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Unique factorization of ideals in the sense of quasi-equality

To Professor Y. Akizuki for celebration of his 60th birthday

By

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Introduction

Throughout this paper, let R be an integral domain, i.e., a commutative ring with an identity and having no proper zero-divisors, and let K be the field of quotients of R. By an R-module, we shall mean in this paper an R-module contained in K. Let A and B be R-modules, then the set of all elements x in K such that xb is in A for every element b of B is denoted by A/B. In the special case that $A=R$, R/B is often denoted by B^{-1}, and we write $(B^{-1})^n$ by B^{-n}, for brevity. By an ideal of R, we mean a non-zero *fractional* ideal of R. If $A\subseteqq R$, then we say that A is an *integral* ideal of R. If $(A^{-1})^{-1}=A$, then we say that A is a *V-ideal* of R. If $(A^{-1})^{-1}=A$ and $AA^{-1}=A$, then we say that A is an *F-ideal* of R. It is known (cf. Mori [1]) that an F-ideal is an integral ideal and is characterized by the properties that (1) A^{-1} is a ring containing R and (2) A is a V-ideal. If $A^{-1}=B^{-1}$, then we say that A is *quasi-equal to* B and write $A\sim B$.

In this paper we shall prove the following theorem.

Theorem. *The following three conditions are equivalent to each other:*

(1) *Any ideal A of R satisfies a quasi-equality of the following type:*

$$A \sim \mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n},$$

where $\mathfrak{p}_i$ $(i=1, 2, \cdots, n)$ are prime ideals in R and r_i $(i=1, 2, \cdots, n)$

are integers, and $\mathfrak{p}_i$, r_i $(i=1, 2, \cdots, n)$ *are uniquely determined up to the order and factors which are quasi-equal to R.*

(2) *R is completely integrally closed in K and satisfies the ascending chain condition for integral V-ideals.*

(3) *R is a Krull ring*[1].

A result of this kind was stated in Krull's "Idealtheorie" (Ergeb. der Math. 4 No. 3, Julius Springer, Berlin, 1935), p. 119, without proof. The assertion is that the condition (3) is equivalent to a little stronger condition than (1), i.e., the condition (1) with additional assumption that all $\mathfrak{p}_i$ are of height 1.

As is well known, for a Noetherian integral domain, integrally closedness implies completely integrally closedness. Therefore one may ask a question if the condition of completely integrally closedness may be replaced by the integrally closedness in (2) because of the presence of the maximum condition for integral V-ideals. Unfortunately, the answer of this question is negative, and we shall show it by an example at the end of this paper.

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1. Ascending chain condition

In this section we shall obtain some results about an integral domain which satisfies the ascending chain condition for integral V-ideals.

Lemma 1. *The following two conditions for the integral domain R are equivalent to each other:*

(1) *The ascending chain condition, for integral V-ideals, holds in R.*

(2) *For every ideal A of R, there exists an ideal B such that* (i) $A \supseteqq B$, (ii) $A \sim B$ *and* (iii) *B is a finite R-module.*

Proof. At first, we shall assume that the condition (1) holds,

1) We shall follow the definition of Nagata [2]; A Krull ring is an "endliche diskrete Hauptordnung" in the sense of Krull.

and let A be a given ideal. Denote by $\mathfrak{S}=\{B_\lambda\}$ the set of all finitely generated ideals B_λ contained in A. Since every B_λ^{-1} is a V-ideal and contains A^{-1}, there is a minimal member, say B_m^{-1} ($B_m \in \mathfrak{S}$), of the set $\{B_\lambda^{-1}\}$ (cf. Nishimura [4]). If $B_m^{-1} \supsetneqq A^{-1}$, then there exists an element b of A which is not contained $(B_m^{-1})^{-1}$. Put $B_m+bR=B'$, then $B'(\subseteqq A)$ is a finite R-module and $B'^{-1} \subsetneqq B_m^{-1}$. This is a contradiction. Hence $B_m^{-1}=A^{-1}$, and the condition (2) is satisfied.

