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Square integrable normal differentials on Riemann surfaces*

Dedicated to Professor Y. Akizuki on his 60th birthday

By

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Introduction. In this article we shall study some properties on periods of abelian differentials square integrable on open Riemann surfaces. In § 1 we shall show the relation of canonical differentials defined in [4] and Ahlfors' reproducing differentials [2], which play a fundamental role in the theory of abelian differentials. In § 2 we shall prove a general periods theorem, namely, let Γ_0 $(\subset\Gamma_a)$ be a Hilbert space of square integrable analytic differentials and γ be a set of cycles for which corresponding reproducing differentials in Γ_0 are linearly independent, then Theorem 1 gives a necessary and sufficient condition in order that given complex numbers should be the periods of a differential of Γ_0 along γ. As its corollary we know the existence of three kinds of normal differentials, in particular, normal differentials of the first kind, for which in § 3 some properties on corresponding Riemann matrix are shown. Theorem 3 is partly identical with Oikawa's results in his unpublished paper studying normal differentials under Ahlfors' and Accola's method.

§ 1

Let R be an arbitrary Riemann surface of genus g $(0\leq g\leq\infty)$

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and $\{A_n, B_n\}_{n=1,\cdots,g}$ and $\{C_\nu\}_{\nu=1,\cdots,p}$ $(0\leq p\leq\infty)$ be a canonical homology basis on R such that (i) any cycle in R is homologous to a finite sum $\sum(p_nA_n+q_nB_n)+\sum r_\nu C_\nu$ where p_n, q_n and r_ν are integers; (ii) the intersection numbers are characterized by $A_i\times B_j=\delta_{ij}$, $A_i\times A_j=B_i\times B_j=0$ for $i, j=1,\cdots,g$; (iii) any dividing cycle is homologous to a finite sum $\sum r_\nu C_\nu$.

Let Γ_a be the Hilbert space of analytic differentials square integrable over R where the scalar product and norm are defined by

$$(1)\qquad (\phi, \psi)=\frac{i}{2}\int_R \phi\wedge\bar{\psi}, \quad \|\phi\|^2=(\phi, \phi).$$

For $\phi=du+i^*du$, $\psi=du'+i^*du'$ one can write

$$(1')\qquad (\phi, \psi)=D_R(du, du')-iD_R(du, {}^*du'),$$

where $D_R(du, du')=\int_R du\wedge{}^*du'=\int_R(u_xu'_x+u_yu'_y)\,dx\,dy$ stands for the mixed Dirichlet integral. Let Γ_{ae} and Γ_{ase} denote the subspaces of Γ_a consisting of exact and semi-exact differentials respectively. Now Virtanen's decomposition theorem can be expressed ([4]) as

$$\Gamma_a=\Gamma_{ase}\oplus\Gamma_C, \quad \Gamma_{ase}=\Gamma_{ae}\oplus\Gamma_{AB},$$

where Γ_C, orthogonal complement of Γ_{ase}, is spanned by analytic differentials $\{\varphi_{C_\nu}\}$ derived from generalized harmonic measures associated with dividing cycles $\{C_\nu\}$ and Γ_{AB} is spanned by semi-exact canonical differentials $\{\varphi_{A_n}, \varphi_{B_n}\}$ such that $Re\,\varphi_{A_n}$, $Re\,\varphi_{B_n}$ have no periods except along B_n resp. A_n where

$$(2)\qquad Re\int_{B_n}\varphi_{A_n}=-Re\int_{A_n}\varphi_{B_n}=1.$$

PROPOSITION 1. *For any $\phi\in\Gamma_{ase}$ we have*

$$(3)\qquad (\phi, \varphi_{A_n})=i\int_{A_n}\phi, \quad (\phi, \varphi_{B_n})=i\int_{B_n}\phi.$$

