J. Math. Kyoto Univ.
3-1 (1963) 37-58.

The $(n+20)$-th homotopy groups of n-spheres

Dedicated to Professor Y. Akizuki for his 60-th birthday

By

Mamoru MIMURA and Hirosi TODA

(Received July 10, 1963)

This paper is a continuation of the calculation of the homotopy groups of spheres in [3]. Denote by S^n the unit n-sphere in euclidean $(n+1)$-space and by $\pi_r(S^n)$ the r-th homotopy groups of S^n, then $\pi_{n+20}(S^n)$ can be calculated and our results are stated as follows.

Theorem.

$\pi_{22}(S^2) \cong Z_{132} \oplus Z_2$,
$\pi_{23}(S^3) \cong Z_2 \oplus Z_2$,
$\pi_{24}(S^4) \cong Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$,
$\pi_{25}(S^5) \cong Z_6 \oplus Z_2 \oplus Z_2$,
$\pi_{26}(S^6) \cong Z_{480} \oplus Z_{12}$,
$\pi_{27}(S^7) \cong Z_{24}$,
$\pi_{28}(S^8) \cong Z_{24} \oplus Z_3$,
$\pi_{29}(S^9) \cong Z_{24}$,
$\pi_{30}(S^{10}) \cong Z_{504} \oplus Z_{24}$,
$\pi_{31}(S^{11}) \cong Z_{24} \oplus Z_2 \oplus Z_2$,
$\pi_{32}(S^{12}) \cong Z_{24} \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$,
$\pi_{33}(S^{13}) \cong Z_{24} \oplus Z_2 \oplus Z_2 \oplus Z_2$,
$\pi_{34}(S^{14}) \cong Z_{240} \oplus Z_{24}$,
$\pi_{35}(S^{15}) \cong Z_{24}$,
$\pi_{36}(S^{16}) \cong Z_{24}$,
$\pi_{37}(S^{17}) \cong Z_{24}$,
$\pi_{38}(S^{18}) \cong Z_{24} \oplus Z_{12}$,
$\pi_{39}(S^{19}) \cong Z_{24} \oplus Z_2$,

$$\pi_{40}(S^{20}) \cong Z_{24}\oplus Z_2\oplus Z_2,$$
$$\pi_{41}(S^{21}) \cong Z_{24}\oplus Z_2,$$
$$\pi_{42}(S^{22}) \cong Z_{24},$$
$$G_{20} \cong Z_{24}.$$

More explicit results, for example, generators of 2-primary components are stated in (i) of § XVI. The definitions and the notations of [3] are carried over to the present work and we quote from [3] without any reference. As the continuation of [3], we give the number XV for the first paragraph of this paper which follows from the last chapter XIV of [3]. We shall use the notation $\pi(X, Y)$ in place of the notation $\pi(X\to Y)$ in [3].

In § XV we examine a generator κ_n of $\pi_{n+14}(S^n)$ and calculate some secondary compositions. We show the existence of essential elements $\bar{\kappa}'$ in $\pi_{26}(S^6)$ and $\bar{\kappa}_n$ in $\pi_{n+20}(S^n)$ for $n\geq 7$ in the same section.

The 2-primary components of $\pi_{n+20}(S^n)$, which we denote by π^n_{n+20} according to [3], are calculated in (i) of XVI mainly by means of the following exact sequence:

$$\cdots \longrightarrow \pi^n_i \overset{E}{\longrightarrow} \pi^{n+1}_{i+1} \overset{H}{\longrightarrow} \pi^{2n+1}_{i+1} \overset{\Delta}{\longrightarrow} \pi^n_{i-1} \overset{E}{\longrightarrow} \pi^{n+1}_i \longrightarrow \cdots,$$

where E is a suspension homomorphism and H is a generalized Hopf homomorphism. To complete the calculation of $\pi_{n+20}(S^n)$ we compute the odd primary components of $\pi_{n+20}(S^n)$ in (ii) of § XVI.

§ XV. Some elements of $\pi_{n+20}(S^n)$.