Next, we shall assume that the condition (2) holds. Let

$$A_1 \subseteqq A_2 \subseteqq \cdots \subseteqq A_n \subseteqq \cdots$$

be a sequence of integral V-ideals of R, and set $A=\bigcup_i A_i$. Then there exists a finitely generated ideal B such that $A \supseteqq B$ and $A \sim B$. Let $B=b_1R+b_2R+\cdots+b_kR$, and $b_i \in A_{n_i}$. Put $m=\max(n_1, n_2, \cdots, n_k)$, then $B \subseteqq A_m$. Since A_m is a V-ideal and $B \sim A$, we have $A \subseteqq A_m$. Hence $A_m=A_{m+1}=\cdots$, and the condition (1) is satisfied.

We shall assume throughout hereafter in this section that the ascending chain condition, for integral V-ideals, holds in the integral domain R. Let S be a non-empty subset of R which does not contain zero and which is closed under multiplication. And let $R'=R_S$ be the ring of quotients of R with respect to S.

Lemma 2. *Let A be an ideal of R and set $R/A=B$, then $R'/AR'=BR'$.*

Proof. There exists a finitely generated ideal C such that $C \subseteqq A$ and $C \sim A$. Hence $BR' =$[2] $R'/CR' \supseteqq R'/AR' \supseteqq BR'$. Therefore $R'/AR'=BR'$.

Proposition 1. *If A is a V-ideal of R, then AR' is a V-ideal of R'.*

Proof. Put $R/A=B$, then $AR'=R'/BR'$ by Lemma 2. Hence AR' is a V-ideal of R'.

Corollary. *If $\mathfrak{f}$ is an F-ideal of R, then $\mathfrak{f}R'$ is an F-ideal of R'.*

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2) Cf. Lemma 7 of Nishimura [3] and Nagata [2], (18. 2), (2).

Proof. Put $R/\mathfrak{f}=\tilde{R}$, $\tilde{R}R'(\supseteqq R')$ is a ring and $\mathfrak{f}R'=R'/\tilde{R}R'$. Hence $\mathfrak{f}R'$ is an F-ideal of R'.

Proposition 2. *If A' is a V-ideal of R', then there exists a V-ideal A of R such that $A'=AR'$; if, in this case, A' is an integral ideal of R', then we can choose A to be $A'\cap R$.*

Proof. At first, we shall assume that $A'\subseteqq R'$. Put $A'\cap R=A$ and $R/A=B$. Then BR' is a V-ideal of R' and $R'/A'=R'/AR'=BR'$. Let $R/B=C\supseteqq A$. Then $CR'=R'/BR'=A'=AR'$. Since $AR'\cap R=A$, $C\subseteqq A$. Hence $C=A$, and A is a V-ideal of R. Next, we shall assume that $A'\not\subseteqq R'$. Since A' is a V-ideal of R', there exists an element α of K such that $\alpha A'\subseteqq R'$. On the other hand, $\alpha A'$ is a V-ideal of R' (cf. Nishimura [4]). Hence there exists a V-ideal A of R such that $\alpha A'=AR'$. Therefore $A'=\alpha^{-1}AR'$, and $\alpha^{-1}A$ is also a V-ideal of R.

Corollary. *If $\mathfrak{f}'$ is an F-ideal of R', then there exists an F-ideal $\mathfrak{f}$ of R such that $\mathfrak{f}'=\mathfrak{f}R'$.*

Proof. Put $\mathfrak{f}'\cap R=\mathfrak{f}$ and $R/\mathfrak{f}=A$. Then $\mathfrak{f}$ is a V-ideal of R and $\mathfrak{f}'=\mathfrak{f}R'$, $\mathfrak{f}A\supseteqq\mathfrak{f}$. And $\mathfrak{f}R'\cdot AR'=\mathfrak{f}'AR'=\mathfrak{f}'$. Hence $\mathfrak{f}A\subseteqq\mathfrak{f}$, and $\mathfrak{f}$ is an F-ideal of R.

Lemma 3. *If $\mathfrak{p}$ is a prime ideal of height 1 in R, then $\mathfrak{p}$ is a V-ideal of R.*

Proof. Applying Proposition 2, $R_\mathfrak{p}$ satisfies the ascending chain condition for integral V-ideals. Hence the set of all integral V-ideals ($\neq R_\mathfrak{p}$) of $R_\mathfrak{p}$ has a maximal element. Since a maximal integral V-ideal ($\neq R_\mathfrak{p}$) is a prime ideal, $\mathfrak{p}R_\mathfrak{p}$ is a V-ideal of $R_\mathfrak{p}$. Hence $\mathfrak{p}$ is a V-ideal of R by Proposition 2.