PROOF. Take an exhaustion of compact domains $\{R_m\}$ of R such that each component of the boundary ∂R_m consists of a dividing analytic curve. Then for $\phi=du+i^*du$, $\varphi_{A_n}=du_{A_n}+i^*du_{A_n}$ and R_m containing the cycle A_n we have by the period relation

$$D_{R_m}(du_{A_n}, du) = -\int_{A_n} {}^*du + \int_{\partial R_m} u_{A_n} {}^*du .$$

Since u_{A_n} is single-valued outside R_m, moreover canonical, it follows that $\int_{\partial R_m} u_{A_n} {}^*du \to 0$ for $m \to \infty$ (cf. Lemma 4 [4]). Hence

$$D_R(du_{A_n}, du) = -\int_{A_n} {}^*du .$$

Analogously we have

$$(4) \qquad D_R(du_{A_n}, {}^*du) = -D_R(du, {}^*du_{A_n}) = \int_{A_n} du .$$

By (1′) we get therefore the first formula of (3), similarly the second one. Q. E. D.

We recall [4] that for any $\phi \in \Gamma_a$ and harmonic differential dU with finite Dirichlet integral

$$(5) \qquad (\phi, \varphi_{C_\nu}) = -i \int_{C_\nu} \phi, \quad D_R(dU, {}^*d\omega_{C_\nu}) = \int_{C_\nu} dU ,$$

where $\varphi_{C_\nu} = d\omega_{C_\nu} + i^*d\omega_{C_\nu}$. (3), (4), (5) imply that our canonical differentials φ_{A_n}, φ_{B_n} and φ_{C_ν} are identical, except constant factors, with reproducing analytic differentials in Γ_{ase} resp. Γ_a in the Ahlfors sense [2] and du_{A_n}, du_{B_n} and $d\omega_{C_\nu}$ are also reproducing (real) harmonic differentials in Γ^*_{hse} resp. Γ_h under the Dirichlet norm.

§ 2. General Periods Theorem

Let $\Gamma_0 \subset \Gamma_a$ be any Hilbert space. Now let $P_\gamma = P_\gamma\{\Gamma_0\}$ be a linear mapping of Γ_0 into a complex vector space E such that

$$P_\gamma : \phi \to P_\gamma \phi = \left(\int_{\gamma_1} \phi, \int_{\gamma_2} \phi, \cdots \right), \qquad \gamma = \{\gamma_n\}_{n=1,2,\cdots}$$

We note that the kernel of P_γ is a Hilbert space $\subset \Gamma_0$ while the image of P_γ, say $P_\gamma(\Gamma_0)$, is in general not an l_2-(Hilbert) space, namely we can easily find such an example among hyperelliptic surfaces.

We say a set of cycles $\gamma = \{\gamma_j\}_{j=1,2,\ldots}$ be *admissible* for Γ_0 if the corresponding reproducing differentials $\sigma_{\gamma j}$ in Γ_0 defined by

$$(\phi, \sigma_{\gamma j}) = \int_{\gamma j} \phi \qquad \text{for all } \phi \in \Gamma_0$$

are linearly independent (for any finite pair). $\sigma_{\gamma_j} \in \Gamma_0$ exist, because $\int_{\gamma_j} \phi$ is a bounded linear functional on Γ_0. Let

$$\Gamma_0' = [\sigma_{\gamma_j}] \subset \Gamma_0$$

be a subspace of Γ_0 spanned by σ_{γ_j}, and $\{\sigma_{\gamma_j}'\}$ be an orthonormal system constructed by the Schmidt method:

$$\sigma_{\gamma_j}' = \sum_{k=1}^{j} s_{jk} \sigma_{\gamma_k} \qquad (j=1,2,\cdots). \tag{6}$$

We call the triangular matrix $S_\gamma = S_\gamma\{\Gamma_0\} = (s_{jk})$ with $s_{jk}=0$ $(j<k)$ the matrix (mapping) *associated* with Γ_0 and γ.

