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A. Skorohod's stochastic integral equation for a reflecting barrier diffusion

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1. Introduction. Given a standard Brownian sample path b with $b(0)=0$, let $[\mathfrak{x}, \mathfrak{t}]$ be a solution of

1a) $$\mathfrak{x}(t) = a+\int_0^t \sqrt{c_2[\mathfrak{x}(s)]}\,db+\int_0^t c_1[\mathfrak{x}(s)]\,ds+\mathfrak{t}(t)$$

1b) $$\mathfrak{x}(0) = a \geq 0^2$$

subject to the conditions: a) for each $t\geq 0$, $\mathfrak{x}(t)$ and $\mathfrak{t}(t)$ are Borel functions of the path $b(s): s\leq t$, b) $\mathfrak{x}(t)$ is continuous and non-negative, and c) $\mathfrak{t}(t)$ is continuous, non-negative, increasing, flat outside $\mathfrak{Z}=(t: \mathfrak{x}=0)$, and $\mathfrak{t}(0)=0$. A. SKOROHOD [6] proved that if $c_2>0$ and if

2) $$\begin{aligned}&|c_1(b)-c_1(a)| + |\sqrt{c_2(b)}-\sqrt{c_2(a)}|\\ &\leq \text{constant} \times (b-a) \qquad (0\leq a< b),\end{aligned}$$

then 1) has a unique solution $[\mathfrak{x}, \mathfrak{t}]$ for all but a negligeable class of Brownian paths, $\mathfrak{x}$ being identical in law to the diffusion with generator $\mathfrak{G}u=(c_2/2)u''+c_1u'$ subject to the reflecting barrier condition $u^+(0)=\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}[u(\varepsilon)-u(0)]=0$.

SKOROHOD seems to have overlooked it, but $\mathfrak{t}$ is the local time[3] of $\mathfrak{x}$:

3) $$\mathfrak{t}(s) = c_2(0)\lim_{\varepsilon\downarrow 0}(2\varepsilon^{-1})\text{ measure }(s: \mathfrak{x}(s)<\varepsilon,\ s\leq t),$$

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2 $\int f\,db$ is an Itô stochastic integral.

3 See K. Itô and H. P. McKean, Jr. [4] for information about local times.

and this identification (together with time substitutions and scale changes) can be used to make a simple proof that the reflecting barrier motion $\mathfrak{x}$ associated with $\mathfrak{G}$ and its local time $\mathfrak{t}$ solve 1) for a suitable Brownian motion b under the sole condition that c_1 and $0<c_2$ be piecewise continuous. GIRSANOV [2] proved that $\mathfrak{x}=a+\int_0^t |\mathfrak{x}(s)|^\gamma db$ has several solutions if $0<\gamma<1/2$, so some extra condition such as 2) is needed to ensure that 1) has just one solution.

2. Reflecting BROWNian Motion. P. LÉVY [5] proved that if b is a standard Brownian motion starting at $b(0)=a\geq 0$ and if $\mathfrak{t}^+(t)=-\min_{s\leq t} b(s)\wedge 0$[4], then $b^+=b+\mathfrak{t}^+$ is a reflecting Brownian motion starting at $b^+(0)=a$ and $\mathfrak{t}^+$ is its local time:

$$\mathfrak{t}^+(t) = \lim_{\varepsilon\downarrow 0} (2\varepsilon)^{-1} \text{ measure } (s : b^+(s)<\varepsilon,\ s\leq t) .$$

$[\mathfrak{x}=b^+,\ \mathfrak{t}=\mathfrak{t}^+]$ solves SKOROHOD's problem 1) in this special case $(c_1=0,\ c_2=1)$, and to see that no other solution can exist the best method is to follow SKOROHOD who observed that if two solutions $\mathfrak{x}=b+\mathfrak{t}$ and $\mathfrak{y}=b+\mathfrak{r}$ existed, then $\mathfrak{x}-\mathfrak{y}=\mathfrak{t}-\mathfrak{r}$ could not become >0 (<0) because that would make $\mathfrak{x}>0$ ($\mathfrak{y}>0$), $\mathfrak{t}(\mathfrak{r})$ would be flat, and $\mathfrak{t}-\mathfrak{r}=\mathfrak{x}-\mathfrak{y}$ would be falling (rising).

