J. Math. Kyoto Univ.
3-1 (1963) 137-143.

Note on paracompactness

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(Received May 15, 1963)
(Revised Sept. 3, 1963)

In this note, we shall be concerned with the paracompactness of a subspace of a Tychonoff space (completely regular T_1-space). Generalizing the result previously given in [8, Th. 2. 8], we shall give some necessary and sufficient conditions for a subspace of a Tychonoff space to be paracompact (Theorem 1). As a consequence of this, we shall obtain a new characterization of paracompactness (Theorem 2). In § 2, we shall apply our theorem to a subspace $X\times M$ of $BX\times M$, where BX is a compactification of a paracompact space X and M is a metrizable space, and discuss the paracompactness of the product $X\times M$ (Theorem 3).

Theorem 3 in § 2 stated originally that the product $X\times M$ of a hereditarily paracompact space X and a metrizable space M is paracompact if and only if it is normal and countably paracompact. The author is indepted to Prof. K. Morita for valuable remarks, in revision of Theorem 3.[1]

§ 1. PARACOMPACTNESS OF SUBSPACES

All spaces mentioned in this note will be completely regular and T_1 and all neighborhood will be assumed to be open. Let X

1) In his letter to the author, Prof. K. Morita has recently informed that he obtained the following result, which is slightly weaker than ours, with a theorem giving a necessary and sufficient condition for the product $X\times M$ of a normal space X and a metrizable space M to be normal and countably paracompact (to appear): The product $X\times Y$ of a paracompact space X and a metrizable space M is paracompact if and only if it is normal and countably paracompact. The author expresses his sincere thanks to Prof. K. Morita for his kindness.

be a subspace of a space E. Let x be a point of X. By the term "E-neighborhood of x" we mean an open subset of E containing x. Similarly, X-neighborhood of x is an open subset of X containing x. When no confusion can arise, we use the term "neighborhood" as usual. Let U be a neighborhood of X. If $Cl_E(S_\alpha) \subset U$ for each α, then we shall say that $\mathcal{F}$ is an *U-bounded* family, or *U-bounded* covering of X when $\mathcal{F}$ is a covering of X. A family $\mathcal{F}$ of subsets of X is said to be σ-locally finite [3] if $\mathcal{F}=\bigvee_{n=1}^{\infty}\mathcal{F}_n$, and $\mathcal{F}_n$ is a locally finite family for each n. We call a family $\mathcal{F}=\{S_\alpha\}_{\alpha\in A}$ *linearly locally finite* if there is a linear ordering $<$ of the index set A such that $\mathcal{F}(\alpha)=\{S_\lambda\}_{\lambda\leqq\alpha}$ is locally finite for each $\alpha\in A^*=(A, <)$. It is obvious that every σ-locally finite family is linearly locally finite. The following theorem asserts that a subspace X of E is paracompact if it is the union of linearly locally finite family of closed paracompact subsets whose interiors cover X.

THEOREM 1. *Let X be a subspace of a space E. Then, the following conditions are equivalent.*

a) *X is paracompact.*

b) *For each neighborhood U of X, there is a locally finite U-bounded open covering $\{S_\alpha\}_{\alpha\in A}$ of X such that $Cl_X(S_\alpha)$ is contained in a paracompact subset R_α of E for each $\alpha\in A$.*

c) *For each neighborhood U of X, there is a σ-locally finite U-bounded open covering $\{S_\alpha\}_{\alpha\in A}$ of X such that $Cl_X(S_\alpha)$ is contained in a paracompact subset R_α of E for each $\alpha\in A$.*

d) *For each neighborhood U of X, there is a linearly locally finite U-bounded open covering $\{S_\alpha\}_{\alpha\in A}$ of X such that $Cl_X(S_\alpha)$ is contained in a paracompact subset R_α of E for each $\alpha\in A$.*