PROPOSITION 2. $S_\gamma\{\Gamma_0\} = (s_{mm})$ *has the properties*:

(i) $s_{nn} > 0$;

(ii) s_{mn} *are real if* $(\sigma_{\gamma_j}, \sigma_{\gamma_k})$ *are real for* $j, k = 1, 2, \cdots$;

(iii) $\sum_m |s_{mn}|^2 = \|\phi_n\|^2 < \infty$ *if there exists a* $\phi_n \in \Gamma_0$ *such that* $P_\gamma \phi_n = e_n$ *(unit vector in E with n-th component one).*

(i), (ii) are immediate consequences from the orthogonalization process and (iii) follows easily from the following Theorem 1.

Now S_γ defines a linear mapping of E into itself such that

$$S_\gamma : \alpha \to S_\gamma(\alpha) = (\bar{s}_{11}\alpha_1,\ \bar{s}_{21}\alpha_1 + \bar{s}_{22}\alpha_2 \cdots),$$

that is, $S_\gamma(\alpha)$ is the product of matrix $(\bar{s}_{jk})$ (bar stands for complex conjugate) and the transposed vector ${}^t\alpha$ of $\alpha = (\alpha_1, \alpha_2, \cdots) \in E$. S_γ is one-to-one by (i).

THEOREM 1. *For given* Γ_0 *and* $\gamma = \{\gamma_j\}$ *admissible for* Γ_0, (I) $S_\gamma \circ P_\gamma$ *is a continuous linear mapping of* Γ_0 *onto the complex* l_2-*space*[1] *such that*

$$\|S_\gamma \circ P_\gamma(\phi)\|_{l_2} \leq \|\phi\|, \qquad \phi \in \Gamma_0.$$

(II) $S_\gamma \circ P_\gamma$ *is one-to-one on* Γ_0' *and* $P_\gamma(\Gamma_0) = P_\gamma(\Gamma_0') = S_\gamma^{-1}(l_2)$.

PROOF. Let $\phi \in \Gamma_0$ and ϕ' be the orthogonal projection of ϕ onto Γ_0'. Then ϕ' can be written as $\phi' = \sum \beta_n \sigma_{\gamma_n}'$ with the Fourier coefficients β_n, given by

1) This is just a complex vector space of dimension q if γ consists of $q(<\infty)$ cycles.

$$\beta_n = (\phi', \sigma'_{\gamma_n}) = (\phi, \sigma'_{\gamma_n}) = \sum_{k=1}^{n} \bar{s}_{nk}(\phi, \sigma_{\gamma_k}) = \sum_{k=1}^{n} \bar{s}_{nk} \int_{\gamma_k} \phi .$$

Hence by Parseval's equality we have

$$\sum |\beta_n|^2 = ||S_\gamma \circ P_\gamma(\phi)||^2_{l_2} = ||\phi'||^2 \leq ||\phi||^2 .$$

which shows (I) except "onto" and $P_\gamma(\Gamma_0) \subset S_\gamma^{-1}(l_2) = \{\alpha\, ; S_\gamma(\alpha) \in l_2\}$. Now let $\beta^* = (\beta_1^*, \beta_2^* \cdots) \in l_2$, then there is an α such that $S_\gamma(\alpha) = \beta^*$;

$$(7) \qquad \beta_n^* = \sum_{k=1}^{n} \bar{s}_{nk} \alpha_k$$

Since $\sum |\beta_n^*|^2 < \infty$,

$$(8) \qquad \phi^* \equiv \sum \beta_n^* \sigma'_{\gamma_n} \in \Gamma'_0 \subset \Gamma_0 ,$$

$$(9) \qquad \beta_n^* = (\phi^*, \sigma'_{\gamma_n}) = \sum_{k=1}^{n} \bar{s}_{nk} \int_{\gamma_k} \phi^* .$$

Therefore by (7), (9) and (i) we have successively

$$\int_{\gamma_k} \phi^* = \alpha_k \qquad (k = 1, 2, \cdots) , \text{ i.e., } \alpha = P_\gamma \phi^* ,$$

which implies $S_\gamma^{-1}(l_2) \subset P_\gamma(\Gamma'_0) \subset P_\gamma(\Gamma_0)$, hence they coincide. Since any differential of Γ'_0 orthogonal to every σ_{γ_j} is identically zero, $S_\gamma \circ P_\gamma$ is one-to-one on Γ'_0.

