\iota\in 1)$*

mod. $\mathfrak{m}^n M$ *form its free base.*

Proof. M is the closure of $\Sigma_\iota Ra_\iota$ in M, hence $M/\mathfrak{m}^n M$ is generated by the classes of the a_ι mod. $\mathfrak{m}^n M$. The rest of the proof follows from our definition.

Lemma 3. *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let M be an $\mathfrak{m}$-adic free R-module and $\{a_\iota\}_{\iota\in I}$ be its $\mathfrak{m}$-adic free base. If $\mathfrak{A}$ is an ideal of R with a minimal basis $\gamma_1, \gamma_2, \cdots, \gamma_h$, then $\{\gamma_i a_\iota\}_{\iota\in I, i=1,2,\cdots,h}$ is a free base of $\mathfrak{A}M$ mod. $\mathfrak{m}\mathfrak{A}M$ over $R/\mathfrak{m}$.*

Proof. It follows from Lemma 2 that $\mathfrak{A}M$ is generated by the $\gamma_i a_\iota$ mod. $\mathfrak{m}\mathfrak{A}M$. Hence we have only to show that the $\gamma_i a_\iota$ are linearly independent mod. $\mathfrak{m}\mathfrak{A}M$. Take a finite subset, say $\iota(1), \iota(2), \cdots, \iota(h)$ out of I. Assume that $\sum_{i,j} \xi_{ij}\gamma_i a_{\iota(j)} \equiv 0$ mod. $\mathfrak{m}\mathfrak{A}M$ $(\xi_{ij}\in R)$. It is easily shown that $\sum_{i,j}\xi_{ij}\gamma_i a_{\iota(j)} \in \sum_\iota \mathfrak{m}\mathfrak{A}a_\iota + \mathfrak{m}^n\mathfrak{A}M$ for every positive integer n, by induction on n and using Lemma 2. Hence $\sum_i \xi_{ij}\gamma_i \in \mathfrak{m}\mathfrak{A} + \mathfrak{m}^n$. It follows that $\sum_i \xi_{ij}\gamma_i \in \mathfrak{m}\mathfrak{A}$, because R is a local ring. Hence it holds that $\xi_{ij} \equiv 0$ mod. $\mathfrak{m}$ for all i, j.

Lemma 4. *Besides the assumptions in Lemma* 3, *we assume that $\{b_\lambda\}_{\lambda\in\Lambda}$ is a set of elements of M such that the classes of the b_λ mod. $\mathfrak{m}M$ form a free base of $M/\mathfrak{m}M$ over $R/\mathfrak{m}$. Then $\{\gamma_i b_\lambda\}_{\lambda\in\Lambda, i=1,2,\cdots,h}$ form a free base of $\mathfrak{A}M$ mod. $\mathfrak{m}\mathfrak{A}M$ over $R/\mathfrak{m}$.*

The proof is straightforword by virtue of Lemma 3 because the a_ι form a free base mod. $\mathfrak{m}M$.

Theorem 1. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and let M be an $\mathfrak{m}$-adic free R-module. Then if a set of elements $\{b_\lambda\}_{\lambda\in\Lambda}$ of M is such that the set of classes mod. $\mathfrak{m}M$ of the b_λ form a free base of $M/\mathfrak{m}M$ over $R/\mathfrak{m}$, then $\{b_\lambda\}_{\lambda\in\Lambda}$ is an $\mathfrak{m}$-adic free base of M.*

Proof. We have only to prove that the b_λ are linearly independent mod. $\mathfrak{m}^n M$ over $R/\mathfrak{m}^n$ for every positive integer n. Assume that there exists a relation:

$$(*) \quad \sum_{i=1}^t \beta_i b_{\lambda(i)} \equiv 0 \text{ mod. } \mathfrak{m}^n M, \qquad \text{where } \beta_i \in R,\ \lambda(i)\in\Lambda,$$

$i=1, 2, \cdots, t$ and not all β_i are in $\mathfrak{m}^n$. Then there exists a non-negative integer $m<n$ such that $\beta_i \in \mathfrak{m}^m$ for all $i=1, 2, \cdots, t$ and $\beta_i \in \mathfrak{m}^{m+1}$ for some i. Let $\{\gamma_j\}_{j=1,2,\cdots,h}$ be a minimal basis for $\mathfrak{m}^m$ and let $\beta_i=\sum_{j=1}^{h} \varepsilon_{ij}\gamma_j$ $(i=1, 2, \cdots, t\,;\, \varepsilon_{ij} \in R)$. Then we have from (*):

$\sum_{i,j} \varepsilon_{ij}\gamma_j b_{\lambda(i)} \equiv 0 \mod. \mathfrak{m}^m M$. It follows from Lemma 4 that $\varepsilon_{ij} \equiv 0 \mod. \mathfrak{m}$ for all $i=1, 2, \cdots, t$ and all $j=1, 2, \cdots, h$. Hence $\beta_i \equiv 0 \mod. \mathfrak{m}^{m+1}$. This is a contradiction.

Corollary. *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let M be a Hausdorff $\mathfrak{m}$-adic module. We denote by R^* and M^* the $\mathfrak{m}$-adic completions of R and M respectively and put $\mathfrak{m}^*=\mathfrak{m}R^*$. Then M is $\mathfrak{m}$-adic free if and only if M^* is $\mathfrak{m}^*$-adic free.*

Proof. The only if part is already known. The if part is an immediate consequence of Proposition 2, (b) and Theorem 1.

Proposition 3. *Let R be a ring and let $\mathfrak{m}$ and $\mathfrak{n}$ be two ideals of R such that they have finite bases, $\mathfrak{n}$ contains $\mathfrak{m}$ and R is an $\mathfrak{n}$-adic ring (hence R is an $\mathfrak{m}$-adic ring, too). Let M and M' be two $\mathfrak{m}$-adic free modules with $\mathfrak{m}$-adic free base $\{a_\iota\}_{\iota\in I}$ and $\{b_\tau\}_{\tau\in J}$ respectively. Moreover, let R' be an arbitrary $\mathfrak{m}R'$-adic over-ring of R. Then the following four assertions hold.*

(a) *$M\oplus M'$ is an $\mathfrak{m}$-adic free module with $\mathfrak{m}$-adic free base $\{a_\iota\}_{\iota\in I}\cup\{b_\tau\}_{\tau\in J}$.*

(b) *$M\otimes_R M'/\bigcap_n \mathfrak{m}^n(M\otimes_R M')$ is an $\mathfrak{m}$-adic free module with an $\mathfrak{m}$-adic free base {the class of $a_\iota\otimes b_\tau$ mod. $\bigcap_n \mathfrak{m}^n(M\otimes_R M')\}_{\iota\in I,\tau\in J}$.*

(c) *$M/\bigcap_n \mathfrak{n}^n M$ is an $\mathfrak{n}$-adic free module with an $\mathfrak{n}$-adic free base {the class of a_ι mod. $\bigcap_n \mathfrak{n}^n M\}_{\iota\in I}$.*

(d) *$R'\otimes_R M/\bigcap_n \mathfrak{m}^n(R'\otimes_R M)$ is an $\mathfrak{m}R'$-adic free module with an $\mathfrak{m}$-adic free base {the class of $1\otimes a_\iota$ mod. $\bigcap_n \mathfrak{m}^n(R'\otimes_R M)\}_{\iota\in I}$.*

Proof. (a) is obvious. We put $R_n=R/\mathfrak{m}^n$. Since $M\otimes_R M'/\mathfrak{m}^n(M\otimes_R M')=(M/\mathfrak{m}^n M)\otimes_{R_n}(M'/\mathfrak{m}^n M')$, $M\otimes_R M'/\mathfrak{m}^n(M\otimes_R M')$ is a free R_n-module with a free base {the class of $a_\iota\otimes b_\tau$ mod. $\mathfrak{m}^n(M\otimes_R M')\}_{\iota\in I,\tau\in J}$. (b) follows from this. (c) and (d) are proved by similar resonings.

Remark. In Proposition 3, (d) above, if we assume that R is Noetherian and complete, that M is complete too and that R' is a finite R-module, then it can be proved that $R'\otimes_R M$ is Hausdorff, hence it is $\mathfrak{m}R'$-adic free.

Since we do not use this fact in this paper, we omit the proof.