2. Completely integrally closed domains

In this section we shall state about a completely integrally closed domain. The following lemmas 4~9 are known (cf. Suetsuna [5]), but we recall them for the reader's convenience.

Lemma 4. *The integral domain R is completely integrally closed in K if and only if $AA^{-1}\sim R$ for any ideal A of R.*

Proof. At first, we assume that R is completely integrally closed, and let A be a given ideal. If α is an element of A/A, then $\alpha^n A \subseteqq A$ $(n=1, 2, \cdots)$. Hence α is almost integral over R, and $A/A=R$. Hence $R/AA^{-1}=A^{-1}/A^{-1}=R$. Next, we assume that $AA^{-1}\sim R$ for any ideal A of R, then $A/A\subseteqq R/AA^{-1}=R$ and therefore $A/A=R$. If α is almost integral over R, let $R[\alpha]=B$ be a ring of polynomials in α with coefficients in R, then $\alpha B\subseteqq B$. Hence $\alpha\in B/B$, and $\alpha\in R$. Hence R is completely integrally closed in K.

We shall assume throughout hereafter in this section that R is completely integrally closed in K.

Lemma 5. *Let A and B be integral ideals of R, then $(A^{-1})^{-1} \subseteqq (B^{-1})^{-1}$ if and only if there exists an integral ideal C such that $A\sim BC$.*

Proof. If $(A^{-1})^{-1}\subseteqq(B^{-1})^{-1}$, then we can put $AB^{-1}=C$. Next, if $A\sim BC$, then $B^{-1}A\sim B^{-1}BC\sim C$. Hence AB^{-1} is an integral ideal of R, and $(A^{-1})^{-1}\subseteqq(B^{-1})^{-1}$.

Remark. If $(A^{-1})^{-1}\subsetneqq(B^{-1})^{-1}$, then C is not quasi-equal to R.

Lemma 6. *Let $\mathfrak{p}$ be a prime ideal of R, and let A and B be integral ideals of R. If $\mathfrak{p}\sim AB$ then the one of A, B is quasi-equal to $\mathfrak{p}$ and the other is quasi-equal to R.*

Proof. If $\mathfrak{p}\sim R$, then this lemma is obvious. We shall assume that $\mathfrak{p}$ is not quasi-equal to R. Then there are integral ideals C_1, C_2 such that $C_1\mathfrak{p}=C_2AB$, $C_1\sim R$, $C_2\sim R$. Since $C_2\not\subseteqq\mathfrak{p}$, $AB\subseteqq\mathfrak{p}$. If $A\subseteqq\mathfrak{p}$, then $(A^{-1})^{-1}\subseteqq(\mathfrak{p}^{-1})^{-1}$. By Lemma 5 $(A^{-1})^{-1}\supseteqq(\mathfrak{p}^{-1})^{-1}$. Hence $A\sim\mathfrak{p}$.

Lemma 7. *If the height of a prime ideal $\mathfrak{p}$ in R is greater than* 1, *then $\mathfrak{p}\sim R$.*

Proof. Let $\mathfrak{p}'$ be a prime ideal contained in $\mathfrak{p}$. Then there exists an integral ideal C such that $\mathfrak{p}'\sim C\mathfrak{p}$. Hence $\mathfrak{p}'\sim\mathfrak{p}$ or $\mathfrak{p}\sim R$. Since a prime ideal which is not quasi-equal to R is a V-ideal of R, $\mathfrak{p}\sim R$.

Lemma 8. *Let $A_1, A_2, \cdots, A_r, B_1, B_2, \cdots, B_s$ be integral ideals* *of R. If* $A_1A_2\cdots A_r\sim B_1B_2\cdots B_s$, *then there are integral ideals* $A_{i\mu}$, $B_{j\nu}$ *such that*

$A_i\sim A_{i1}A_{i2}\cdots A_{ik}$, $B_j\sim B_{j1}B_{j2}\cdots B_{jl}$ $(i=1, 2, \cdots, r;\ j=1, 2, \cdots, s)$ *and such that* $\Pi A_{i\mu}=\Pi B_{j\nu}$.