In chapter X of [3], we have given elements

$$\kappa_n = E^{n-7}\kappa_7 \quad (n\geq 7), \qquad \bar{\varepsilon}_n = E^{n-3}\bar{\varepsilon}_3 \quad (n\geq 3),$$

which have the following properties:

(15.1) $\quad 2\kappa_7 \equiv \bar{\nu}_7\circ\nu^2_{15} \quad \bmod 4(\sigma'\circ\sigma_{14})$.

(15.2) $\quad \{\varepsilon_3, 2\iota_{11}, \nu^2_{11}\}_6$ *consists of the single element* $\bar{\varepsilon}_3$.

(15.3) $\quad \eta_n\circ\kappa_{n+1} = \bar{\varepsilon}_n$ *for* $n\geq 6$ *and*

$\quad \kappa_n\circ\eta_{n+14} = \bar{\varepsilon}_n$ *for* $n\geq 9$.

(15.4) $\quad \eta^2_n\circ\kappa_{n+2} = \kappa_n\circ\eta^2_{n+14} = 0$ *for* $n\geq 9$.

The relation (15.1) follows from Lemma 10.1, (10.7) and Lemma 5.14. By (4.7) and Theorem 10.5, $\{\varepsilon_3, 2\iota_{11}, \nu_{11}^2\}_6 \subset \pi_{18}^3 = \{\bar{\varepsilon}_3\} \approx Z_2$. But the proof of Lemma 10.2 shows that $HH\{\varepsilon_3, 2\iota_{11}, \nu_{11}^2\}_6 = \nu_9^3 \neq 0$. Thus $\{\varepsilon_3, 2\iota_{11}, \nu_{11}^2\}_6 \neq 0$ and (15.2) follows. (15.3) is (10.23). (15.4) follows from Lemma 12.10, the relation $\Delta(\nu_{17}^3) = \nu_8 \circ \sigma_{11} \circ \nu_{18}^3$ of page 142 and from the exactness of $\pi_{26}^{17} \xrightarrow{\Delta} \pi_{24}^8 \xrightarrow{E} \pi_{25}^9 \longrightarrow 0$.

It follows from (15.1) that

(15.1)′
$$2\kappa_7 = \bar{\nu}_7 \circ \nu_{15}^2 \quad or$$
$$2(\kappa_7 + 2\sigma' \circ \sigma_{14}) = \bar{\nu}_7 \circ \nu_{15}^2 .$$

Remark that the next relation holds:

(15.3)′
$$\kappa_7 \circ \eta_{21} = \sigma' \circ \bar{\nu}_{14} + \bar{\varepsilon}_7 .$$

Now *we change the definition of* κ_7. We replace κ_7 by $\kappa_7 + 2\sigma' \circ \sigma_{14}$ in the second case of (15.1)′. Then we have

(15.5)
$$2\kappa_7 = \bar{\nu}_7 \circ \nu_{15}^2 \qquad and$$
$$2\kappa_n = 2\iota_n \circ \kappa_n = 0 \qquad for\ n \geq 10 .$$

The second part of the above relation follows from (15.1)′ and that $\bar{\nu}_n \circ \nu_{n+8} = 0$ for $n \geq 10$ (*cf.* (7.22)). It is easily checked that the relations (15.3) and (15.4) still hold for new κ_7 and $\kappa_n = E^{n-7}\kappa_7$, $n \geq 7$.

We shall prove the following lemmas.

Lemma 15.1. *For $n \geq 9$, we have*

$$\{\eta_n, \eta_{n+1} \circ \kappa_{n+2}, 2\iota_{n+16}\}_{n-5} \equiv \nu_n \circ \kappa_{n+3}$$

$$\bmod 2\pi_{n+17}(S^n) + \eta_n \circ E^{n-5}\pi_{22}^6 + \{\eta_n \circ \mu_{n+1} \circ \sigma_{n+10} + \sigma_n \circ \eta_{n+7} \circ \mu_{n+8}\} ,$$

where $\quad 2\pi_{n+17}(S^n) = 0$ *if* $n \neq 14$ $(n \geq 9)$,

$$\eta_n \circ E^{n-5}\pi_{22}^6 + \{\eta_n \circ \mu_{n+1} \circ \sigma_{n+10} + \sigma_n \circ \eta_{n+7} \circ \mu_{n+8}\}$$
$$= \{\eta_n \circ \mu_{n+1} \circ \sigma_{n+10}\} \qquad if\ n \geq 10.$$

