J. Math. Kyoto Univ.
3-1 (1963) 103-110.

Some remarks on algebraic rings

To Professor Y. Akizuki on his 60-th birthday

By

Hiroshi YANAGIHARA

(Communicated by Prof. Nagata, Aug. 5, 1963)

1. In the paper [1], Greenberg called a unitary ring R to be an *algebraic ring defined over* k if the following conditions are satisfied:

1) *R is a union of a finite number of algebraic varieties defined over k.*

2) *R is an algebraic group defined over k as to its additive law.*

3) *The mapping of $R\times R$ onto R, which maps (a, b) onto ab, is an everywhere regular mapping defined over k.*

4) *The unit 1 of R is a rational point of R over k.*

5) *The set U of the units in R is a locally k-closed subset in R.*

6) *The mapping of U onto U, which maps a onto a^{-1}, is an everywhere regular mapping on U.*

In this note we shall remark, first, that if R is an algebraic ring defined over k in the above sense, then the set U of the units in R is a *k-open* subset of R, and that if the characteristic of k is zero, the conditions 5) and 6) can be excluded from the definition of an algebracic ring, i.e., if R satisfies 1), 2), 3) and 4), then R satisfies necessarily 5) and 6). Let R be an algebraic ring defined over k. Then a two-sided ideal I of R will be called an *algebric ideal* of R if I is a closed subset of R. Then we shall construct a residue class ring of R by an algebraic ideal, which is also an algebraic ring. Lastly we shall show that if R is connected, any two-sided ideal of R is a connected algebraic ideal and R is a ring with maximal and minimal conditions for two-sided ideals.

2. Let R be a unitary ring which satisfies the conditions 1), 2),

3) and 4), and U the set of the units in R. Let $R_0, R_1, \cdots, R_{s-1}$ be the components of R, where 1 is contained in R_1. When a is any element of U, let R_i and R_j be the components which contain a and a^{-1} respectively, and let Γ_{ij} be the graph of the mapping f_{ij} of $R_i \times R_j$ into R_1 which maps (a, b) onto ab.

LEMMA 1. *Let R, U, a, Γ_{ij} be as above. Then we have the following*:

(i) *$k(a, a^{-1})$ is purely inseparable over $k(a)$.*

(ii) *$\Gamma_{ij} \cap (R_i \times R_j \times 1)$ has only one component, which is the locus of $(x, x^{-1}, 1)$ over $\bar{k}$, where x is any generic point of R_i over k.*

(iii) *$R_i \cap U$ is a dense subset of R_i.*

PROOF. If (i) is not true, there is a point b different from a^{-1} such that b is a generic specialization of a^{-1} over $k(a)$. Then $ab = f_{ij}(a, b) = f_{ij}(a, a^{-1}) = aa^{-1} = 1$, since f_{ij} is defined over k. This is a contradiction. Therefore (i) is true. Since $(a, a^{-1}, 1)$ is a point of $\Gamma_{ij} \cap (R_i \times R_j \times 1)$, this set is not empty. Let n be the dimension of R. Then it is easy to see that the dimension of any component C of $\Gamma_{ij} \cap (R_i \times R_j \times 1)$ is not less than n, since $R_i \times R_j \times R_1$ is non-singular. If $(x, y, 1)$ is a generic point of C over $\bar{k}$, y is equal to x^{-1}. Therefore $k(x, y)$ is purely inseparable over $k(x)$ by (i), and x is a generic point of R_i over k. This means that $\Gamma_{ij} \cap (R_i \times R_j \times 1)$ has only one component and that $R_i \cap U$ contains any generic point of R_i over k. Therefore (ii) and (iii) are also satisfied.
q.e.d.

PROPOSITION 1. *Let R be an algebraic ring defined over k. Then the set U of the units of R is a k-open subset of R.*

PROOF. Let R_i be a component of R which intersects with U. Then by Lemma 1 $R_i \cap U$ is a dense subset of R_i. On the other hand U is a locally closed subset of R, and hence $R_i \cap U$ must be a k-open subset of R_i. Therefore U is a k-open subset of R, since R is a disjoint sum of $R_0, R_1, \cdots, R_{s-1}$. q.e.d.

PROPOSITION 2. *Let R be a unitary ring which satisfies the conditions* 1), 2), 3) *and* 4). *Let R_1 be the component containing* 1, *and x a generic point of R_1 over k. Then if $k(x^{-1})$ is equal to $k(x)$,*

R is an algebraic ring defined over k, i.e., R satisfies the conditions 5) *and* 6).