Proof. To prove the implication (a)$\Rightarrow$(b), let U_x be a X-neighborhood of $x\in X$ such that $Cl_E(U_x)\subset U$ and consider a covering $\{U_x\}_{x\in X}$ of X. Take a locally finite open refinement $\{U_\alpha\}$ of $\{U_x\}$, then $\{U_\alpha\}$ is the desired open covering of X, since $Cl_X(U_\alpha)$ is paracompact for each α. Implications (b)$\Rightarrow$(c) and (c)$\Rightarrow$(d) are obvious. Assuming (d), we now prove that $X\times C$ is normal for any compact space C, which will imply (a), by virtue of the theorem due to the author [7] asserting that the normality of $X\times\beta X$

implies paracompactness of X.[2] Let C be any compact space and let F_1, F_2 be two disjoint closed subsets of $X\times C$. Put $Cl_{E\times C}(F_1)\frown Cl_{E\times C}(F_2)=K$ and $H=Pr_E(K)$, where Pr_E denotes the projection of $E\times C$ onto E. Since C is compact, Pr_E is a closed mapping and hence H is a closed subset of E contained in $E-X$. Put $U=E-H$, then U is a neighborhood of X. There is by (d) a linearly locally finite U-bounded open covering $\{S_\alpha\}_{\alpha\in A}$ such that $Cl_X(S_\alpha)$ is contained in a paracompact subset R_α of E for each α. We may assume without loss of generality that $Cl_E(R_\alpha)\subset U$ for each α. In fact, if we put $R'_\alpha=Cl_E(S_\alpha)\frown R_\alpha$, then $R'_\alpha\supset Cl_X(S_\alpha)$ and R'_α is a paracompact subset of E for which $Cl_E(R'_\alpha)\subset Cl_E(S_\alpha)\subset U$. Now, let us put $F_1(\alpha)=Cl_{E\times C}(F_1)\frown(R_\alpha\times C)$ and $F_2(\alpha)=Cl_{E\times C}(F_2)\frown(R_\alpha\times C)$, then $F_1(\alpha)$ and $F_2(\alpha)$ are disjoint closed subsets of $R_\alpha\times C$ as we now verify:

$$\begin{aligned}F_1(\alpha)\frown F_2(\alpha) &= Cl_{E\times C}(F_1)\frown Cl_{E\times C}(F_2)\frown(R_\alpha\times C)\\ &= K\frown(R_\alpha\times C)\subset(H\times C)\frown(R_\alpha\times C) = (H\frown R_\alpha)\times C\\ &= \emptyset\,, \quad \text{because} \quad R_\alpha\subset U\,.\end{aligned}$$

Since $R_\alpha\times C$ is normal (paracompact), there are disjoint open subsets $U'_1(\alpha)$ and $U'_2(\alpha)$ of $R_\alpha\times C$ containing $F_1(\alpha)$ and $F_2(\alpha)$ respectively. Put $U_1(\alpha)=U'_1(\alpha)\frown(S_\alpha\times C)$ and $U_2(\alpha)=U'_2(\alpha)\frown(S_\alpha\times C)$, then $U_1(\alpha)$ and $U_2(\alpha)$ are disjoint open subsets of $X\times C$. We put $U_1(\alpha)=\emptyset$ ($U_2(\alpha)=\emptyset$) if $F_1(\alpha)=\emptyset$ (resp. $F_2(\alpha)=\emptyset$). Since $Cl_X(S_\alpha)\subset R_\alpha$, we can see without difficulty that $Cl_{X\times C}(U_1(\alpha))\frown F_2=\emptyset$.
In fact,

$$\begin{aligned}Cl_{X\times C}(U_1(\alpha)) &= Cl_{X\times C}(U'_1(\alpha)\frown(S_\alpha\times C))\\ &\subset Cl_{E\times C}(U'_1(\alpha))\frown(Cl_X(S_\alpha)\times C)\\ &\subset Cl_{E\times C}(U'_1(\alpha))\frown(R_\alpha\times C) = Cl_{R_\alpha\times C}(U'_1(\alpha))\,.\end{aligned}$$

Therefore, we have

$$\begin{aligned}Cl_{X\times C}(U_1(\alpha))\frown F_2 &\subset Cl_{E\times C}(U'_1(\alpha))\frown(R_\alpha\times C)\frown F_2\\ &= Cl_{R_\alpha\times C}(U'_1(\alpha))\frown F_2(\alpha) = \emptyset\,.\end{aligned}$$

Similarly, we have $Cl_{X\times C}(U_2(\alpha))\frown F_1=\emptyset$ for each α. Since $\{S_\alpha\}_{\alpha\in A}$ is a linearly locally finite covering of X, both $\{U_1(\alpha)\}_{\alpha\in A}$ and

2) Cf. [5] or [8] for simple proof.