COROLLARY. *Suppose s_{jk} are all, except finite numbers, real (or pure imaginary) and let $P_\gamma\phi = \alpha + i\beta$, $\phi \in \Gamma_0$ where α, β are real vectors then there exist ϕ_1, $\phi_2 \in \Gamma'_0$ such that*

$$P_\gamma \phi_1 = \alpha , \quad P_\gamma \phi_2 = \beta .$$

Here we refer to important special cases. For canonical basis $\{A_n, B_n\}$ and $\{C_\nu\}$ we know by (3), (5) that

$$(10) \qquad \begin{array}{ll} \sigma_{A_n} = i\varphi_{A_n} , \quad \sigma_{B_n} = i\varphi_{B_n} & \text{for } \Gamma_0 = \Gamma_{ase} \\ \sigma_{C_\nu} = -i\varphi_{C_\nu} & \text{for } \Gamma_0 = \Gamma_a \end{array}$$

and $(\sigma_{A_j}, \sigma_{A_k})$, $(\sigma_{B_j}, \sigma_{B_k})$ and $(\sigma_{C_j}, \sigma_{C_k})$ are real, but $(\sigma_{A_j}, \sigma_{B_k})$ are not necessarily real; moreover, $\{C_\nu\}' = \{C_\nu \in \{C_\nu\}\, ; \sigma_{C_\nu} \not\equiv 0\}$ is admissible for Γ_a, and $\{A_n\}$, $\{B_n\}$ are admissible for Γ_{ase}. In fact, let $\{\omega_j\}$ denotes any one of the systems $\{\sigma_{A_j}\}$, $\{\sigma_{B_j}\}$ and $\{\sigma_{C_j}\}$

($C_j \in \{C_\nu\}'$), then $\{\omega_j\}$ are linearly independent in the real field, hence for finite number of real x_j the quadratic form

$$\sum_{i,j} x_i x_j D_R(Re\,\omega_i,\ Re\,\omega_j) = D_R(\sum_j x_j \omega_j) \geq 0$$

is positive definite, i.e.

$$det\,(D_R(Re\,\omega_i,\ Re\,\omega_j)) > 0$$

Noting that $(\omega_j,\ \omega_j) = D_R(Re\,\omega_i,\ Re\,\omega_j)$ (=real) we find easily that $\{\omega_j\}$ are linearly independent in the complex field.

Virtanen [7] observed first the case $\Gamma_0 = \Gamma_a$, $\gamma = \{A_n\}$ on Riemann surfaces of parabolic type (where $\Gamma_a = \Gamma_{ase} = \Gamma_{AB}$). For general surfaces, I [4] treated the case $\Gamma_0 = \Gamma_C$, $\gamma = \{C_\gamma\}'$ and Sainouchi did independently the case $\Gamma_0 = \Gamma_{ase}$, $\gamma = \{A_n\}$ in his recent paper [6]. While in the case where γ contains mixed basis it is generally difficult to decide whether γ is admissible or not. For instance if R is closed or parabolic surface of finite genus g, $\Gamma_a = \Gamma_{AB} = [\sigma_{A_n}, \sigma_{B_n}]_{n=1,\cdots,g}$ reduces g-dimensional. If R is a bordered surface with p contours, $\{A_n,\ B_n,\ C_\nu\}_{n=1,\cdots,g;\nu=1,\cdots,p-1}$ is admissible for $\Gamma_0 = \Gamma_{aS} = \Gamma_{AB} \oplus \Gamma_C$ and P_γ gives a one-to-one mapping of $\Gamma_0 = \Gamma_0'$ to $(2g+p-1)$-dimensional vector space by (II) (Ahlfors [1]).