Lemma 15.2. *For $n \geq 12$, we have*

$$\{\eta_n, 2\iota_{n+1}, \kappa_{n+1}\}_{n-9} \equiv 0 \quad \bmod \eta_n \circ \rho_{n+1} .$$

Proof of Lemma 15.1. Let n be sufficiently large such that the homotopy groups in the following discussions are stable. (e.g., $n\geq 19$.) The secondary composition $\{\nu_n^2, 2\iota_{n+6}, \nu_{n+6}^2\}$ is defined and it is a subset of π_{n+13}^n. Since $\pi_{n+13}^n=0$ for $n\geq 15$ (Theorem 7.7), $\{\nu_n^2, 2\iota_{n+6}, \nu_{n+6}^2\}=0$. We have also $\{2\iota_n, \nu_n^2, 2\iota_{n+6}\}\equiv\nu_n^2\circ\eta_{n+6}=0$ by Corollary 3.7 and (5.9).

Let a cell complex $K=S^n\cup e^{n+7}$ has the characteristic class ν_n^2 of e^{n+7}. By Proposition 1.7, there exists an extension $\mathrm{Ext}(2\iota_n)\in\pi(K, S^n)$ of $2\iota_n$ and a coextension $\mathrm{Coext}\,(2\iota_{n+6})\in\pi_{n+7}(K)$ of $2\iota_{n+6}$, such that $\mathrm{Ext}(2\iota_n)\circ\mathrm{Coext}(2\iota_{n+6})=0$. By Proposition 1.8, $\mathrm{Coext}(2\iota_{n+6})\circ\nu_{n+7}^2\in i^*\{\nu_n^2, 2\iota_{n+6}, \nu_{n+6}^2\}=0$ where i is the injection of S^n into K. Thus the following secondary composition is defined.

$$\{\mathrm{Ext}\,(2\iota_n), \mathrm{Coext}\,(2\iota_{n+6}), \nu_{n+7}^2\}\subset\pi_{n+14}(S^n)$$

Consider the composition with η_{n-1}, then

$$\begin{aligned}
&\eta_{n-1}\circ\{\mathrm{Ext}\,(2\iota_n), \mathrm{Coext}\,(2\iota_{n+6}), \nu_{n+7}^2\} & \\
&\subset\{\eta_{n-1}\circ\mathrm{Ext}\,(2\iota_n), \mathrm{Coext}\,(2\iota_{n+6}), \nu_{n+7}^2\} & \text{by Proposition 1.2}\\
&\subset\{p^*\{\eta_{n-1}, 2\iota_n, \nu_n^2\}, \mathrm{Coext}\,(2\iota_{n+6}), \nu_{n+7}^2\} & \text{by Proposition 1.9}\\
&=\{p^*(\varepsilon_{n-1}+\{\eta_{n-1}\circ\sigma_n\}), \mathrm{Coext}\,(2\iota_{n+6}), \nu_{n+7}^2\} & \text{by (6.1)}\\
&\subset\{\varepsilon_{n-1}+\{\eta_{n-1}\circ\sigma_n\}, p_*\mathrm{Coext}\,(2\iota_{n+6}), \nu_{n+7}^2\} & \text{by Proposition 1.2}\\
&\subset\{\varepsilon_{n-1}, 2\iota_{n+7}, \nu_{n+7}^2\}+\{\eta_{n-1}\circ\sigma_n, 2\iota_{n+7}, \nu_{n+7}^2\} & \text{by (1.18),}
\end{aligned}$$