4. Semi-finite module. Let R be a ring and M be an R-module.

Definition 2. *We say that M is a semi-finite module, if M is the direct sum of a free module and a finite module.*

Proposition 4. *If R is a Noetherian ring and M is an R-module, then the following three conditions are equivalent to each other:*

(a) *M is semi-finite,*

(b) *M is a residue module of a free module modulo its finite submodule,*

(c) *M has a free submodule N such that M/N is a finite module.*

Proof. It is easy to see that (b) and (c) follow from (a), and (a) follows from (b). Therefore we have only to prove that (b) follows from (c). Let $\{t_\iota\}_{\iota\in I}$ be a free base of N and $f_1, f_2, \cdots, f_h$ be elements of M such that their residue classes $\bar{f}_1, \bar{f}_2, \cdots, \bar{f}_h$ modulo N generate M/N. We have only to prove that there are only a finite number of t_ι which appear in some linear relations among the t_ι and the f_i. We shall use induction on the number h of the f_i. Assume that there are only a finite number of t_ι, say $t_{\iota(1)}, t_{\iota(2)}, \cdots, t_{\iota(l)}$ which appear in some linear relations among the t_ι and $f_1, f_2, \cdots, f_{h-1}$. Let $M'=N+Rf_1+Rf_2+\cdots+Rf_{h-1}$ and $\mathfrak{c}=\{r: r\in R,\ rf_h\in M'\}$. Then $\mathfrak{c}$ is an ideal of R. If $\mathfrak{c}=0$, then we have nothing to prove. Therefore we assume that $\mathfrak{c}\neq(0)$. Let $c_1, c_2, \cdots, c_r$ be a basis for $\mathfrak{c}$ and $c_i f_h=\sum_{u=1}^{l+k}\alpha_{iu}t_{\iota(u)}+\sum_{j=1}^{h-1}\beta_{ij}f_j$ $(i=1, 2, \cdots, r)$. If $\xi f_h=\sum_{j=1}^{h-1}\gamma_j f_j+\sum_{\iota\in\Omega}\varepsilon_\iota t_\iota$ is a relation among the f_i and the t_ι, where Ω is a finite subset of I and $\varepsilon_\iota\neq 0$, then $\xi\in\mathfrak{c}$ and ξ is expressed in the form $\xi=\sum_i\eta_i c_i$ $(\eta_i\in R)$. Hence $0=\xi f_h-\sum_i\eta_i c_i f_h$ $=\sum_{j=1}^{h-1}(\gamma_j-\sum_i\eta_i\beta_{ij})f_j+\sum_{\iota\in\Omega}\varepsilon_\iota t_\iota-\sum_{u=1}^{l+k}(\sum_i\eta_i\alpha_{iu})t_{\iota(u)}$. This is a relation among the t_ι and $f_1, \cdots, f_{h-1}$. Hence $\Omega \leq \{\iota(1), \iota(2), \cdots, \iota(l+k)\}$, which proves our assertion.

The following corollary is an immediate consequence of (b).

Corollary. *Let R be a Noetherian ring. Let M be an R-module and let N be its finite submodule. Then M/N is a semi-finite module, if and only if M is a semi-finite module.*

Theorem 2. *Let R be a local ring with maximal ideal* $\mathfrak{m}$ *and let M be a semi-finite R-module. If M is an* $\mathfrak{m}$*-adic free R-module, then it is a free module.*

Proof. Let $M=N\oplus F$, where N is a free R-module and F is a finite R-module and let $\{t_\iota\}_{\iota\in I}$ be a free base of N and $f_1, f_2, \cdots, f_h$ be a minimal basis of F. Then the residue classes of the t_ι and the f_j modulo $\mathfrak{m}M$ form a basis for $M/\mathfrak{m}M$ over $R/\mathfrak{m}$. Hence by Theorem 1, they are an $\mathfrak{m}$-adic free base of M and have no linear relations. Hence M is a free module.

When R is a Zariski ring, some properties in the theory of finite R-modules are extended to the case of semi-finite modules. For instance :

Proposition 5. *Let R be an* $\mathfrak{m}$*-adic Zariski ring and let M be a semi-finite R-module. Then the following three assertions hold.*

(a) *M is Hausdorff, every finite submodule N of M is closed and the* $\mathfrak{m}$*-adic topology of N is induced by the* $\mathfrak{m}$*-adic topology of M.*

Let R^* *and* M^* *be the* $\mathfrak{m}$*-adic completions of R and M respectively.*

(b) *If elements* x_i *of M are linearly independent over R, then they are linearly independent over* R^*.

(c) *If R is an integral domain and M has no torsion, then those elements of* R^* *which are zero-divisors in* M^* *are zero-divisors in* R^*.

Proof. This follows directly from the corresponding results in the theory of finite modules (see, for instance, [2], Exposé 18).

Part II

1. Basic notions on modules of differentials. Let R and S be two rings and let R be an S-algebra, that is, there is a ring homomorphism f of S into R such that $f(1)=1$. We denote by $D_S(R)$ the module of S-differentials in R, that is, an R-module characterized by the universal mapping property with respect to the S-derivations of R into an arbitrary R-module, and denote by d_S^R the S-differential operator of R, that is, the canonical derivation map of R into $D_S(R)$ (see [4] or [8])[1]. If S is the subring of R generated by the identity, they are called simply the module of differentials in R and the differential operator of R, and denoted simply by $D(R)$ and d^R. Moreover when R is an $\mathfrak{m}$-adic ring, we have introduced the notion of $\mathfrak{m}$-adic differentials in [9]. The module of $\mathfrak{m}$-adic S-differentials in R is characterized as follows. It is a Hausdorff $\mathfrak{m}$-adic module and satisfies the universal mapping property with respect to the S-derivations of R into an arbitrary Hausdorff $\mathfrak{m}$-adic module. We denote it by $\hat{D}_S(R)$. It is identified with the residue module of $D_S(R)$ modulo its submodule $\bigcap_n \mathfrak{m}^n D_S(R)$. We denote by $\hat{d}_S^R$ the canonical derivation map of R into $\hat{D}_S(R)$ and call it the $\mathfrak{m}$-adic S-differential operator of R. When S is the subring of R generated by the identity, we omit S in the notation and terminology above, as in the former case.

Lemma 1. *Let T be a ring and let R and S be two rings both of which are T-algebras. Then it holds that*:

$$D_T(R\otimes_T S) = (D_T(R)\otimes_T S)\oplus(R\otimes_T D_T(S)).$$

Proof. It is easy to see that $d=d_T^R\otimes 1+1\otimes d_T^S$ is a T-derivation map of $R\otimes_T S$ into $(D_T(R)\otimes_T S)\oplus(R\otimes_T D_T(S))$. Put $d'=d_T^{R\otimes S}$. Then by definition, there exists a T-homomorphism φ of $D_T(R\otimes_T S)$ into $(D_T(R)\otimes_T S)\oplus(R\otimes_T D_T(S))$ such that:

(1) $\quad d = \varphi\circ d'$.

If we denote by ρ the natural homomorphism of R into $R\otimes_T S$,

1) $D(R)$ and $D_S(R)$ are denoted by $M(R)$ and $M\left(\frac{R}{S}\right)$ in [4].

then $d'\circ\rho$ is a T-derivation map of R into $D_T(R\otimes_T S)$. Hence there exists an R-homomorphism ψ_1 of $D_T(R)$ into $D_T(R\otimes_T S)$ such that $d'\circ\rho=\psi_1\circ d_T^R$. Since $D_T(R\otimes_T S)$ is an $R\otimes_T S$-module, ψ_1 induces an $R\otimes_T S$-homomorphism ψ_1' of $D_T(R)\otimes_T S$ into $D_T(R\otimes_T S)$ such that $\psi_1'((d_T^R r)\otimes s)=(1\otimes s)d'(r\otimes 1)$ $(r\in R,\ s\in S)$. In the same manner, we have an $R\otimes_T S$-homomorphism ψ_2' of $R\otimes_T D_T(S)$ into $D_T(R\otimes_T S)$. Put $\psi'=\psi_1'+\psi_2'$. Then for every $r\in R$ and $s\in S$

$$\begin{aligned}\psi'\circ d(r\otimes s) &= (\psi_1'+\psi_2')\circ(d_T^R\otimes 1+1\otimes d_T^S)\ (r\otimes s)\\ &= \psi_1'((d_T^R r)\otimes s)+\psi_2'(r\otimes d_T^S s)\\ &= (1\otimes s)d'(r\otimes 1)+(r\otimes 1)d'(1\otimes s)\\ &= d'((1\otimes s)(r\otimes 1))\\ &= d'(r\otimes s)\,.\end{aligned}$$

Hence it holds that

$$(2)\qquad d'=\psi'\circ d\,.$$

Since we know that $D_T(R\otimes_T S)$ and $(D_T(R)\otimes_T S)\oplus(R\otimes_T D_T(S))$ are generated by $d(R\otimes_T S)$ and $d'(R\otimes_T S)$ respectively, (1) and (2) show that φ and ψ' are isomorphisms.