PROOF. Let $R_0, R_1, \cdots, R_{s-1}$ be the components of R. Let f_{ij} and Γ_{ij} be the same as in Lemma 1. By the assumptions, the locus V of (x, x^{-1}) over k on $R_1 \times R_1$ defines a birational correspondence g between R_1 and itself. If a is any point of $R_1 \cap U$, a^{-1} is also in $R_1 \cap U$. Then $(a, a^{-1}, 1)$ is in $\Gamma_{11} \cap (R_1 \times R_1 \times 1)$ and hence (a, a^{-1}) is in V by (ii) of Lemma 1. Moreover it is easy to see that if (a, b) is in V, b is equal to a^{-1}. Therefore g is regular at a by Zariski's Main Theorem, since R_1 is nonsingular. Similarly g^{-1} is regular at a^{-1}. On the other hand if a is any point of R_1 such that (a, b) is in V for some b in R_1, a is in $R_1 \cap U$ by (ii) of Lemma 1. Therefore $R_1 \cap U$ is the set of the points at which g is biregular and hence it is a k-open subset of R_1. Next let R_i be a component of R which intersects with U. Then any generic point x_i of R_i is in U by (iii) of Lemma 1. Assume that x_i^{-1} is in R_j, and let x_j be a generic point of R_j over $k(x_i)$. Then $x_i x_j$ is in R_1 and hence $R_1 \cap U$. Therefore we have $k(x_i, x_j) \supset k(x_i x_j) = k((x_i x_j)^{-1})$. On the other hand, since $x_j(x_i x_j)^{-1} = x_i^{-1}$, we have $k(x_i, x_j) \supset k(x_i^{-1}, x_j)$ and hence $k(x_i) \supset k(x_i^{-1})$. Similarly we have $k(x_i^{-1}) \supset k(x_i)$. Therefore $k(x_i)$ is equal to $k(x_i^{-1})$, and we can define a birational correspondence g_{ij} between R_i and R_j. By the quite similar way to the above, we can see that $R_i \cap U$ is a k-open subset of R_i and g_{ij} is biregular at any point of $R_i \cap U$. This means that the conditions 5) and 6) are slso satisfied on R. q.e.d.

COROLLARY. *Let R be a unitary ring which satisfies the conditions* 1), 2), 3) *and* 4). *Then if the characteristic of k is zero, R is an algebraic ring defined over k.*

PROOF. This is a direct consequence of Proposition 2 and Lemma 1.

3. Now we shall call an algebraic ring S to be a *residue class ring* of an algebraic ring R by an algebraic ideal I of R (see 1) if S satisfies the following condition:

There is a separable homomorphism φ of R onto S, whose kernel is equal to I.

Remark: S is a factor group of R by I with respect to additive law in the sense of [2]. It is easy to see that the residue class ring of R by I is uniquely determined within a biregular isomorphism.

THEOREM 1. *Let R be an algebraic ring defined over k and I an algebraic ideal of R whose components are defined over k. Then there exists always a residue class ring of R by I, which is defined over k.*

