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On existence of a resolved surface of a singular surface

To Professor Y. Akizuki on his 60-th birthday

By

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Throughout this paper, we shall mean by a surface either a projective surface or a complete surface. For a surface whose singularities are only isolated singular points, the notion of its resolved surface was introduced by J. E. Reeve and J. A. Tyrrell (see [2]), and they stated that the existence is not proved yet. Though the writer thinks that the existence is known practically by many people, because of the refered statement and also because of the lack of the published proof to the writer's knowledge, the writer likes to give an explicit proof of the existence. As a corollary we shall answer affirmatively a question raised by J. E. Tyrrell which asks if the following is true: when V_1 and V_2 are resolved surface of a given surface V, there is a resolved surface V_3 of V which dominates both V_1 and V_2.

Let V be a surface, and let V' be a non-singular surface which is birationally equivalent to V and dominates V. We denote by T the anti-regular map from V onto V'. If V' satisfies the following conditions 1)$\sim$4) then we shall say that V' is a *resolved* surface of V.

Let Ω^* be the set of all points of V' which correspond to singular points of V, then Ω^* is a closed set of V'.

1) T is biregular at every simple point of V.

2) Ω^* is pure of dimension one and each irreducible component of Ω^* is non-singular.

3) Two components of Ω^* make a normal crossing at any common point.

4) No three components of Ω^* have a common point.

Lemma 1. *Let V be a surface then there is a non-singular surface V' which dominates V, such that the correspondence is biregular at every simple points of V.*

Proof. As is known (see Zariski [3] [4], Abhyankar [1]), in order to resolve singularities of a surface, we have only to repeat normalization and quadratic dilatation. If such a sequence of procedure is given, we can omit quadratic dilatations with simple centers. Then we get the desired surface obviously.

Lemma 2. *Let V be a normal surface. For a non-singular surface V' which dominates V, such that all simple points of V is mapped in V' biregularly, let $\Omega^*(V')$ be the set of all points of V' which correspond to singular points of V. Then there is a V' such that $\Omega^*(V')$ is pure of dimension one and each irreducible component of $\Omega^*(V')$ is non-singular.*

Proof. If, for a given V', $\Omega^*(V')$ has components which are isolated point, say $P'_1, \cdots, P'_t$, then we consider quadratic dilatation V'' of V' with centers $P'_1, \cdots, P'_t$. Then every component of $\Omega^*(V'')$ is of dimension one, hence we may assume that $\Omega^*(V')$ is of pure dimension one. Let $C'=C'_1, \cdots, C'_t$ be components of $\Omega^*(V')$ which have singular points (as curves). Let P' a singular point on C'. By a quadratic dilatation $V'\to V''$, C' is mapped to a curve C'', then the transformation $C'\to C''$ is birational and is not biregular at P'. By such dilatations we can get a finite chain $C'\to C''$ $\to C'''\to\cdots$. This sequence will come to an end by its derived normal curve, and then we can get a non-singular curve $C^{(n)}$. Then, on the final surface $V^{(n)}$, $\Omega^*(V^{(n)})$ is the union of proper transforms of C'_i and non-singular curves, because (1) the total transform of a simple point P of a surface by the quadratic dilatation with center P is a non-singular curve and (2) the proper transform of a non-singular curve on the same surface under the same transformation is also non-singular. Therefore we complete the proof by induction on t.

Lemma 3. *Under the same notation as above, there is a V' such that two components of $\Omega^*(V')$ make a normal crossing at any common point and no three components of $\Omega^*(V')$ have a common point.*

Proof. By lemma 2, we can assume that the components of $\Omega^*(V')$ are non-singular curves. Let $C'_1, \cdots, C'_n$ be the components of $\Omega^*(V')$. As is known (see Zariski [3]), by successive quadratic dilations with centers within transforms of the C'_i, we get a non-singular surface V'' such that any two of the proper transforms $C''_1, \cdots, C''_n$ of the $C'_1, \cdots, C'_n$ have no common point. Then we see that any two components of $\Omega^*(V'')$ make a normal crossing at any common point. Now, if three components of $\Omega^*(V'')$ have a common point P then they have different tangents at P, hence by the quadratic dilatation with center P, we can eliminate such a situation.

Thus, we can now get the main theorem:

Theorem 1. *A resolved surface V' of a given surface V exists.*

Next we shall show the following.

Theorem 2. *Let $\mathfrak{F}$ be the set of all resolved surfaces of V. If $\mathfrak{F}$ have two elements V_1 and V_2, then it have a third element V_3 which dominates V_1 and V_2.*

Proof. Let the join of V_1 and V_2 be $J(V_1, V_2)$. We know that $J(V_1, V_2)$ dominates V_1 and V_2. Moreover since V_1 and V_2 are resolved surfaces of V, the transform $V \rightarrow J(V_1, V_2)$ is biregular at every simple point of V. Therefore if we apply the proof of theorem 1 to $J(V_1, V_2)$ we can get the desired surface.

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