Lemm 2. *Let S be a ring and let R be an $\mathfrak{m}$-adic ring which is an S-algebra. We denote by R^* the $\mathfrak{m}$-adic completion of R. Then $\hat{D}_S(R)$ is contained in $\hat{D}_S(R^*)$ as a submodule and as a dense subspace.*

Proof. By the universality of $\hat{d}_S^R$ there exists an R-homomorphism φ of $\hat{D}_S(R)$ into $\hat{D}_S(R^*)$ such that $\varphi\circ\hat{d}_S^R x=\hat{d}_S^{R^*}x$ for each $x\in R$. On the other hand, since $\hat{d}_S^R$ is a continuous map, $\hat{d}_S^R$ can be extended to a derivation map of R^* into the $\mathfrak{m}$-adic completion $(\hat{D}_S(R))^*$ of $\hat{D}_S(R)$. Hence by the universality of $\hat{d}_S^{R^*}$, there exists an R-homomorphism ψ of $\hat{D}_S(R^*)$ into $(\hat{D}_S(R))^*$ such that $\psi\circ\hat{d}_S^{R^*}(\lim_{n\to\infty} x_n)=\lim_{n\to\infty}\hat{d}_S^R x_n$ where $\{x_n\}$ is a Cauchy sequence in R. Since $\psi\circ\varphi\circ\hat{d}_S^R x=\psi\circ\hat{d}_S^{R^*}x=\hat{d}_S^R x$ for all $x\in R$, φ is an injective map and the restriction of ψ on $\varphi(\hat{D}_S(R))$ is an injective map. Evidently $\varphi(\hat{D}_S(R))$ is dense in $\hat{D}_S(R^*)$. Hence by Lemma 1 in Part I we get our lemma.

Let R be a ring with a unique maximal ideal $\mathfrak{m}$ and let P

be a subring of R. We denote by K the residue field of R with respect to $\mathfrak{m}$ and by k the quotient field of the residue ring of P with respect to $\mathfrak{m}\cap P$. Let $\mathfrak{q}$ be an ideal of R generated by the elements of $\mathfrak{m}\cap P[R^p]$, where p is the characteristic of K.

The following lemma 1 is stated in 4, §2 as "Rangsatz", when some finiteness conditions are assumed. But it holds in general without these assumptions. For the proof of it, see [4], §2 and [2], Exposé 17, 3.

Lemma 3.[1] *With the notations as above, the following exact sequence holds*:

$$0 \to \mathfrak{m}/\mathfrak{m}^2+\mathfrak{q} \to (R/\mathfrak{m})\otimes_R D_P(R) \to D_K(K) \to 0\,.$$

Let R be an $\mathfrak{m}$-adic ring. Then, since $\hat{D}_P(R)=D_P(R)/\bigcap_n \mathfrak{m}^n D_P(R)$, it holds that $(R/\mathfrak{m})\otimes_R \hat{D}_P(R)=(R/\mathfrak{m})\otimes_R D_P(R)$. Hence the following lemma 4 holds.

Lemma 4. *If R is an $\mathfrak{m}$-adic ring, $D_P(R)$ can be replaced by $\hat{D}_P(R)$ in Lemma* 3.

In some cases these exact sequences are simplified. For the proof of Lemma 3′, see [2], Exposé 17, 3 and [9], §5.

Lemma 3′. *Besides the assumptions in Lemma* 3, *we assume that K is separable over k.*

(a) *If P is a field, then the following exact sequence holds*:

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to (R/\mathfrak{m})\otimes_R D_P(R) \to D_k(K) \to 0\,.$$

(b) *If P is a discrete valuation ring with a prime element u, then the following exact sequence holds*:

$$0 \to \mathfrak{m}/\mathfrak{m}^2 +uR \to (R/\mathfrak{m})\otimes_R D_P(R) \to D_k(K) \to 0\,.$$

Lemma 4′. *If R is an $\mathfrak{m}$-adic ring, $D_P(R)$ can be replaced by $\hat{D}_P(R)$ in Lemma* 3′.

2. Differentials of localities. The following theorem is

1) Added in Proof. Prof. Y. Nakai has kindely communicated to me that as for the complete proof of Lemma 3, we should refer to Satz 1 in R. Berger and E. Kunz: Über die Struktur der Differentialmoduln von discreten Bewertunksringen, Math. Zeit. 77, 314–338 (1961). In Satz 1 in that paper, our lemma 3′ was also mentioned in a more general form.

essentially due to Kunz [5]. We restate it as an application of Lemma 1.

Theorem 1. *Let K be a field. Let k be a subfield of K and let $S=K[x_1, \cdots, x_n]$ be an affine integral domain generated by $x_1, \cdots, x_n$ over K. Then $D_k(S)$ is a semi-finite module.*

Proof. Let k' be a subfield of K which is a finitely generated extension of k such that K and $S'=k'[x_1, \cdots, x_n]$ are linearly disjoint over k' (see, [13], Chap. 1, §6). Then $S=K\otimes_{k'}S'$. Hence $D_{k'}(S)=D_{k'}(K)\otimes_{k'}S'\oplus K\otimes_{k'}D_{k'}(S')$ by virtue of Lemma 1. Therefore $D_{k'}(S)$ is a semi-finite module. On the other hand, $D_{k'}(S)$ is the residue module of $D_k(S)$ modulo its submodule which is the natural homomorphic image of $S\otimes_{k'}D_k(k')$, and $D_k(k')$ is a finite module. Hence by the corollary to Proposition 4 in Part 1, we get our conclusion.

Corollary. *Let, K, k and S be those as in Theorem 1. Let R be a locality which is a quotient ring with respect to a prime ideal in S. Then $\hat{D}_k(R)$ coincides with $D_k(R)$.*

Proof. This is a direct consequence of Theorem 1 and Proposition 5, (a) in Part I.

3. Unramified extension. Let P be a ring and let R be an over-ring of P. Let $\mathfrak{P}$ be a prime ideal in R and let $\mathfrak{p}=\mathfrak{P}\cap P$. Then we say that $\mathfrak{P}$ is unramified if the following conditions are satisfied ([8], §5).

(U_1) $\mathfrak{P}R_{\mathfrak{P}}=\mathfrak{p}R_{\mathfrak{P}}$,

(U_2) $R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$ is a separably algebraic finite extension of $P_{\mathfrak{p}}/\mathfrak{p}P_{\mathfrak{p}}$.

It is known that when R is a finite P-module, then $\mathfrak{P}$ is unramified if and only if $D_P(R_{\mathfrak{P}})=0$ ([8], §5).

The condition (U_2) is divided into two parts ($U_2^{(1)}$) and ($U_2^{(2)}$):

($U_2^{(1)}$) $R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$ is a separably algebraic extension of $P_{\mathfrak{p}}/\mathfrak{p}P_{\mathfrak{p}}$,

(U_2) $R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$ is a finitely generated extension of $P_{\mathfrak{p}}/\mathfrak{p}P_{\mathfrak{p}}$.

Theorem 2. *We use the notations as above. If $\mathfrak{P}$ satisfies the conditions (U_1) and ($U_2^{(1)}$), then it holds that $\hat{D}_P(R_{\mathfrak{P}})=0$. Conversely,* *if we assume that* R *is a Noetherian ring and satisfies one of the following conditions* (1) *and* (2), *then from* $\hat{D}_P(R_{\mathfrak{P}})=0$ *the conditions* (U_1) *and* $(U_2^{(1')})$ *follows.*

(1) $\mathfrak{P}$ *satisfies the condition* $(U_2^{(2')})$, (2) $R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$ *is separable over* $P_{\mathfrak{p}}/\mathfrak{p}P_{\mathfrak{p}}$.