PROOF. By Theorem 4 in [2], there exists a factor group $S=R/I$ of R by I, defined over k, with respect to additive law. Let φ be the natural homomorphism of R onto S with the kernel I. We shall show that S has a structure of an algebraic ring defined over k, which satisfies the condition of a residue class ring of R by I. Let S_1 and S_2 be two components of S (S_1 may be equal to S_2), and let R_1 and R_2 be components of R which are mapped onto S_1 and S_2 by φ respectively. If x and y are independent generic points of R_1 and R_2 over k respectively, $\varphi(x)$ and $\varphi(y)$ are independent generic points of S_1 and S_2 over k respectively. Now we show that $k(\varphi(x), \varphi(y))$ contains $k(\varphi(xy))$. In fact $k(\varphi(xy), \varphi(x), \varphi(y))$ is separable over $k(\varphi(x), \varphi(y))$, since $k(x, y)$ contains $k(\varphi(xy))$ and $k(x, y)$ is separable over $k(\varphi(x), \varphi(y))$. If $k(\varphi(x), \varphi(y))$ does not contain $k(\varphi(xy))$, there exists a generic specialization z, different from $\varphi(xy)$, of $\varphi(xy)$ over $k(\varphi(x), \varphi(y))$, which can be extended to a generic specialization (x', y') of (x, y). Then we have $z=\varphi(x'y')$, $\varphi(x')=\varphi(x)$ and $\varphi(y')=\varphi(y)$. From these, we conclude that $x-x'$ and $y-y'$ are both in I and hence that $xy-x'y'$ is in I. This means that $z=\varphi(x'y')=\varphi(xy)$, and we have a contradiction. Therefore we have a rational mapping h' of $S_1\times S_2$ into a component S_3 of S which maps $(\varphi(x), \varphi(y))$ onto $\varphi(xy)$. Assume that xy is in a component R_3 of R, and let h be the rational mapping of $R_1\times R_2$ into R_3 which maps (x, y) onto xy. Let (a', b') be any point of $S_1\times S_2$. Then the specialization $(\varphi(x), \varphi(y)) \overset{k}{\rightarrow} (a', b')$ can always be extended to a specialization $(x, y) \overset{k}{\rightarrow} (a, b)$. Then $(xy, \varphi(xy))$ has a uniquely determined specialization $(ab, \varphi(ab))$ over the above specialization. Moreover we can easily see that $\varphi(ab)$ depends only on (a', b'), but not on (a, b). From this fact we can see, using Zariski's Main Theorem, that h'. is regular at (a', b'). Therefore we easily see that S has a structure of a ring whose multiplication is defined naturally by that of R. Then φ is a ring homomorphism of R onto S, and S satisfies the conditions 1), 2), 3) and 4). Now let R_1 (resp. S_1) be the component of R (resp. S) containing 1. Then $\varphi(R_1)$ is equal to S_1. Let x be a generic point of R_1. Then we have $k(x)=k(x^{-1})\supset k(\varphi(x^{-1}))=k(\varphi(x)^{-1})$. On the other hand $k(x)$ is separable over $k(\varphi(x))$ and $k(\varphi(x), \varphi(x)^{-1})$ is purely inseparable over $k(\varphi(x))$ by Lemma 1. Therefore we have $k(\varphi(x))=k(\varphi(x)^{-1})$ and hence S is an algebraic ring defined over k by Proposition 2.

q.e.d.

Remark: Let R be an algebraic ring and I its algebraic ideal. If the residue class ring R/I is connected, the set U of the units in R is mapped onto the set U' of the units in R/I. In fact let R_1 be the component of R containing 1. Then $\varphi|_{U\cap R_1}$ is a rational homomorphism of $U\cap R_1$ into U', which is defined over k and generically surjective. This means that φ maps $U\cap R_1$ onto U'.

4. In the following we shall study the structure of algebraic ideals of a connected algebraic ring.

PROPOSITION 3. *Let R be an algebraic ring defined over k and I an algebraic ideal of R. Then the component I_0 of I containing 0 is also an algebraic ideal of R. In particular the component R_0 of R containing 0 is an algebraic ideal of R.*

PROOF. Let R_i be any component of R and k' a field containing k over which all the components of I are defined. Let x_i and y be independent generic points of R_i and I_0 over k'. It is sufficient to show that x_iy and yx_i are both in I_0. Since I is an algebraic ideal of R, x_iy and yx_i are in components of I. On the other hand y is specialized to 0 over k' and hence x_iy (resp. yx_i) is specialized to 0 over k'. Therefore x_iy are yx_i must be in I_0.

q.e.d.

COROLLARY. *Let R be an algebraic ring defined over k. Then R is connected if and only if the components containing* 0 *and* 1 *are the same.*

PROPOSITION 4. *Let R be a connected algebraic ring defined over k. Then any algebraic ideal of R is also connected.*

PROOF. Let I be an algebraic ideal of R, and let k' be a field containing k over which all the components of I are defined. Let I_i be any component of I. If x and y_i are independent generic points of R and I_i, xy_i is in a component I_j of I. However x has 1 as a specialization over k', xy_i is in I_i, i.e., I_j is equal to I_i. On the other hand x has 0 as a specialization over k', xy_i is in the component I_0 of I containing 0, i.e., I_j is equal to I_0. This means that I_i is equal to I_0. q.e.d.

COROLLARY. *Let R be a connected algebraic ring. Then the length m of a sequence of proper algebraic ideals of R such that*

$$I_1 \subsetneqq I_2 \subsetneqq \cdots \subsetneqq I_m$$

is less than the dimension of R.