Proof. We prove the assertion by induction on r. If $r=1$, the assertion is trivial. We shall assume that $r\geqq 2$, and put $A_1+B_1=C_1$, then $A_1\sim C_1A_1'$, $B_1\sim C_1B_1'$, and $A_1'+B_1'\sim R$. Put $A_1'+B_2=C_2$, then $A_1'\sim C_2A_1''$, $B_2\sim C_2B_2'$, and $A_1''+B_2'\sim R.\cdots$Put $A_1^{(s-1)}+B_s=C_s$, then $A_1^{(s-1)}\sim C_sA_1^{(s)}$, $B_s\sim C_sB_s'$, and $A_1^{(s)}+B_s'\sim R$. Hence $C_1C_2\cdots C_sA_1^{(s)}A_2\cdots A_r\sim C_1B_1'C_2B_2'\cdots C_sB_s'$, and $A_1^{(s)}+B_1'B_2'\cdots B_s'\sim R$. Since $A_1^{(s)}(R+A_2\cdots A_r)\sim R$, $A_1^{(s)}\sim R$. Therefore $A_2\cdots A_r\sim B_1'B_2'\cdots B_s'$.

Lemma 9. *If R satisfies the ascending chain condition for integral V-ideals, then any integral ideal A in R which is not quasi-equal to R satisfies a quasi-equality of the following type*:

$$A\sim\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n},$$

where $\mathfrak{p}_i$ $(i=1, 2, \cdots, n)$ *are prime ideals of height* 1 *in R and* r_i $(i=1, 2, \cdots, n)$ *are positive integers, and* $\mathfrak{p}_i$, r_i $(i=1, 2, \cdots, n)$ *are uniquely determined up to the order.*

Proof. The height of a prime ideal which is not quasi-equal to R is 1 by Lemma 7. We shall assume that A is not quasi-equal to any finite products of prime ideals which are not quasi-equal to R. Then there is an integral ideal A_1 which is not quasi-equal to R such that $(A^{-1})^{-1}\subsetneqq(A_1^{-1})^{-1}$. Hence $A\sim A_1A_1'$ where $(A_1'^{-1})^{-1}\supsetneqq(A^{-1})^{-1}$ and A_1' is not quasi-equal to R. Hence either A_1 or A_1' is not quasi-equal to any finite products of prime ideals which are not quasi-equal to R. Thus we get an infinite sequence of integral V-ideals

$$(A^{-1})^{-1}\subsetneqq(A_1^{-1})^{-1}\subsetneqq(A_2^{-1})^{-1}\subsetneqq\cdots.$$

This is a contradiction. The uniqueness of prime ideals follows from Lemma 8.

Lemma 10. *Let A be an ideal of R. If* $A\sim\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n}$, *then* $A^{-1}\sim\mathfrak{p}_1^{-r_1}\mathfrak{p}_2^{-r_2}\cdots\mathfrak{p}_n^{-r_n}$.

Proof. $\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n}\mathfrak{p}_1^{-r_1}\mathfrak{p}_2^{-r_2}\cdots\mathfrak{p}_n^{-r_n}\sim R\sim\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n}A^{-1}$. Hence $\mathfrak{p}_1^{-r_1}\mathfrak{p}_2^{-r_2}\cdots\mathfrak{p}_n^{-r_n}\sim A^{-1}$.

3. Main theorems

In this section, we shall obtain main results about an integral domain which is completely integrally closed and satisfies the ascending chain condition for integral V-ideals.

Theorem 1. *The following three conditions are equivalent to each other:*

(1) *R is completely integrally closed in K and satisfies the ascending chain condition for integral V-ideals.*

(2) *Any ideal A of R which is not quasi-equal to R satisfies a quasi-equality of the following type:*

$$A\sim\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n},$$

where $\mathfrak{p}_i$ $(i=1,2,\cdots,n)$ are prime ideals of height 1 in R and r_i $(i=1,2,\cdots,n)$ are non-zero integers, and $\mathfrak{p}_i$, r_i $(i=1,2,\cdots,n)$ are uniquely determined, up to the order.