3. A Time Substitution. Given piecewise continuous $c_2>0$, if $\mathfrak{f}(t)=\int_0^t ds/c_2(b^+)$, then $\mathfrak{x}=b^+(\mathfrak{f}^{-1})$[5] is identical in law to the reflecting barrier motion associated with $\mathfrak{G}u=(c_2/2)u''$, $\mathfrak{t}=\mathfrak{t}^+(\mathfrak{f}^{-1})$ is its local time:

$$\begin{aligned} & c_2(0) \lim_{\varepsilon\downarrow 0} (2\varepsilon)^{-1} \text{ measure } (s : \mathfrak{x}(s)<\varepsilon,\ s\leq t) \\ &= c_2(0) \lim_{\varepsilon\downarrow 0} (2\varepsilon)^{-1} \int_0^t e_{0\varepsilon}(\mathfrak{x})ds\ ^{6} \\ &= c_2(0) \lim_{\varepsilon\downarrow 0} (2\varepsilon)^{-1} \int_0^{\mathfrak{f}^{-1}} e_{0\varepsilon}(b^+)ds/c_2(b^+) \\ &= \mathfrak{t}^+(\mathfrak{f}^{-1}) , \end{aligned}$$

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4 $a\wedge b$ means the smaller of a and b.

5 $\mathfrak{f}^{-1}$ is the inverse function of $\mathfrak{f}$; for information about time substitutions, see B. Volkonskii [7].

6 e_{ab} is the indicator function of $[a, b]$.

and $\mathfrak{x}=a+\int_0^t\sqrt{c_2(\mathfrak{x})}\,db^\bullet+\mathfrak{t}$ with a new standard Brownian motion $b^\bullet$ as is clear from $\mathfrak{x}=b^+(\mathfrak{f}^{-1})=b(\mathfrak{f}^{-1})+\mathfrak{t}^+(\mathfrak{f}^{-1})$ and the following indications.

Given $d>0$, change c_2 so as to have $c_2(a)\equiv c_2(d)$ $(a\geq d)$. DOOB's optional sampling recipie [1 : 373] tells us that under the time substitution $t\to\mathfrak{f}^{-1}$, the martingales b and b^2-t go over into new martingales. Also, $\mathfrak{f}^{-1}(t_1)=\mathfrak{m}$ is a Brownian stopping time, and introducing the field $\boldsymbol{B}_{\mathfrak{m}+}$ of Brownian events B such that $B\cap(\mathfrak{m}<t)$ depends upon $b(s): s\leq t$ alone for each $t\geq 0$, it develops that

$$\begin{aligned}&E(|b(\mathfrak{f}^{-1}(t_2))-b(\mathfrak{f}^{-1}(t_1))|^2|\boldsymbol{B}_{\mathfrak{m}+})\\ =\;&E(\mathfrak{f}^{-1}(t_2)-\mathfrak{f}^{-1}(t_1)|\boldsymbol{B}_{\mathfrak{m}+})\\ =\;&E\left(\int_{t_1}^{t_2}c_2(\mathfrak{x})\,ds\,|\boldsymbol{B}_{\mathfrak{m}+}\right).\end{aligned}$$

DOOB [1 : 449] now tells us that $b(\mathfrak{f}^{-1})=a+\int_0^t\sqrt{c_2(\mathfrak{x})}\,db^\bullet$ with a new standard Brownian motion $b^\bullet$, and it is obvious that the provisional modification of c_2 is not needed for the correctness of the final formula.

4. A Scale Change. Bring in a new scale $l=l(a)$ based on c_2 and another piecewise continuous function c_1:

$$a=\int_0^l db\exp\left(-2\int_0^b c_1/c_2\right),$$

let $\mathfrak{x}$ be the motion described above but based on b and $c_3=c_2(l)\exp\left(-4\int_0^l c_1/c_2\right)$ instead of on $b^\bullet$ and c_2 $(d\mathfrak{x}=\sqrt{c_3}\,db+dt)$, and let $\mathfrak{y}$ be the scaled motion $\mathfrak{y}=l(\mathfrak{x})$. K. ITÔ's rule of stochastic differentiation [3 : 59] tells us that

4a) $$d\mathfrak{y}=l'(\mathfrak{x})d\mathfrak{x}+(1/2)l''(\mathfrak{x})(d\mathfrak{x})^2,$$

	db	dt
db	dt	0
dt	0	0

and computing $(d\mathfrak{x})^2$ with the aid of the indicated multiplication table gives

4b) $$d\mathfrak{y}=\sqrt{c_2(\mathfrak{y})}\,db+c_1(\mathfrak{y})dt+d\mathfrak{t}.$$

$\mathfrak{t}$ is the local time of $\mathfrak{y}$, and it follows from $\mathfrak{y}=l(\mathfrak{x})$

that $\mathfrak{b}$ is identical in law to the reflecting barrier motion associated with $\mathfrak{G}u=(c_2/2)u''+c_1u'$. It can happen that $l(\infty)<\infty$; in this case $\mathfrak{b}$ explodes to ∞ at a time $t=\mathfrak{e}<\infty$, and 4b) holds just up to time $\mathfrak{e}$.

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