$\{U_2(\alpha)\}_{\alpha\in A}$ are linearly locally finite families of open subsets of $X\times C$ (with respect to the same linear ordering of the index set A). Let $<$ denote the linear ordering of A with respect to which $\mathcal{S}(\alpha)=\{S_\lambda\}_{\lambda\leqq\alpha}$ is locally finite for each $\alpha\in A^*=(A,<)$. Then, $\mathcal{F}_1(\alpha)=\{U_1(\lambda)\}_{\lambda\leqq\alpha}$ and $\mathcal{F}_2(\alpha)=\{U_2(\lambda)\}_{\lambda\leqq\alpha}$ are locally finite families of open subsets of $X\times C$ and therefore we have

$$Cl_{X\times C}(\bigcup_{\lambda\leqq\alpha}U_1(\lambda))=\bigcup_{\lambda\leqq\alpha}Cl_{X\times C}(U_1(\lambda))\subset(X\times C)-F_2\,,$$
$$Cl_{X\times C}(\bigcup_{\lambda\leqq\alpha}U_2(\lambda))=\bigcup_{\lambda\leqq\alpha}Cl_{X\times C}(U_2(\lambda))\subset(X\times C)-F_1\,,$$

for each $\alpha\in A^*$. Put $V_1(\alpha)=U_1(\alpha)-Cl_{X\times C}(\bigcup_{\lambda\leqq\alpha}U_2(\lambda))$ and $V_2(\alpha)=U_2(\alpha)-Cl_{X\times C}(\bigcup_{\lambda\leqq\alpha}U_1(\lambda))$, then $V_1(\alpha)$ and $V_2(\alpha)$ are disjoint open subset of $X\times C$ containing $F_1\cap(S_\alpha\times C)$ and $F_2\cap(S_\alpha\times C)$ respectively. Obviously, $V_1(\alpha)\cap V_2(\alpha')=\emptyset$ for each paire (α,α') of members of A^* and therefore $W_1=\bigcup_{\alpha\in A}V_1(\alpha)$ and $W_2=\bigcup_{\alpha\in A}V_2(\alpha)$ are disjoint open subsets of $X\times C$. On the other hand, we have $W_1\supset F_1$ and $W_2\supset F_2$, since $\bigcup_{\alpha\in A}S_\alpha=X$. It follows that $X\times C$ is normal. The proof is completed.

Let us call a subset R of X relatively paracompact if $Cl_X(R)$ is a paracompact subspace of X. Then, we have the following

COROLLARY. *If X is the union of linearly locally finite family of relatively paracompact open subsets, then X is paracompact.*

In case where E is paracompact, Theorem 1 can be stated as follows.

THEOREM 1*. *Let X be a subspace of a paracompact space E. Then, the following conditions are equivalent.*

a) *X is paracompact.*

b) *For each neighborhood U of X, there is a locally finite U-bounded open covering of X.*

c) *For each neighborhood U of X, there is a σ-locally finite U-bounded open covering of X.*

d) *For each neighborhood U of X, there is a linearly locally finite U-bounded open covering of X.*

We now give a new characterization of paracompactness.

THEOREM 2. *A space is paracompact if and only if every open covering has a linearly locally finite open refinement.*

Proof. The necessity of the condition is clear. To prove the sufficiency, let BX be a compactification of X and consider the space X as a subspace of $E=BX$. Now, let U be any neighborhood of X, and put $BX-U=C$. For each point $x\in X$, let U_x be a X-neighborhood of x such that $Cl_{BX}(U_x)\cap C=\emptyset$ and consider a covering $\mathcal{U}=\{U_x\}_{x\in X}$ of X. Take a linearly locally finite open refinement $\mathcal{V}=\{S_\alpha\}_{\alpha\in A}$ of $\mathcal{U}$ and put $R_\alpha=Cl_{BX}(S_\alpha)$ for each $\alpha\in A$. Since R_α is compact and hence paracompact, $\mathcal{V}$ is a covering of X satisfying (d) of Theorem 1. It follows that X is paracompact.