§ 3. Normal Differentials

Hereafter we fix the canonical basis $\{A_n,\ B_n\}$ once for all and define a Hilbert space Γ_A^0 by

$$\Gamma_A^0 = \left\{\phi \in \Gamma_{ase},\ \int_{A_n} \phi = 0,\ n = 1, 2, \cdots\right\},$$

i.e., the kernel of $P_\gamma\{\Gamma_{ase}\}$, $\gamma = \{A_n\}$. Let Γ_A be the orthogonal complement of Γ_A^0 in Γ_{ase}. Obviously

$$\Gamma_{ae} \subset \Gamma_A^0 \subset \Gamma_{ase},\quad \Gamma_A \subset \Gamma_{AB}.$$

Every differential of Γ_A is by definition uniquely determined by its A-periods. Moreover among differentials of Γ_{ase} having the same A-periods with $\phi \in \Gamma_{ase}$ the orthogonal projection of ϕ onto Γ_A has the minimum norm.

PROPOSITION 3. $\Gamma_A = [\varphi_{A_n}] = [\sigma_{A_n}] \qquad (n = 1, \cdots, g)$, This is an immediate consequence of Proposition 1.

Since for general surfaces Γ_A does not necessarily coincide with Γ_{AB}, we must confine ourselves to the space Γ_A to achieve the complex normalization about A-periods. Now we take $\Gamma_0=\Gamma_{ase}$, $\gamma=\{A_n\}$, then $\Gamma_0'=\Gamma_A$ by Proposition 3 and $S_\gamma=(s_{mn})$ is real. For $\varphi_{B_n}\in\Gamma_0$ $\int_{A_m}-\varphi_{B_n}=\delta_{mn}+i(\ )$ hence by Corollary of Theorem 1 we have the following theorem [8].

THEOREM 2. *There exist normal differentials $\omega_{A_n}\in\Gamma_A$ such that*

$$\int_{A_m}\omega_{A_n}=\delta_{mn}\qquad(m, n=1,\cdots,g).$$

The ω_{A_n} can be written by (8) *as*

$$\omega_{A_n}=\sum_m s_{mn}\sigma'_{A_m}=\sum_m s_{mn}\Big(\sum_{l=1}^{m}s_{ml}\sigma_{A_l}\Big),\tag{11}$$

where (s_{mn}) is a real matrix associated with Γ_{ase} and $\{A_n\}$ for which

$$\sum_m|s_{mn}|^2=\|\omega_{A_n}\|^2<\infty\ .\tag{12}$$

For the following purposes we note that

$$s_{mn}=-Re\,b_{mn}\,,\qquad\text{where } b_{mn}=-i\int_{B_n}\overline{\sigma'_{A_m}}\,.\tag{13}$$

To see this, let $\varphi^0_{B_n}$ be the orthogonal projection φ_{B_n} to Γ_A, then it can be written as

$$\varphi^0_{B_n}=\sum b_{mn}\sigma'_{A_m}\,,\tag{14}$$

where $b_{mn}=(\varphi^0_{B_n},\sigma'_{A_m})=(\varphi_{B_n},\sigma'_{A_m})=-i\int_{B_n}\overline{\sigma'_{A_m}}=-\int_{B_n}\sum_{k=1}^{m}s_{mk}\bar{\varphi}_{A_k}$,

hence (13) holds. Now set

$$\int_{B_j}\omega_{A_k}=\sigma_{jk}+i\tau_{jk}\,,\qquad(j, k=1,\cdots,g),$$
$$\boldsymbol{\sigma}=(\sigma_{jk})\,,\quad\boldsymbol{\tau}=(\tau_{jk})\,.$$

We say an (infinite) quadratic form $\sum\xi_i\bar{\xi}_j a_{ij}$ is *convergent* if for given $\varepsilon>0$ there exists an integer N such that $|\sum_{i,j=n}^{m}\xi_i\bar{\xi}_j a_{ij}|<\varepsilon$

for $m>n>N$. Now our Riemann matrix has the following properties.