Proof. We denote by k and K the fields $P_{\mathfrak{p}}/\mathfrak{p}P_{\mathfrak{p}}$ and $R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$ respectively. Since $\hat{D}_P(R_{\mathfrak{P}})$ is a Hausdorff $\mathfrak{P}R_{\mathfrak{P}}$-adic module, we see by (4) in Part 1, §1 that $\hat{D}_P(R_{\mathfrak{P}})=0$ is equivalent to $(R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}})\otimes_R\hat{D}_P(R_{\mathfrak{P}})=0$. Hence our first assertion follows from Lemma 4. Assme that $\hat{D}_P(R_{\mathfrak{P}})=0$. Then $D_k(K)=0$ by virtue of Lemma 4. This is equivalent to say that K is separably algebraic over k in the case (1) ([2], éposé 13). Hence in both cases (1) and (2), we can use Lemma 4′ and we prove our second assertion.

When R is a finite P-module, this theorem is nothing but the above mentioned result in [8].

Remark. In the second assertion of Theorem 2, the condition (1) or (2) is not removable. In the following example, $\hat{D}_P(R)=0$ induces neither of the conditions (U_1) and $(U_2^{(1')})$.

Example. Let k_0 be a perfect field of characteristic $p\neq 0$. Let u be an independent element over k_0. We denote by K' the field $k_0(u, u^{p^{-1}}, u^{p^{-2}}, \cdots)$. Let R be a commutative K'-algebra $K'+K'x$ with a defining relation $x^2=0$. Then R is a local ring with maximal ideal $\mathfrak{P}=xR$. Let P be the subfield of R which is generated by $u+x$ over k_0. Let K be the residue field of R and let k be the image of P modulo $\mathfrak{P}$. It is evident that (U_1) does not hold. Since K is purely inseparable over k, $(U_2^{(1')})$ does not hold. On the other hand, since $R^P=K'^P=K'$, it holds that $\hat{d}_P^R u=0$. Hence $\hat{d}_P^R x=0$. Therefore we have $\hat{D}_P(R)=0$.

Corollary 1. *Let* R *be a Noetherian* $\mathfrak{m}$*-adic ring and let* P *be a subring of* R. *We assume that* $R/\mathfrak{m}$ *is finitely generated over* $P/\mathfrak{m}\cap P$ *and the module of* $\mathfrak{m}$*-adic differentials* $\hat{D}_P(R)$ *is a semi-finite module. Let* $\mathfrak{A}$ *be the annihilator of* $\hat{D}_P(R)$ *in* R. *Then a prime ideal* $\mathfrak{P}$ *in* R *containing* $\mathfrak{m}$ *is unramified if and only if* $\mathfrak{P}$ *does not contain* $\mathfrak{A}$.

Proof. By our assumption and Theorem 2, $\mathfrak{P}$ is unramified if and only if $\hat{D}_P(R_{\mathfrak{P}})=0$. We know that $D_P(R_{\mathfrak{P}})=R_{\mathfrak{P}}\otimes_R D_P(R)$ holds (by [8], Proposition 10). We denote by L and N the submodules $\bigcap_n \mathfrak{P}^n(R_{\mathfrak{P}}\otimes_R D_P(R))$ and $R_{\mathfrak{P}}\otimes(\bigcap_n \mathfrak{m}^n D_P(R))$ of $D_P(R_{\mathfrak{P}})$ respectively. Since $\mathfrak{P}^n(R_{\mathfrak{P}}\otimes_R D_P(R)) = R_{\mathfrak{P}}\otimes\mathfrak{P}^n D_P(R)$ and $R_{\mathfrak{P}}\otimes\mathfrak{P}^n D_P(R)\supset R_{\mathfrak{P}}\otimes \mathfrak{m}^n D_P(R)\supset R_{\mathfrak{P}}\otimes(\bigcap_n \mathfrak{m}^n D_P(R))$, it holes that $L\supset N$. On the other hand, we know that $\hat{D}_P(R_{\mathfrak{P}})=D_P(R)/L$ and that $R_{\mathfrak{P}}\otimes_R\hat{D}_P(R)=R_{\mathfrak{P}}\otimes_R(D_P(R)/N)$. Hence it holds that $\hat{D}_P(R_{\mathfrak{P}})=R_{\mathfrak{P}}\otimes_R\hat{D}_P(R)/\bigcap_n \mathfrak{P}^n(R_{\mathfrak{P}}\otimes_R\hat{D}_P(R))$. Since $\hat{D}_P(R)$ is a semi-finite module, it follows from this that $\hat{D}_P(R_{\mathfrak{P}})=R_{\mathfrak{P}}\otimes_R\hat{D}_P(R)$. Hence the annihilator of $\hat{D}_P(R_{\mathfrak{P}})$ is $\mathfrak{A}R_{\mathfrak{P}}$. This proves our assertion.

When R contains a field of characteristic 0, the condition (2) in Theorem 2 is automatically satisfied. The following corollary is proved in the same way as in Corollary 1.

Corollary 2. *Let R be a Noetherian $\mathfrak{m}$-adic ring, containing a field of characteristic 0. Let P be a subring of R. We assume that the module of $\mathfrak{m}$-adic differentials $\hat{D}_P(R)$ is a semi-finite mofule. Let $\mathfrak{A}$ be the annihilator of $\hat{D}_P(R)$. Then a prime ideal $\mathfrak{P}$ in R containing $\mathfrak{m}$ satisfies the conditions (U_1) and $(U_2^{(1)})$, if and only if $\mathfrak{P}$ does not contain $\mathfrak{A}$.*

4. Characterizations of regular local ring (1) (equal characteristic case). In this section, we shall extend Kunz's Satz 1 in [5] and the results in [9], §6. In the proof of Theorem 3 we shall use a similar technique as the one used in the proof of Theorem 7 in [9] with some modifications. Throughout this section we always denote by p the characteristic of a local ring under consideration.

Theorem 3. *Let R be a local ring of equal characteristic with maximal ideal $\mathfrak{m}$. If R is a regular local ring, then $\hat{D}(R)$ is an $\mathfrak{m}$-adic free R-module. Conversely, if R satisfies one of the following two conditions* (a) *and* (b) *and if $\hat{D}(R)$ is an $\mathfrak{m}$-adic free module, then R is a regular local ring.*

(a) $p=0$. (b) $p\neq 0$ *and R is analytically unramified.*

Proof. By the corollary to Theorem 1 in Part I and by Lemma 2, we may assume that R is complete. If R is a regular local ring, then R is a power series ring $K[[X_1, \cdots, X_h]]$ over a field K. Let $A=K[X_1, \cdots, X_h]$. Let k_0 be the prime field contained in K and put $B=k_0[X_1, \cdots, X_h]$. Then $A=K\otimes_{k_0}B$. Hence by Lemma 1, $D(A)$ is a free A-module. Therefore it follows that $\hat{D}(R)$ is $\mathfrak{m}$-adic free by virtue of Lemma 2 and the corollary to Theorem 1 in Part I. Conversely, assume that R satisfies one of the conditions (a) and (b) and that $\hat{D}(R)$ is an $\mathfrak{m}$-adic free R-module. Let K be a coefficient field of R. Let $\{a_\iota\}_{\iota\in I}$ be, in the case (a), a transcendental base of K over the prime field and, in the case (b), a p-independent base of K over K^p. Let $(x_1, \cdots, x_h)$ be a minimal basis for $\mathfrak{m}$. Then by Lemma 4′ we have the following exact sequence:

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to (R/\mathfrak{m})\otimes_R\hat{D}(R) \to D(R/\mathfrak{m}) \to 0\,.$$