§2. Paracompactness of product spaces

In his paper [8], the author proved that *if Y is a space such that $X\times Y$ is normal for any paracompact space X, then $X\times Y$ is paracompact for any paracompact space X.* However, the normality of the product space $X\times Y$ of a paracompact space X and a space Y does not always imply the paracompactness of $X\times Y$. In fact, the product $X\times Y$ of a compact metric space X and a normal countably paracompact space Y is normal [1] but is not necessarily paracompact. A question arise here: *Does the normality of the product $X\times Y$ of a paracompact space X and a space Y imply the paracompactness of $X\times Y$, in case where Y is a metrizable space?*[3] As we can see from Theorem 3 below, the negative answer to this question implies the negative answer to Dowker's problem on the countable paracompactness of a normal space [1]. The product of a paracompact space and a metrizable space need not be normal. E. Michael [4] has given an example of a non-normal product space of a hereditarily paracompact space and a separable metric space. On the other while, K. Morita [6] has proved that *if X is a paracompact space such that $X\times M$ is normal for any metrizable space M, then $X\times M$ is paracompact for any metrizable space M.* However, the question stated above remains open.

Theorem 3. *Let X be a paracompact space and M a metrizable space. Then, the product $X\times M$ is paracompact if and only if it is countably paracompact.*

3) Note that the answer is positive if Y is a compact metrizable space.

Proof. To prove the non-trivial part of the theorem, let us consider the product $X\times M$ as a subspace of $BX\times M$, where BX is a compactification of X. By virtue of Theorem 1*, it will suffice to show that for any closed subset C of $(BX\times M)-(X\times M)$ there is a σ-locally finite open covering $\mathcal{W}$ of $X\times M$ such that $Cl_{BX\times M}(W)\cap C=\emptyset$ for each $W\in\mathcal{W}$, since $BX\times M$ is paracompact. Let $\mathcal{V}=\bigcup_{n=1}^{\infty}\mathcal{V}^n$, $\mathcal{V}^n=\{V_{\alpha_n}\}_{\alpha_n\in A_n}$, be a σ-discrete open base of M. Let Λ_k be the collection of *maximal* sets $\{(\alpha_i, \alpha_j, \cdots, \alpha_k)\}$ of indices $\alpha_i\in A_i$, $\alpha_j\in A_j$, $\cdots$, $\alpha_k\in A_k$, with $1\leq i<j<\cdots<k$, and $V_{\alpha_i}\cap V_{\alpha_j}\cap\cdots\cap V_{\alpha_k}\neq\emptyset$, and put $\Lambda=\bigcup_{k=1}^{\infty}\Lambda_k$. We put $N_\lambda=V_{\alpha_i}\cap V_{\alpha_j}\cap\cdots\cap V_{\alpha_k}$ for $\lambda=(\alpha_i, \alpha_j, \cdots, \alpha_k)$ and choose a point $P_\lambda\in N_\lambda$ for each $\lambda\in\Lambda$. It is easy to see that $\{N_\lambda\}_{\lambda\in\Lambda}$ is a σ-locally finite open base of M. For each point $z\in X\times M$, there is a neighborhood (in $X\times M$) of the form $0(z)=U\times V_{\alpha_n}$ whose closure taken in $BX\times M$ does not intersect C. Let Pr_M denote the projection of $BX\times M$ onto M and let $S(\alpha_n)$ be the set of $z\in X\times M$ such that $Pr_M[0(z)]=V_{\alpha_n}$. Let A_n' be the subset of A_n consisting of all α_n such that $S(\alpha_n)\neq\emptyset$ and put $0(\alpha_n)=\bigcup_{z\in S(\alpha_n)}0(z)$ for each $\alpha_n\in A_n'$. Put $0(n)=\bigcup_{\alpha_n\in A_n'}0(\alpha_n)$, $0_m=\bigcup_{n=1}^{m}0(n)$, and put $F_m=(X\times M)-0_m$, then $\{F_m\}$ is a countable descending chain of closed subsets of $X\times M$ with empty intersection[4]. There is a countable descending chain $\{U_m\}$ of open subsets of $X\times M$ such that $U_m\supset F_m$ for each m and $\bigcap_{m=1}^{\infty}Cl_{X\times M}(U_m)=\emptyset$ by virtue of Ishikawa's characterization of countable paracompactness [2]. Put $H_m=(X\times M)-Cl_{X\times M}(U_m)$, then we have $Cl_{X\times M}(H_m)\subset 0_m$ for each m and $\bigcup_{m=1}^{\infty}H_m=X\times M$. Thus, $\{H_m\}$ is a countable open covering of $X\times M$. Let Pr_X denote the projection of $X\times M$ onto X. For each $\lambda\in\Lambda_k$, let us put $F_\lambda=Pr_X[(X\times\{P_\lambda\})\cap Cl_{X\times M}(H_k)]$, and consider the family $\{F_\lambda\times N_\lambda\}_{\lambda\in\Lambda}$ of subsets of $X\times M$, which is a covering of $X\times M$ as we now verify. Let z be any point of $X\times M$. There is a neighborhood of z of the form $U\times N_\lambda$ contained in some H_m. We may take λ to be a member of Λ_k for $k>m$. Then, we have $U\times N_\lambda\subset H_k$ and it follows that $F_\lambda=Pr_X[(X\times\{P_\lambda\})\cap(Cl_{X\times M}(H_k))]\supset U\ni Pr_X[z]$ and consequently