THEOREM 3. 1°) *τ is symmetric and positive definite; more precisely, for any finite or infinite numbers of complex numbers $\xi_1, \xi_2, \cdots$ not simultaneously zero*

$$\sum \xi_i \bar{\xi}_j \tau_{ij} > 0$$

provided that the quadratic form is convergent.

2°) *$\tau = {}^tS_\gamma \cdot S_\gamma$ (multiplication in the obvious sense) where S_γ is a triangular real matrix associated with Γ_{ase} and $\{A_n\}$.*

3°) *if $\Gamma_A^0 = \Gamma_{ae}$, equivalently $\Gamma_A = \Gamma_{AB}$, then σ is symmetric.*

PROOF. From (10), (11)

$$\tau_{jk} = \sum_m s_{mk} \int_{Bj} Im\, \sigma'_{A_m} = \sum_m s_{mk} \sum_{l=1}^m s_{ml} \int_{Bj} Re\, \varphi_{A_l} \tag{15}$$
$$= \sum_m s_{mk} s_{mj}\,.$$

The series on the right-hand side is absolutely convergent by (12) and Schwarz' inequality. This formula shows 2°) and the symmetry of τ. On the other hand

$$(\omega_{Aj}, \omega_{Ak}) = \lim_{N\to\infty} \left(\sum_{m=1}^N s_{mj}\sigma'_{A_m}, \sum_{n=1}^N s_{nk}\sigma'_{A_n} \right) = \sum_m s_{mj} s_{mk}\,.$$

Hence we have

$$(\omega_{Aj}, \omega_{Ak}) = \tau_{jk}\,,$$

consequently for any finite pair $(\xi_1, \cdots, \xi_n) \neq (0, \cdots, 0)$

$$\|\xi_1\omega_{A_{\nu_1}} + \cdots + \xi_n\omega_{A_{\nu_n}}\|^2 = \sum_{i,j=1}^n \xi_i\bar{\xi}_j \tau_{\nu_i\nu_j} > 0\,.$$

Now let $\Omega_n = \sum_{i=1}^n \xi_i \omega_{A_i}$ $(n=1, 2, \cdots)$ and quadratic form $\sum \xi_i\bar{\xi}_j\tau_{ij}$ be convergent, then for given $\varepsilon>0$

$$\|\Omega_m - \Omega_n\|^2 = \sum_{i,j=n+1}^m \xi_i\bar{\xi}_j\tau_{ij} < \varepsilon \qquad (m>n>N)\,.$$

Therefore there is an $\Omega \in \Gamma_A$ such that $\|\Omega_n - \Omega\| \to 0$ $(n\to\infty)$ and

$$||\Omega||^2 = \lim_{n\to\infty} ||\Omega_n||^2 = \sum \xi_i \bar{\xi}_j \tau_{ij} \geq 0 .$$

If the last quadratic form vanishes, $\Omega = 0$ and since Ω_n converge to Ω uniformly on every compact set on R, we have

$$0 = \int_{Aj} \Omega = \lim_{n\to\infty} \int_{Aj} \Omega_n = \xi_j \qquad (j = 1, 2, \cdots) .$$

Next we prove 3°). By (13), (15)

$$\sigma_{jk} = -\sum_m (Re\, b_{mk}) \int_{Bj} Re\, \sigma'_{Am} = -\sum_m (Re\, b_{mk})(Im\, b_{mj})$$
$$= -\sum_m [b_{mk} b_{mj} - \bar{b}_{mk} \bar{b}_{mj} - 2i\, Im\, (b_{mj} \bar{b}_{mk})]/4i .$$

Hence the symmetry of σ is equivalent to the condition

$$\sum_m Im\, (b_{mj} \bar{b}_{mk}) = 0 .$$

On the other hand we have by (14)

$$(\varphi^0_{Bj}, \varphi^0_{Bk}) = \sum b_{mj} \bar{b}_{mk} .$$

Therefore if $\Gamma_A = \Gamma_{AB}$, then $\varphi_{Bj} = \varphi^0_{Bj}$, $\varphi_{Bk} = \varphi^0_{Bk}$ and we know $(\varphi^0_{Bj}, \varphi^0_{Bk}) = (\varphi_{Bj}, \varphi_{Bk})$ are real. Q. E. D.