By this fact and Theorem 1 in Part I, we see that $\hat{d}^R x_i$ $(i=1, \cdots, h)$ and $\hat{d}^R a_\iota$ form an $\mathfrak{m}$-adic free base of $\hat{D}(R)$. Let A^* be a formal power series ring with h variables $X_1, \cdots, X_h$ over K and let $\bar{\mathfrak{m}}$ be its maximal ideal. Let φ be a K-homomorphism of A^* onto R such that $\varphi(X_i)=x_i$ $(i=1, \cdots, h)$ and let $\mathfrak{A}$ be the kernel of φ. We have only to prove that $\mathfrak{A}=0$. Let $f(x)=\sum_{j=0}^{\infty} f_j(X)$ be an element of $\mathfrak{A}$ where $f_j(X)$ is a homogeneous polynomial of degree j. Since $0=f(x)=\sum_{j=0}^{\infty} f_j(x)$ and $\sum_{j=1}^{n} f_j(x)=-\sum_{j=n+1}^{\infty} f_j(x)\in\mathfrak{m}^{n+1}$, it holds that $\sum_{j=0}^{n}\hat{d}^R f_j(x)\equiv 0 \mod \mathfrak{m}^n\hat{D}(R)$. Therefore

$$\sum_{k=1}^{h}\left(\sum_{j=0}^{n}\frac{\partial f_j(x)}{\partial x_k}\right)\hat{d}^R x_k+\sum_{\iota}\left(\sum_{j=0}^{n}\frac{\partial f_j(x)}{\partial a_\iota}\right)\hat{d}^R a_\iota\equiv 0 \qquad \mod \mathfrak{m}^n\hat{D}(R)$$

where $\dfrac{\partial f_j(x)}{\partial a_\iota}=0$ except for a finite number of ι. Since the $\hat{d}^R x_k$ and the $\hat{d}^R a_\iota$ form an $\mathfrak{m}$-adic free base of $\hat{D}(R)$, it follows that

$$\sum_{j=0}^{n}\frac{\partial f_j(x)}{\partial x_k}\equiv 0 \qquad \mod \mathfrak{m}^n \text{ for each } k \text{ and } \sum_{j=0}^{n}\frac{\partial f_j(x)}{\partial a_\iota}\equiv 0$$

mod. $\mathfrak{m}^n$ for each ι. Hence $\dfrac{\partial f(x)}{\partial x_k}=\sum_{j=0}^{\infty}\dfrac{\partial f_j(x)}{\partial x_k}=0$ and $\dfrac{\partial f(x)}{\partial a_\iota}=\sum_{j=0}^{\infty}\dfrac{\partial f_j(x)}{\partial a_\iota}=0$. Therefore it holds that

(1) $\dfrac{\partial f(X)}{\partial X_k} \in \mathfrak{A}$ for every $k=1, 2, \cdots, h$ and (2) $\dfrac{\partial f(X)}{\partial a_\iota} \in \mathfrak{A}$

for every $\iota \in I$. From (1) it follows, with the same reasoning as in the proof of Theorem 7 in [9], that $\mathfrak{A}=0$ in the case (a), and that $\mathfrak{A}$ is generated by elements of $\mathfrak{A} \cap K[[X_1^p, \cdots, X_h^p]]$ in the case (b). Henceforce we shall treat only the case (b). Let f be an element of $\mathfrak{A} \cap K[[X_1^p, \cdots, X_h^p]]$. Since every coefficient of f is expressed uniquely as a polynomial of the a_ι over K^p in which the degree of each a_ι is less than p, f is expressed in the form (in the sence of convergence in $\overline{\mathfrak{m}}$-adic topology)

$$(3) \qquad f = \Sigma f_{\lambda_1(i_1)\cdots\lambda_t(i_t)} a_{\lambda_1}^{i_1} \cdots a_{\lambda_t}^{i_t},$$

where $f_{\lambda_1(i_1)\cdots\lambda_t(i_t)} \in A^{*p}$, $0 \leq i_\alpha \leq p-1$ and there are only a finite number of terms whose coefficients $f_{\lambda_1(i_1)\cdots\lambda_t(i_t)}$ does not belong to $\overline{\mathfrak{m}}^n$ for each non-negative integer n. If we prove that each $f_{\lambda_1(i_1)\cdots\lambda_t(i_t)}$ belongs to $\mathfrak{A}+\overline{\mathfrak{m}}^n$ for any given non-negative integer n, then we have $f_{\lambda_1(i_1)\cdots\lambda_t(i_t)} \in \mathfrak{A}$ because A^* is a local ring. Hence f is the limit of a sequence of elements of $(\mathfrak{A} \cap A^{*p})A^*$. Therefore it holds than $\mathfrak{A}=(\mathfrak{A} \cap A^{*p})A^*$. Then if $\mathfrak{A} \neq (0)$, we may take an element $g(\neq 0)$ of the lowest degree among all the elements of $\mathfrak{A}$ from A^{*p}. Then $g^{\frac{1}{p}} \in A^*$, $g^{\frac{1}{p}} \notin \mathfrak{A}$ and $(g^{\frac{1}{p}})^p \in \mathfrak{A}$. This contradicts to our assumption that R has no nilpotent element. Therefore we have only to prove that each $f_{\lambda_1(i_1)\cdots\lambda_t(i_t)}$ belongs to $\mathfrak{A}+\overline{\mathfrak{m}}^n$ for any given natural number n. We divide the sum of (3) into two parts as follows:

$$f = \Sigma' f_{\lambda_1(i_1)\cdots\lambda_t(i_t)} a_{\lambda_1}^{i_1} \cdots a_{\lambda_t}^{i_t} + \Sigma'' f_{\mu_1(j_1)\cdots\mu_s(j_s)} a_{\mu_1}^{j_1} \cdots a_{\mu_s}^{j_s},$$

where Σ'' sums all terms in (3) which belong to $\overline{\mathfrak{m}}^n$ and Σ' sums the rest. Then Σ' consists of a finite number of terms. We denote these sums by f_1 and f_2. Since for any set of natural numbers $(e_1, \cdots, e_v)$ and any subset $(\nu_1, \cdots, \nu_v)$ of I the partial derivative

$$\frac{\partial^{e_1+\cdots+e_v}}{\partial a_{\nu_1}^{e_1} \cdots a_{\nu_v}^{e_v}} f = \frac{\partial^{e_1+\cdots+e_v}}{\partial a_{\nu_1}^{e_1} \cdots a_{\nu_v}^{e_v}} f_1 + \frac{\partial^{e_1+\cdots+e_v}}{\partial a_{\nu_1}^{e_1} \cdots \gamma a_{\nu_v}^{e_v}} f_2$$

belongs to $\mathfrak{A}$ by (2) and since $\dfrac{\partial^{e_1+\cdots+e_v}}{\partial a_{\nu_1}^{e_1} \cdots \gamma a_{\nu_v}^{e_v}} f_2$ belongs to $\overline{\mathfrak{m}}^n$, we see that $\dfrac{\partial^{e_1+\cdots+e_v}}{\partial_{a_{\nu_1}^{e_1}\cdots\gamma a_{\nu_v}^{e_v}}} f_1$ belongs to $\mathfrak{A}+\overline{\mathfrak{m}}^n$. Hence we see that every coefficient in Σ' belongs to $\mathfrak{A}+\overline{\mathfrak{m}}^n$ by the same reasoning as in the proof of Theorem 7 in [9].

Remark. In the condition (b) in Theorem 3, the assumption that R is analytically unramified is not removable, which is shown by the following counter example.

Example. Let K be a prime field of characteristic $p \neq 0$. Let $R=K[[X, Y]]$ be a formal power series ring in two variables over K. Then $R=R/X^pR$ is not a regular local ring but $\hat{D}(R)$ is a free module of rank 2.

Corollary 1. *Let R be a locality over a field. Then R is a regular local ring if and only if $D(R)$ is a free module.*

Proof. Since every locality is analytically unramified, our assertion follows immediately from Theorem 2 in Part I, Theorem 1 and Theorem 3.

The proof of Theorem 3 is also valid for the following corollary.

Corollary 2. *Let R be a local ring of equal characteristic with maximal ideal $\mathfrak{m}$. Let k be a field contained in R. If R is a regular local ring, then $\hat{D}_k(R)$ is an $\mathfrak{m}$-adic free R-module. Conversely, if R satisfies one of the following two conditions* (a) *and* (b) *and if $\hat{D}_k(R)$ is an $\mathfrak{m}$-adic free module, then R is a regular local ring.*

(a) $p=0$. (b) $p\neq 0$, *R is analytically unramified and k is a perfect field.*

Remark. Let R and $\mathfrak{m}$ be those as above. Let k be a subfield of R such that $R/\mathfrak{m}$ is separable over k. We assume that $\hat{D}_k(R)$ is a finite R-module. Then we can prove easily by Lemma 4′ that R is a regular local ring, if and only if rank $\hat{D}_k(R)=$ rank $R+$rank $D_k(R/\mathfrak{m})$.

5. Some results on the structures of local rings. We shall give in Proposition 1 and Proposition 2 below some precise statements of a structure theorem of complete local rings. Throughout this section, we mean by a p-ring a discrete valuation ring of characteristic 0 with a prime number p as a prime element.