(3) *Any ideal A of R satisfies a quasi-equality of the following type:*

$$A\sim\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n},$$

where $\mathfrak{p}_i$ $(i=1,2,\cdots,n)$ are prime ideals in R and r_i $(i=1,2,\cdots,n)$ are integers, and $\mathfrak{p}_i$, r_i $(i=1,2,\cdots,n)$ are uniquely determined up to the order and factors which are quasi-equal to R.

Proof. Lemma 9 shows that (2) follows from (1) in the case of an integral ideal A. Let A be an ideal not contained in R. Then there exists an element a of R such that $aA\subseteqq R$. Hence $aA\sim\mathfrak{p}_1^{h_1}\mathfrak{p}_2^{h_2}\cdots\mathfrak{p}_s^{h_s}$ and by Lemma 10 $a^{-1}R\sim\mathfrak{p}_1'^{h_1'}\mathfrak{p}_2'^{h_2'}\cdots\mathfrak{p}_t'^{h_t'}$. Hence $A\sim\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n}$, where $\mathfrak{p}_i$ $(i=1,2,\cdots,n)$ are prime ideals of height 1 in R and r_i $(i=1,2,\cdots,n)$ are non-zero integers. Now we shall show that $\mathfrak{p}_i$, r_i $(i=1,2,\cdots,n)$ are uniquely determined. We suppose that we have another representation $A\sim\mathfrak{p}_1^{r_1'}\cdots\mathfrak{p}_n^{r_n'}\cdots\mathfrak{p}_p^{r_p'}$, where r_i' $(i=1,2,\cdots,n)$ are integers and r_j' $(j=n+1,\cdots,p)$ are non-zero integers. Then $\mathfrak{p}_1^{r_1'-r_1}\mathfrak{p}_2^{r_2'-r_2}\cdots\mathfrak{p}_n^{r_n'-r_n}\cdots\mathfrak{p}_p^{r_p'}\sim R$. If $r_1'-r_1<0$, then

$\mathfrak{p}_2^{r'_2-r_2}\cdots\mathfrak{p}_n^{r'_n-r_n}\cdots\mathfrak{p}_p^{r'_p}\sim\mathfrak{p}_1^{-(r'_1-r_1)}$. Hence $\mathfrak{p}_{i_1}^{k_1}\cdots\mathfrak{p}_{i_u}^{k_u}\sim\mathfrak{p}_{j_1}^{l_1}\cdots\mathfrak{p}_{j_v}^{l_v}$ where $\mathfrak{p}_{i_\mu}\neq\mathfrak{p}_{j_\nu}$ ($\mu=1, 2, \cdots, u$: $\nu=1, 2, \cdots, v$) and $k_i>0$, $l_j>0$ ($i=1, 2, \cdots u$; $j=1, 2, \cdots, v$). This contradicts to the uniqueness of factorization of integral ideals. Hence $\mathfrak{p}_i$, r_i ($i=1, 2, \cdots, n$) are uniquely determined. Thus (1) implies (2). (2) implies (3) obviously.

Next, we shall show that the condition (3) implies (1). We assume that there exists an F-ideal $\mathfrak{f}$ ($\neq R$) of R. Then

$$\mathfrak{f}\sim\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n},\quad R/\mathfrak{f}=\tilde{R}\sim\mathfrak{p}_1'^{t_1}\mathfrak{p}_2'^{t_2}\cdots\mathfrak{p}_m'^{t_m}.$$