4) We assume that $F_m\neq\phi$ for each m. Otherwise, the prochect $X\times M$ is easily shown to be paracompact.

$z\in F_\lambda\times N_\lambda$. Therefore $\{F_\lambda\times N_\lambda\}_{\lambda\in\Lambda}$ is a covering of $X\times M$. In view of the definition of $0(n)$, $0(n)$ is the union of open subsets of the form $U\times V_{\alpha_i}$ $(i\leq n)$ whose closures taken in $BX\times M$ are disjoint from C. Therefore $(X\times N_\lambda)\cap 0_k$, where $\lambda\in\Lambda_k$, is the union of open subsets of the form $U\times N_\lambda$ with the same property. Now, let us fix a $\lambda\in\Lambda_k$ and put $U_\lambda=Pr_X[(X\times N_\lambda)\cap 0_k]$, then $(X\times N_\lambda)\cap 0_k=U_\lambda\times N_\lambda$. Since $Cl_{X\times M}(H_k)\subset 0_k$, we see that F_λ is a closed subset of X contained in U_λ. For each $x\in F_\lambda$, there is a neighborhood $U_x\subset U_\lambda$ such that $Cl_{BX\times M}(U_x\times N_\lambda)\cap C=\emptyset$. Consider an open covering of X consisting of $X-F_\lambda$ and $\{U_x\}_{x\in F_\lambda}$ and let $\mathcal{U}_1^\lambda$ be a locally finite open refinement of the covering. Let $\mathcal{U}^\lambda=\{U_\alpha\}_{\alpha\in A(\lambda)}$ be the subfamily of $\mathcal{U}_1^\lambda$ consisting of all members intersecting F_λ. Then $\mathcal{W}^\lambda=\{U_\alpha\times N_\lambda\}_{\alpha\in A(\lambda)}$ is a locally finite family of open subsets of $X\times M$ such that $Cl_{BX\times M}(U_\alpha\times N_\lambda)\cap C=\emptyset$ and $\bigcup_{\alpha\in A(\lambda)}(U_\alpha\times N_\lambda)\supset F_\lambda\times N_\lambda$. Constructing $\mathcal{W}^\lambda$ for each $\lambda\in\Lambda$ in this fashion, we have a σ-locally finite open covering $\mathcal{W}=\bigcup_{\lambda\in\Lambda}\mathcal{W}^\lambda$ of $X\times M$ such that $Cl_{BX\times M}(W)\cap C=\emptyset$ for each $W\in\mathcal{W}$. (The σ-local finiteness of $\mathcal{W}$ follows from the σ-local finiteness of $\{N_\lambda\}_{\lambda\in\Lambda}$.) It follows that $X\times M$ is paracompact by Theorem 1*.

Theorem 1 has another consequence: If X is a paracompact space which is the union of linearly locally finite family of compact sets, then $X\times Y$ is paracompact for any paracompact space Y.

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