Lemma 5. *Let Z be the ring of rational integers and let $X_1, \cdots, X_s$ be s indeterminates. If p is a prime number, then for any integers n and k such that $0 \leqq k \leqq n$, it holds that*

$$p^{n-k}(X_1+X_2+\cdots+X_s)^{p^{n+k}} \equiv p^{n-k}(X_1^{p^{n+k}}+X_2^{p^{n+k}}+\cdots+X_s^{p^{n+k}}) + \sum_{j=0}^{k-1} p^{n-j} f_j(X_1^{p^{n+j}}, X_2^{p^{n+j}}, \cdots, X_s^{p^{n+j}}) \quad mod.\, p^n,$$

where the f_j are polynomials with coefficients in Z.

This can be proved elementarily by induction on s. Hence we omit the proof.

Proposition 1. *Let R be a complete local ring of characteristic p^m, where p is a prime number and m is a natural number. Let $\mathfrak{m}$ be a maximal ideal of R. Assume that there exists an integer n such that $\mathfrak{m}^n=(0)$. Let $\{c_\tau\}_{\tau\in\Gamma}$ be an arbitrary set of elements of R such that the classes of the c_τ mod. $\mathfrak{m}$ form a p-independent base of $R/\mathfrak{m}$ over $(R/\mathfrak{m})^p$ and let A be the set of p^{2n}-th powers of all the elements of R. Then the following assertions* (1) *and* (2) *hold.*

(1) *Let S be the set of polynomials of the c_τ with coefficients in A and of degrees less than p^{2n} on each $c_\tau^{p^n}$. Then the set of elements $J=S+pS+\cdots+p^{n-1}S$ forms a coefficient ring of R.*

(2) *Let S' be the set of polynomials of the $c_\tau^{p^n}$ with coefficients in A and of degrees less than p^n on each $c_\tau^{p^n}$. Then the set of elements $J'=S'+pS'+\cdots+p^{n-1}S'$ forms a subring of J, it is a residue ring of a p-ring mod. p^m and J'/pJ' is naturally identified with $(R/\mathfrak{m})^{p^n}$.*

Proof. (1) was already proved in Narita [10]. (2) can be proved by a similar method as in [10][1], using Lemma 5 above. In fact, in order to prove (2), it is sufficient to prove that J' is a ring. And for the purpose, it is sufficient to prove that J' is additively closed. For the brevity, we put $d_\tau=c_\tau^{p^n}$. Let f and g

1) Strictly speaking, we refer to Nagata [7], Chap. V, because in [10], the detailed proofs were omitted.

be two elements of $p^r S'$ $(0 \leq r \leq n-1)$. These are expressed as follows:

$$f = p^r \sum a^{p^{2n}} d_{\tau_1}^{e_1} \cdots d_{\tau_\nu}^{e_\nu}, \quad g = p^r \sum b^{p^{2n}} d_{\tau_1}^{e_1} \cdots d_{\tau_\nu}^{e_\nu} \quad (0 \leq e_i \leq p^n - 1).$$

By Lemma 5 we see that $(a+b)^{p^{2n}} - (a^{p^{2n}} + b^{p^{2n}})$ is a sum of elements of the forms $p^{n-j}(a^{p^{n+j}})^u (b^{p^{n+j}})^v$ $(0 \leq j < n)$. Since every element of the form $h^{p^{n+j}}$ is uniquely determined by the class of h mod. $\mathfrak{m}$, we can replace a and b by elements of S, that is, $a^{p^{n+j}}$ and $b^{p^{n+j}}$ can be expressed as p^{n+j}-th powers of polynomials of the c_τ with coefficients in A. Then applying Lemma 5 repeatedly, we see that $(a+b)^{p^{2n}} - (a^{p^{2n}} + b^{p^{2n}})$ is expressed as a sum of elements of $pS' + \cdots + p^{n-1}S'$. Hence we get an expression:

$$f + g = p^r\left(\sum (a+b)^{p^{2n}} d_{\tau_1}^{e_1} \cdots d_{\tau_\nu}^{e_\nu}\right) + h$$

where h is a sum of elements in $p^{r+1}S' + \cdots + p^{n-1}S'$. Using this fact our conclusion can be reached easily by induction on r.

Corollary. *Let J be the residue ring of a p-ring mod. p^m $(m \geq 1)$. Let $\{c_\tau\}_{\tau \in \Gamma}$ be an arbitrary subset of J such that the residue classes of the c_τ mod. pJ form a p-independent base of J/pJ over $(J/pJ)^p$. Let J' be the subring of J, generated by p^{2m}-th powers of all the elements of J and the $c_\tau^{p^m}$. Then it holds that:*

(1) *J' is a residue ring of a p-ring mod. p^m and J'/pJ' is naturally identified with $(J/pJ)^{p^m}$,*

and

(2) *J is J'-isomorphic to the residue ring of the polynomial ring $J'[X_\tau]_{\tau \in \Gamma}$ modulo its ideal $(X_\tau^{p^m} - c_\tau^{p^m})_{\tau \in \Gamma}$.*

This follows directly from Lemma 6.

The following lemma 6 can be proved by Narita's method of constructing coefficient rings of local rings in [10] with some modifications, and was proved by Rutsch [11].

Lemma 6. *Let R be a complete local ring with maximal ideal $\mathfrak{m}$. Let I be a local subring of R whose maximal ideal is generated by p, where we denote by p the characteristic of the field $R/\mathfrak{m}$. We assume that $R/\mathfrak{m}$ is separable over I/pI. Then there exists a coefficient ring J of R which contains I.*

This lemma can be generalized as follows.

Proposition 2. *Let R be a complete local ring with maximal ideal $\mathfrak{m}$. Let I be a local ring dominated by R such that*:

(1) *the maximal ideal of I is a principal ideal uI $(u \in I)$, and*

(2) *$R/\mathfrak{m}$ is separable over I/uI.*

Then there exists a local subring J of R with the same residue field with that of R, such that J contains I and such that uJ is the maximal ideal. In fact; let $\bar{I}$ be a coefficient ring of the completion I^* of I and let $\bar{J}$ be a coefficient ring of R which contains $\bar{I}$. Then $J=\bar{J}[[u]]$ is a complete local ring with maximal ideal uJ and contains I. If I is a discrete valuation ring, then J is a discrete valuation ring with a prime element u.[1]

Proof. When $\bar{I}$ is a field, then $I^*=\bar{I}[[u]]$. Hence $J=\bar{J}[[u]]$ contains I and its maximal ideal is uJ. If u is not an algebraically independent element over $\bar{J}$, then there exists a natural number m such that $u^m=0$. If I is a discrete valuation ring, then u is an algebraically independent element over $\bar{I}$. Hence u is an algebraically independent element over $\bar{J}$ and J is a discrete valuation ring with a prime element u. Henceforce we assume that $\bar{I}$ is not a field. Then $\bar{I}$ and $\bar{J}$ have maximal ideals $p\bar{I}$ and $p\bar{J}$ respectively, where we denote by p the characteristic of the field $R/\mathfrak{m}$. Since there exists a natural number r such that $p=u^r h$ where h is a unit in I and since $\bar{I}$ is complete, we see that I^* is a finite $\bar{I}$-module generated by $1, u, u^2, \cdots, u^{r-1}$ over $\bar{I}$. Hence J contains I and $uJ=(u, p)J$ is the maximal ideal of J. If I is a discrete valuation ring, then $\bar{I}$ and $\bar{J}$ are p-rings. Hence rank $\bar{J}=1$ and rank $J=1$ because J is a finite $\bar{J}$-module. Therefore J is a discrete valuation ring with a prime element u (see [7], (12. 1)).

6. The modules of differentials in discrete valuation rings.

Lemma 7. *Let J and I be p-rings and let I be a subring of J. If J/pJ is separable over I/pI, then $\hat{D}_I(J)$ is a pJ-adic free module.*

1) Since the maximal ideal of I is a principal ideal uI, I (with uI-adic topology) is a subspace of R (with m-adic topolygy). Hence there exists a completion of I in R.