Hence $\mathfrak{f}\tilde{R}=\mathfrak{f}\sim\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n}\mathfrak{p}_1'^{t_1}\mathfrak{p}_2'^{t_2}\cdots\mathfrak{p}_m'^{t_m}$. This is a contradiction. Since R has no F-ideal ($\neq R$), R is completely integrally closed (cf. Mori [1]). Now, we assume that R does not satisfy the ascending chain condition for integral V-ideals, then there is an infinite sequence of integral V-ideals $A_0\subsetneqq A_1\subsetneqq\cdots\subsetneqq A_n\subsetneqq\cdots$. Let $a\in A_0$ and $aR\sim\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_s^{r_s}$, $r_1+r_2+\cdots+r_s=m$. By Lemma 5 $aR\sim A_1B_1$, $A_1\sim A_2B_2, \cdots, A_{m-1}\sim A_mB_m$, where B_i ($i=1, 2, \cdots, m$) are integral ideals of R and not quasi-equal to R. Hence $aR\sim B_1B_2\cdots B_mA_m$, and A_m, B_i ($i=1, 2, \cdots, m$) are quasi-equal to finite products of prime ideals. This contradicts to the uniqueness of factorization. Hence R satisfies the ascending chain condition for integral V-ideals. Q.E.D.

Proposition 3. *If R is completely integrally closed in K and satisfies the ascending chain condition for integral V-ideals, then $R=\bigcap_{\mathfrak{p}} R_{\mathfrak{p}}$ where $\mathfrak{p}$ runs over all prime ideals of height 1 in R.*

Proof.[3] Put $\bigcap_{\mathfrak{p}} R_{\mathfrak{p}}=\tilde{R}$. It is clear that $\tilde{R}\supseteqq R$. Let $\alpha\in\tilde{R}$, then $\alpha R\sim\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_n^{r_n}$, where $\mathfrak{p}_i$ are prime ideals of height 1 in R. We consider $\alpha R_{\mathfrak{p}_i}$. Then $\alpha R_{\mathfrak{p}_i}\sim\mathfrak{p}_i^{r_i}R_{\mathfrak{p}_i}$ by virtue of Lemma 2. Since $\alpha\in\tilde{R}$, $\alpha R_{\mathfrak{p}_i}$ is an integral ideal of $R_{\mathfrak{p}_i}$. Hence $r_i\geqq 0$. Therefore αR is an integral ideal of R. Hence $\alpha\in R$ and $\tilde{R}=R$.

Lemma 11. *Let R be completely integrally closed in K and satisfy the ascending chain condition for integral V-ideals, and let $\mathfrak{p}$ be a prime ideal of height 1 in R. Then $\mathfrak{p}R_{\mathfrak{p}}$ is a principal ideal of*

3) We shall prove this proposition by a method of Prof. Nagata.

$R_{\mathfrak{p}}$, *and* $R_{\mathfrak{p}}$ *is a principal ideal domain. If* $0 \neq a \in R$, *then there are only a finite number of prime divisors* $\mathfrak{p}$ *of* aR *such that height* $\mathfrak{p}=1$.

Proof. Put $R_{\mathfrak{p}}=R'$ and $\mathfrak{p}R'=\mathfrak{p}'$. Then any integral principal ideal of R' is quasi-equal to $\mathfrak{p}'^r$ where r is a positive integer. Let $\mathfrak{S}$ be a set of all principal ideals contained in $\mathfrak{p}'$, and A' be a maximal element of $\mathfrak{S}$. Let $p \in \mathfrak{p}'$ then $pR' \subseteqq A'$. Hence $\mathfrak{p}'=A'$, hence R' is a principal ideal domain. A prime ideal $\mathfrak{p}$ of height 1 in R is a prime divisor of aR if and only if $\mathfrak{p}$ happens to appear in the factorization $aR \sim \mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2} \cdots \mathfrak{p}_n^{r_n}$, as is easily seen by considering $aR_{\mathfrak{p}}$. Therefore the last half is also proved.

Theorem 2. *An integral domain R is a Krull ring if and only if R is completely integrally closed in K and satisfies the ascending chain condition for integral V-ideals.*

Proof. At first, we assume that R is completely integrally closed in K and satisfies the ascending chain condition for integral V-ideals. Then we see that R is a Krull ring, by Proposition 3 and Lemma 11 (cf. Nagata [2], (33.3)).