PROPOSITION 5. *Let R be a connected algebraic ring defined over k. Let $a_1, a_2, \cdots, a_s$ be rational points of R over k. Then the two-sided ideal generated by $a_1, a_2, \cdots, a_s$ is a connected algebraic ideal of R, whose component is defined over k.*

PROOF. Let $x_1, \cdots, x_s, y_1, \cdots, y_s$ be independent generic points of R over k, and put $z = x_1a_1y_1 + \cdots + x_sa_sy_s$. Then $k(z)$ is a regular extension over k. Let W be the locus of z over k. Let $z_1, \cdots, z_i$ be independent generic points of W over k and W_i the locus of $z_1 + \cdots + z_i$ over k. Since W contains 0, we have $W_i \subset W_{i+1}$ and hence there exists a positive integer N such that $W_N = W_i$ for any $i \geqq N$. Then it is easy to see that W_N is an algebraic subgroup of R as an additive group (cf. Propositions 3 and 5 in [3]) and that W_N contains the ideal I generated by $a_1, \cdots, a_s$ as a dense subset. Since W_N is an algebraic subgroup of R, any point in W_N is the sum of two generic points of W_N over k and hence is a point of the ideal I. Therefore W_N is equal to I, i.e., I is an algebraic ideal. q.e.d.

We shall say that an algebraic ideal I of R is *defined over* k, if k is a field of definition of R and all the components of I.

COROLLARY. *Let R be a commutative algebraic ring defined over an algebraically closed field k. If R is not a field, then R has a proper algebraic ideal defined over k.*

PROOF. If R is not connected, the component of R containing 0 is a proper algebraic ideal defined over k by Proposition 3. If R is connected, there is a rational point a over k such that a is not a unit of R. Then aR is a proper algebraic ideal defined over k by Proposition 5. q.e.d.

THEOREM 2. *Let R be a connected algebraic ring defined over an algebraically closed field k. Then for any algebraic ideal I of R defined over k, there are rational points $a_1, \cdots, a_s$ such that $I=(a_1, \cdots, a_s)$. Moreover the number s of the points can be less than the dimension of R.*

PPOOF. Let $a_1, \cdots, a_i$ be any rational points of I over k. Then $(a_1, \cdots, a_i)$ is in I. Therefore we can choose a sequence of rational points $a_1, a_2, \cdots, a_i, \cdots$, of I, such that if $(a_1, \cdots, a_{i-1})$ is not equal to I, a_i is not in $(a_1, \cdots, a_{i-1})$. This is possible, since the set of the rational points in I over k is dense in I. This means that our assetion is true by Corollary of Proposition 4. q.e.d.

THEOREM 3. *Let R be a connected algebraic ring defined over k. Then any ideal of R is an algebraic ideal defined over a field containing k and R is a ring with maximal and minimal conditions for two-sided ideals.*

PROOF. First notice that, for any finite set of points $a_1, \cdots, a_t$ in R, $(a_1, \cdots, a_t)$ is an algebraic ideal of R defined over $k(a_1, \cdots, a_t)$ by Corollary of Proposition 5. From this we can see that R is a ring with maximal condition for two-sided ideals. In fact if it is not so, there exists an ideal I of R which is not finitely generated. Then we can choose a_i $(i=1, 2, \cdots)$ inductively so that a_i is an element of I not contained in $I_{i-1}=(a_1, \cdots, a_{i-1})$. On the other hand I_j is a subvariety of R defined over $k(a_1, \cdots, a_i)$ for $j \leqq i$. Therefore by Corallary of Proposition 4 there exists a positive integer N such that $I_i = I_N$ for $i \geqq N$. This is a contradiction. Since any two-sided ideal I of R is finitely generated, I is an algebraic ideal of R defined over a certain field containing k by Corollary of Proposition 5 and hence the length of any sequence of proper two-sided ideals such that

$$I_1 \supsetneqq I_2 \supsetneqq \cdots \supsetneqq I_m$$

is less than the dimension of R by Corollary of Proposition 4. q.e.d.

Remark: Although we treat only two-sided ideals of connected algebraic rings in Theorems 2 and 3, similar results can be obtained in the case of left (or right) ideals. In other words any left (or right) ideals of a connected algebraic ring R is closed in a Zariski topology and R is a ring with maximal and minimal conditions for left (or right) ideals. In fact we can easily see that all results in this section are modified, taking closed left (or right) ideals instead of algebraic ideals.

FACULTY OF SCIENCE,
HIROSHIMA UNIVERSITY.

REFERENCES

[1] Greenberg, M. J., "Schemata over local rings", Ann. of Math., Vol. 73 (1961), pp. 624-648.

[2] Rosenlicht, M., "Some basic theorems on algebraic groups", Amer. J. of Math., Vol. 78 (1956), pp. 401-443.

[3] Weil, A., "Variétés abéliennes et courbes algébriques", Actualités sci. et ind. no. 1064 (1948), Paris.