Proof. By our assumptions, we see that $(J/pJ)^p$ and I/pI are linearly disjoint over $(I/pI)^p$. Hence there exist sets of elements $\{b_\lambda\}_{\lambda\in\Lambda}$ and $\{a_\omega\}_{\omega\in\Omega}$ of J such that the residue classes of the b_λ mod. pJ form a p-independent base of I/pI over $(I/pI)^p$ and the residue classes of the b_λ and the a_ω mod. pJ form a p-independent base of J/pJ over $(J/pJ)^p$. We put $J_m=J/p^mJ$, $I_m=I/p^mI$ and $d_m=d_{I_m}^{J_m}$. We denote by $b_\lambda^{(m)}$ and $a_\omega^{(m)}$ the classes of b_λ and a_ω mod. p^mJ respectively. Let J'_m and I'_m be the subrings of J_m and I_m respectively such as given in the corollary to Proposition 1, that is, J'_m (resp. I'_m) is a ring generated by all of the p^{2m}-th powers of the elements of J_m (resp. I_m) and the $b^{(m)p^m}$ and the $a^{(m)p^m}$ (resp. $b^{(m)p^m}$). Then $J_m=J'_m[X_\omega, Y_\lambda]_{\omega\in\Omega,\lambda\in\Lambda}/(X_\omega^{p^m}-a_\omega^{(m)p^m}, Y_\lambda^{p^m}-b_\lambda^{(m)p^m})_{\omega\in\Omega,\lambda\in\Lambda}$ and $I_m=I'_m[Y_\lambda]_{\lambda\in\Lambda}/(Y_\lambda^{p^m}-b_\lambda^{(m)p^m})_{\lambda\in\Lambda}$. Hence $J_m=J'_m[I_m][X_\omega]_{\omega\in\Omega}/(X_\omega^{p^m}-a_\omega^{(m)p^m})_{\omega\in\Omega}$. Therefore $D_{I_m}(J_m)$ is a free J_m-module with a free base $\{d_m a_\omega^{(m)}\}_{\omega\in\Omega}$. On the other hand, $D_{I_m}(J_m)=D_I(J)/p^mD_I(J)+Jd_I^J(p^mJ)=D_I(J)/p^mD_I(J)=\hat{D}_I(J)/p^m\hat{D}_I(J)$ and $d_m a_\omega^{(m)}$ is the class of $\hat{d}_I^J a_\omega$ mod. $p^m\hat{D}_I(J)$. Hence we get our result.

Proposition 3. *Let J and I be discrete valuation rings with a common prime element u and such that I is a subring of J. We assume that J/uJ is separable over I/uI. Then $\hat{D}_I(J)$ is a uJ-adic free module.*

Proof. We may assume that I and J are complete by virtue of the corollary to Theorem 1 in Part I and Lemma 2. Let $\bar{I}$ be a coefficient ring of I. Then there exists a coefficient ring $\bar{J}$ of J such that $\bar{J}$ contains $\bar{I}$ by virtue of Lemma 6. First we treat the case where $\bar{I}$ is a field. In this case, I and J are formal power series rings $\bar{I}[[u]]$ and $\bar{J}[[u]]$ respectively. Let $\{b_\lambda\}_{\lambda\in\Lambda}$ be a p-independent base of $\bar{I}$ over $\bar{I}^p$ and let $\{b_\lambda, a_\omega\}_{\lambda\in\Lambda,\omega\in\Omega}$ be a p-independent base of $\bar{J}$ over $\bar{J}^p$. Since J is a regular local ring of equal characteristic, $\hat{D}(J)$ is a uJ-adic free module by Theorem 3 and we see that the $\hat{d}^Jb_\lambda$, the $\hat{d}^Ja_\omega$ and $\hat{d}^Ju$ are a uJ-adic free base of $\hat{D}(J)$ by Theorem 1 in Part I and Lemma 4′. Similarly, we see that $\hat{D}(I)$ is a uI-adic free module with a uI-adic free base the $\hat{d}^Ib_\lambda$ and $\hat{d}^Iu$. By Proposition 5 in [9], it holds that $\hat{D}_I(J)=\hat{D}(J)$ mod. the uJ-adic closure of $\hat{d}^J(I)$. Hence $\hat{D}_I(J)$ is a uJ-adic free module with a uJ-adic free base $\{\hat{d}_I^J a_\omega\}_{\omega\in\Omega}$. Next we treat the case where $\bar{I}$ is a p-ring. Let $f(X)$ be a monic irreducible polynomial in $\bar{I}[X]$ such that $f(u)=0$. Then $f(X)$ is irreducible also in $\bar{J}[X]$, and $I=\bar{I}[X]/(f(X))$ and $J=\bar{J}[X]/(f(X))$. It is easy to see that $D_I(\bar{I}[X])=\bar{I}[X]d_{\bar{I}}^{\bar{I}[X]}(X)$ and $D_{\bar{I}}(\bar{J}[X])=D_{\bar{I}}(\bar{J})\otimes_{\bar{J}}\bar{J}[X]\oplus \bar{J}[X]d_{I}^{\bar{J}[X]}(X)$ where $\bar{I}[X]d_{\bar{I}}^{\bar{I}[X]}(X)$ and $\bar{J}[X]d_{\bar{I}}^{\bar{J}[X]}(X)$ are free modules. Hence applying Proposition 9 in [8], we see that $D_{\bar{I}}(I)$ $=D_{\bar{I}}(\bar{I}[X])/f(X)D_{\bar{I}}(\bar{I}[X])+\bar{I}[X]d_{\bar{I}}^{\bar{I}[X]}(f(X))=I\,d_{\bar{I}}^{I}(u)$ and $D_{\bar{I}}(J)=$ $D_{\bar{I}}(\bar{J}[X])/f(X)D_{\bar{I}}(\bar{J}[X])+\bar{J}[X]d_{\bar{I}}^{\bar{J}[X]}(f(X))=D_{\bar{I}}(\bar{J})\otimes_{\bar{J}}I\oplus J d_{\bar{I}}^{J}(u)$. From these and from Proposition 1 in [8] it follows that $D_I(J)=$ $D_{\bar{I}}(J)/Jd_{\bar{I}}^{J}(I)\simeq D_{\bar{I}}(\bar{J})\otimes_{\bar{J}}J$. Hence $\hat{D}_I(J)\simeq\hat{D}_{\bar{I}}(\bar{J})\otimes_{\bar{J}}J/\bigcap_n u^n(\hat{D}_{\bar{I}}(\bar{J})\otimes_{\bar{J}}J)$. Since $\hat{D}_{\bar{I}}(\bar{J})$ is a $p\bar{J}$-adic free module by Lemma 7, $\hat{D}_I(J)$ is a uJ-adic free module by Proposition 3, (d) and (c) in Part I.

7. Characterizations of regular local rings (II) (unequal characteristic case).

Theorem 4. *Let R be a local ring of characteristic 0 with maximal ideal $\mathfrak{m}$ and with a residue field of prime characteristic p. Then R is an unramified regular local ring, if and only if $\hat{D}(R)$ is an $\mathfrak{m}$-adic free module.*

Proof. We may assume that R is complete by the corollary to Theorem 1 in Part I and by Lemma 2. Let J be a coefficient ring of R. Then J is a p-ring. We shall assume that R is an unramified regular local ring. Then R is a power series ring $J[[X_1,\cdots,X_s]]$. We put $A=J[X_1,\cdots,X_s]$ and $\mathfrak{n}=(p, X_1,\cdots,X_s)A$. Then R is the $\mathfrak{n}$-adic completion of A. Hence in order to prove the only if part in our theorem, it is sufficient to prove that the module of $\mathfrak{n}$-adic differentials $\hat{D}(A)$ is an $\mathfrak{n}$-adic free A-module by virtue of the corollary to Theorem 1 in Part I and Lemma 2. Let J_0 be the ring of p-adic rational integers. We put $B=J_0[X_1,\cdots,X_s]$. Then $A=J\otimes_{J_0}B$, and by lemma 1 it holds that $D(A)=(D(J)\otimes_{J_0}B)$ $\oplus(J\otimes_{J_0}D(B))$. Hence

$$(1) \qquad \hat{D}(A) = (D(J)\otimes_{J_0}B)/\bigcap_n \mathfrak{n}^n(D(J)\otimes_{J_0}B) \oplus(J\otimes_{J_0}D(B))/\bigcap_n \mathfrak{n}^n(J\otimes_{J_0}D(B))\,.$$