Next, we shall assume that R is a Krull ring. It is clear that R is completely integrally closed in K. We assume that R does not satisfy the ascending chain condition for integral V-ideals. Let $A_0 \subsetneqq A_1 \subsetneqq \cdots \subsetneqq A_j \subsetneqq \cdots$ be an infinite sequence of integral V-ideals in R. Let $a \in A_0$, and let $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_n$ be prime ideals of height 1 which contain aR. Since $R_{\mathfrak{p}_i}$ is Noetherian, integrally closed, $aR_{\mathfrak{p}_i}=(\mathfrak{p}_i R_{\mathfrak{p}_i})^{r_i}$ by Lemma 11. Hence $aR=\mathfrak{p}_1^{(r_1)} \cap \mathfrak{p}_1^{(r_2)} \cap \cdots \cap \mathfrak{p}_n^{(r_n)}$. Let $A=\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2} \cdots \mathfrak{p}_n^{r_n}$, $r_1+r_2+\cdots+r_n=m$. Then

$$A \subseteqq aR \subseteqq A_0 \subsetneqq A_1 \subsetneqq \cdots \subsetneqq A_j \subsetneqq \cdots .$$

By the same way as the proof of Theorem 1, $A \sim B_1 B_2 \cdots B_m A_m$, where $B_1, B_2, \cdots, B_m, A_m$ are integral ideals and not quasi-equal to R. Hence $\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2} \cdots \mathfrak{p}_n^{r_n} \sim B_1 B_2 \cdots B_m A_m$. This is a contradiction by virtue of Lemma 6 and Lemma 8. Hence R satisfies the ascending chain condition for integral V-ideals. Q.E.D.

At last, we shall show an example of an integral domain which is integrally closed and satisfies the ascending chain condition for integral V-ideals, and which is not completely integrally closed.

Example. Let K be a field and let x, y be independent variables over K. Then the polynomial ring $K[x, y]$ is Noetherian, integrally closed and unique factorization ring. Put $R=K+xK[x,y]$. Since $R/K[x, y]=xK[x, y]$, $K[x, y]$ is the complete integral closure of R in its quotient field. Hence R is not completely integrally closed. Let z be integral over R and $z\notin R$, then we can write

$$z = y^m+b_1y^{m-1}+ \cdots +b_{m-1}y+b_m,$$
with $b_i\in K$ $(i = 1, 2, \cdots, m-1)$, $b_m\in R$, and
$$z^n+g_1z^{n-1}+ \cdots +g_n=0, \qquad \text{with } g_i\in R.$$ Hence
$$y^{mn}+h_1y^{mn-1}+ \cdots +h_{mn}\equiv 0 \pmod{x}, \text{ with } h_i\in K.$$

This is a contradiction. Hence $z\in R$, and R is integrally closed. Next, we shall show that R satisfies the ascending chain condition for integral V-ideals. Let $A=\{f_\lambda\}$ be an integral V-ideal of R and $d(\in K[x, y])$ be a maximal common divisor of all elements of A. Put $f_\lambda=df'_\lambda$.

(i) If $\{f'_\lambda\}\subseteqq R$, then $A\cdot\frac{R}{d}\subseteqq R$. Let $\alpha\in A^{-1}$, then $\alpha=\frac{r}{d}=\frac{r_1}{f_1}=\cdots=\frac{r_\lambda}{f_\lambda}$, where $r\in K[x, y]$, $r_i\in R$. Hence $rf'_\lambda\in R$ and $r\in R$. Hence $A^{-1}=\frac{R}{d}$. Therefore $A=dR$.

(ii) If $\{f'_\lambda\}\not\subseteqq R$, then $A\cdot\frac{xK[x, y]}{d}\subseteqq R$. Let $\alpha\in A^{-1}$, then $\alpha=\frac{r}{d}=\frac{r_1}{f_1}=\cdots=\frac{r_\lambda}{f_\lambda}$, where $r \in K[x, y]$, $r_i \in R$. Hence $rf'_\lambda \in R$ and $r\in xK[x,y]$. Hence $A^{-1}=\frac{xK[x, y]}{d}$. Therefore $A=dx^{-1}(R/K[x,y])=dK[x, y]$.

Hence there exists a correspondence between A and $dK[x, y]$. Since $K[x, y]$ is Noetherian, R satisfies the ascending chain condition for integral V-ideals.

Remark. Moreover, this example shows that the following two conditions are not equivalent to each other:

(1) An integral domain R satisfies the ascending chain condition for integral V-ideals.

(2) Every integral V-ideal of R is a finite R-module.

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