REMARK. In case of $\Gamma_A = \Gamma_{AB}$, σ_{jk} are given by

$$\sigma_{jk} = -\frac{1}{2} Im \sum_m b_{mj} b_{mk} = \frac{1}{2} D(du_{Bj}, {}^*d\tilde{u}_{Bk}) .$$

where

$$\sum_n \bar{b}_{mk} \sigma'_{Am} = d\tilde{u}_{Bk} + i^* d\tilde{u}_{Bk} .$$

For the sake of convenience we reformulate a result contained in the above proof as follows.

PROPOSITION 4. *Let* $\{\alpha_n\}$ *be given complex numbers for which* $\sum \alpha_i \bar{\alpha}_j \tau_{ij}$ *is convergent, then there exists in* Γ_A *a unique differential* Ω *with A-periods* $\{\alpha_n\}$, *which is expressed as*

(16) $$\Omega = \sum \alpha_n \omega_{A_n}, \quad ||\Omega||^2 = \sum \alpha_i \bar{\alpha}_j \tau_{ij} < \infty ,$$

and the convergence is uniform on every compact set on R.

It is desirable that every differential $\phi \in \Gamma_A$ can be written as (16). Of course it is true in case of finite genus. As for general case we know $\sum_n |\sum_{k=1}^{n} s_{nk}\alpha_k|^2<\infty$ for $\alpha=P_\gamma(\phi)$ while by (15) $\sum \alpha_i\bar{\alpha}_j\tau_{ij}$ means the interchange of above summation. I have no answer about this problem. Here we shall give only some criteria about the convergence of hermitian form $J=\sum \alpha_i\bar{\alpha}_j\tau_{ij}$.

a) Since $\tau_{ij}^2<\tau_{ii}\tau_{jj}$, *J is convergent if*

$$\sum |\alpha_i|\sqrt{\tau_{ii}}=\sum |\alpha_i|\,||\omega_{A_i}||<\infty$$

(cf. [5], [6], [7]).

b) As τ_{ij} (or s_{ij}) are complicatedly dependent on the structure of R, the following criterion by geometric quantities seems to be useful.

PROPOSITION 5. *Let $\lambda(\gamma)$ denote the extremal length of curve family homologous to a cycle γ. Then J is convergent if*

$$\sum_i |\alpha_i|\sqrt{\lambda(B_i)}<\infty\,. \tag{17}$$

PROOF. We note [3] that for $\omega\in\Gamma_a$

$$\left|\int_\gamma \omega\right|^2\leq\lambda(\gamma)\,||\omega||^2\,. \tag{18}$$

On account of symmetry of τ_{ij} it suffices to treat the case of real α_i. Applying (18) for $\omega=\Omega_n^m=\sum_{i=n}^{m}\alpha_i\omega_{A_i}$, $\gamma=B_j$ we have

$$|\sum_{i=n}^{m}\alpha_i\tau_{ji}|^2\leq\left|\int_{B_j}\Omega_n^m\right|^2\leq\lambda(B_j)||\Omega_n^m||^2\,.$$

Multiplying by $|\alpha_j|$ and summing from $j=n$ to m, then

$$|\sum_{i,j=n}^{m}\alpha_i\alpha_j\tau_{ij}|\leq\sum_j|\alpha_j|\,|\sum_i\alpha_i\tau_{ji}|\leq||\Omega_n^m||\sum_{j=n}^{m}|\alpha_j|\sqrt{\lambda(B_j)}\,.$$

Since $||\Omega_n^m||=(\sum_{i,j=n}^{m}\alpha_i\alpha_j\tau_{ij})^{1/2}$ we have our conclusion.

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