If we prove that both components in the right hand side of (1) are $\mathfrak{n}$-adic free modules, we see that $\hat{D}(A)$ is $\mathfrak{n}$-adic free module by Proposition 3, (a) in Part I. Since $D(B)$ is a free B-module, $J\otimes_{J_0}D(B)=A\otimes_B D(B)$ is a free A-module. Hence $J\otimes_{J_0}D(B)/\bigcap_n \mathfrak{n}^n(J\otimes_{J_0}D(B))=J\otimes_{J_0}D(B)$ and it is a free A-module, hence an $\mathfrak{n}$-adic free A-module. Next, it is easily shown that $D(J)\otimes_{J_0}B/\bigcap_n \mathfrak{n}^n(D(J)\otimes_{J_0}B)=\hat{D}(J)\otimes_{J_0}B/\bigcap_n \mathfrak{n}^n(\hat{D}(J)\otimes_{J_0}B)$ (where $\hat{D}(J)$ is the module of pJ-adic differentials of J) as in the proof of Corollary 1 to Theorem 2. Since $\hat{D}(J)\otimes_{J_0}B=\hat{D}(J)\otimes_J A$ and since $\hat{D}(J)$ is a pJ-adic free module by virtue of Lemma 7, it holds that $\hat{D}(J)\otimes_{J_0}B/\bigcap_n p^n(\hat{D}(J)\otimes_{J_0}B)$ is an pA-adic free module by virtue of Proposition 3, (d) in Part I. Hence we see that $\hat{D}(J)\otimes_{J_0}B/\bigcap_n \mathfrak{n}^n(\hat{D}(J)\otimes_{J_0}B)$ is $\mathfrak{n}$-adic free by virtue of (c) in the same proposition. Now, we shall prove the if part of our theorem. Assume that $\hat{D}(R)$ is an $\mathfrak{n}$-adic free R-module. Let $\{a_\iota\}_{\iota\in I}$ be a set of elements of J such that the classes of the a_ι mod. $\mathfrak{m}$ form a p-independent base of $R/\mathfrak{m}$ over $(R/\mathfrak{m})^p$ and let $x_1, \cdots, x_r$ be a minimal basis of $\mathfrak{m}$. First we shall show that $p\notin\mathfrak{m}^2$. In the contrary case, the following exact sequence holds by Lemma 4′;

$$(2) \qquad 0\to\mathfrak{m}/\mathfrak{m}^2\to(R/\mathfrak{m})\otimes_R\hat{D}(R)\to D(R/\mathfrak{m})\to 0\,.$$

We put $d=\hat{d}^R$ and $d'=\hat{d}^J$. Then there exists an R-homomorphism φ of $R\otimes_J\hat{D}(J)$ into $\hat{D}(R)$ such that $d=\varphi\circ(1\otimes d')$. We see that the da_ι and the dx_i form an $\mathfrak{m}$-adic free base of $\hat{D}(R)$ by (2) and by Theorem 1 in Part I. Let f be an element of the power series ring $J[[X_1, \cdots, X_r]]$ and let f_j be the sum of the terms of degrees $\leqq j$ in f. Then $df(x)=\lim df_j(x)$. We shall divide $df_j(x)$ into two parts as follows:

$$(3) \qquad df_j(x) = \sum_i \frac{\partial f_j(x)}{\partial x_i}dx_j+\delta f_j(x)\,.$$

Since it is easily seen that each of the two terms in the right hand side of (3) converges when j increases infinitely, $df(x)$ can be expressed as follows:

$$(4) \qquad df(x) = \sum_i \frac{\partial f(x)}{\partial x_i} dx_i + \lim_{j\to\infty} \delta f_j(x).$$

Since $\delta f_j(x)$ belongs to the image of $R\otimes_J \hat{D}(J)$ by φ, $\lim_{j\to\infty} \delta f_j(x)$ belongs to the $\mathfrak{m}$-adic closure of the submodule of $\hat{D}(R)$ generated by the da_ι. On the other hand, the first term in the right hand side of (4) belongs to the submodule of $\hat{D}(R)$ generated by the dx_i. Therefore if $f(x)=0$, then both of the two terms in the right hand side of (4) are reduced to zero, hence $\frac{\partial f(x)}{\partial x_i}=0$ for every i. Hence by the same reasoning as in the proof of Theorem 3, we see that R is a formal power series ring with coefficients in J. Hence $p \notin \mathfrak{m}^2$. This is a contradiction. Henceforth we shall assume that $p \notin \mathfrak{m}^2$. Then we may put $p=x_1$. Applying Lemma 4′, we get the following exact sequence:

$$0 \to \mathfrak{m}/\mathfrak{m}^2+pR \to (R/\mathfrak{m})\otimes_R \hat{D}(R) \to D(R/\mathfrak{m}) \to 0.$$

Hence we see that $dx_2, \cdots, dx_r$ and the da_ι form an $\mathfrak{m}$-adic free base of $\hat{D}(R)$. Hence by the same reasoning as above, we can prove our assertion.

The proof of Theorem 4 is also valid for the following corollary, by virtue of Proposition 3.

Corollary. *Let R be a local ring of characteristic* 0 *with maximal ideal* $\mathfrak{m}$. *Let I be a discrete valuation ring with a prime element u. Assume that $R/\mathfrak{m}$ is separable over I/uI. Then R is a regular local ring and $u \notin \mathfrak{m}^2$, if and only if $\hat{D}_I(R)$ is an $\mathfrak{m}$-adic free module.*

8. Derivation of Nagata. A derivation ∂ of a ring R in the sense of M. Nagata [6] is an additive endomorphism of the total quotient ring L of R which satisfies the following conditions:

1) $\partial(xy) = x\partial y + y\partial x$ for $x, y \in L$

and

2) there exists an element a of R which is not a zero-divisor such that $a\partial x \in R$ for $x \in R$.

Let P be a subring of R. If $\partial a = 0$ for every $a \in P$, then ∂ is called a derivation (in the sense of Nagata) over P.

Since L is a quotient ring of R, ∂ is uniquely determined by the restriction map ∂' of ∂ on R. If $\bar{\partial}'$ is a derivation of R into L in the usual sense, then $\bar{\partial}'$ can be uniquely extended to the derivation $\bar{\partial}$ of L into L. However $\bar{\partial}$ cannot always satisfy the condition 2) above. The following proposition shows that when R is an $\mathfrak{m}$-adic ring and $\hat{D}_P(R)$ is a finite R-module, then the totality of the derivations in the sense of Nagata is the dual of $L \otimes_R \hat{D}_P(R)$.

Proposition 4. *Let R be an $\mathfrak{m}$-adic ring, let L be the total quotient ring of R and let P be a subring of R. Let $\hat{d}$ be the natural derivation map of R into $L \otimes_R \hat{D}_P(R)$. If ∂ is a derivation of R over P in the sense of Nagata and let ∂' be the restriction of ∂ on R, then there exists an L-homomorphism φ of $L \otimes_R \hat{D}_P(R)$ into L such that $\partial' = \varphi \circ \hat{d}$. Conversely, if $\hat{D}_P(R)$ is a finite R-module and if $\bar{\varphi}$ is an L-homomorphism of $L \otimes_R \hat{D}_P(R)$ into L, then $\bar{\partial}' = \bar{\varphi} \circ \hat{d}$ induces a derivation of R over P in the sense of Nagata.*

Proof. Let a be an element of R which is not a zero-divisor such that $a\partial x \in R$ for $x \in R$. Then ∂' is a derivation map (in the usual sense) of R into an R-module $R\frac{1}{a}$. Since $R\frac{1}{a}$ is a Hausdorff R-module, ∂' is decomposed as follows: $\partial' = \psi \circ \hat{d}_P^R$ where ψ is an R-homomorphism of $\hat{D}_P(R)$ into $R\frac{1}{a} (\subset L)$. Let φ be the induced homomorphism of $L \otimes_R \hat{D}_P(R)$ into L. Then it holds that $\partial' = \varphi \circ d$, proving our first assertion. We shall prove the second assertion. From our assumption it follows that $\bar{\varphi}(1 \otimes \hat{D}_P(R))$ is a finite R-module. Hence there exists an element a of R which is not a zero-divisor such that $a\bar{\varphi}(1 \otimes \hat{D}_P(R)) \subset R$. Therefore $a\bar{\partial}'(R) \subset R$.

In the condition 2), if a can be chosen to be 1, ∂ is called an integral derivation of R (in the sense of Nagata). The totality of the integral derivation of R is the dual of $\hat{D}_P(R)$.

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