where $p: K\to S^{n+7}$ is a mapping which shrinks S^n to a point. Here we have

$$\begin{aligned}
\{\varepsilon_{n-1}, 2\iota_{n+7}, \nu_{n+7}^2\} &\supset E^{n-4}(-1)^n\{\varepsilon_3, 2\iota_{11}, \nu_{11}^2\}_6 & \text{by Proposition 1.3}\\
&\ni E^{n-4}((-1)^n\bar{\varepsilon}_3)=\bar{\varepsilon}_{n-1} & \text{by definitions,}\\
\{\eta_{n-1}\circ\sigma_n, 2\iota_{n+7}, \nu_{n+7}^2\} &=\{\sigma_{n-1}\circ\eta_{n+6}, 2\iota_{n+7}, \nu_{n+7}^2\} & \\
&\supset\sigma_{n-1}\circ\{\eta_{n+6}, 2\iota_{n+7}, \nu_{n+7}^2\} & \text{by Proposition 1.2}\\
&\ni\sigma_{n-1}\circ\varepsilon_{n+6} & \text{by (7.1)}\\
&=0 & \text{by Lemma 10.7.}
\end{aligned}$$

These secondary compositions are cosets of sums of $\eta_{n-1}\circ\sigma_n\circ\pi_{n+14}^{n+7}=\eta_{n-1}\circ\sigma_n\circ\{\sigma_{n+7}\}$, $\varepsilon_{n-1}\circ\pi_{n+14}^{n+7}=\varepsilon_{n-1}\circ\{\sigma_{n+7}\}$ and $\pi_{n+8}^{n-1}\circ\nu_{n+8}^2=\{\nu_{n-1}^3, \mu_{n-1}, \eta_{n-1}\circ\varepsilon_n\}\circ\nu_{n+8}^2$, which are zeros by the relations (14.1):

$$\eta_{n-1}\circ\sigma_n^2=0,\quad \varepsilon_{n-1}\circ\sigma_{n+7}=0,\quad \nu_{n-1}^3\circ\nu_{n+8}^2=0,\quad \mu_{n-1}\circ\nu_{n+8}^2=0$$

and $\eta_{n-1}\circ\varepsilon_n\circ\nu_{n+8}^2=0$.

Therefore we obtain

$$\eta_{n-1}\circ\{\mathrm{Ext}\,(2\iota_n),\ \mathrm{Coext}\,(2\iota_{n+6}),\ \nu_{n+7}^2\} = \bar{\varepsilon}_{n-1} = \eta_{n-1}\circ\kappa_n .$$

As the kernel of $(\eta_{n-1})_*: \pi_{n+14}(S^n)\to\pi_{n+14}(S^{n-1})$ is generated by σ_n^2, so κ_n or $\kappa_n+\sigma_n^2$ belongs to $\{\mathrm{Ext}\,(2\iota_n),\ \mathrm{Coext}\,(2\iota_{n+6}),\ \nu_{n+7}^2\}$. Since $\sigma_n^2\circ\nu_{n+14}=0$ by (7.20), we have

$$\begin{aligned}
\kappa_n\circ\nu_{n+14} &\in \{\mathrm{Ext}\,(2\iota_n),\ \mathrm{Coext}\,(2\iota_{n+6}),\ \nu_{n+7}^2\}\circ\nu_{n+14} \\
&\subset \{\mathrm{Ext}\,(2\iota_n),\ \mathrm{Coext}\,(2\iota_{n+6}),\ \nu_{n+7}^3\} && \text{by Proposition 1.2}\\
&= \{\mathrm{Ext}\,(2\iota_n),\ \mathrm{Coext}\,(2\iota_{n+6}),\ \bar{\nu}_{n+7}\circ\eta_{n+15}\} && \text{by Lemma 6.}\\
&\subset \{\mathrm{Ext}\,(2\iota_n),\ \mathrm{Coext}\,(2\iota_{n+6})\circ\bar{\nu}_{n+7},\ \eta_{n+15}\} && \text{by Proposition 1.2}\\
&= \{\mathrm{Ext}\,(2\iota_n),\ -\mathrm{Coext}\,(2\iota_{n+6})\circ\bar{\nu}_{n+7},\ \eta_{n+15}\} .
\end{aligned}$$

Here

$$\begin{aligned}
-\mathrm{Coext}\,(2\iota_{n+6})\circ\bar{\nu}_{n+7} &= i^*\{\nu_n^2,\ 2\iota_{n+6},\ \bar{\nu}_{n+6}\} && \text{by Proposition 1.8}\\
&= i_*\bar{\varepsilon}_n && \textit{cf. } \text{page 111}\\
&= i_*(\eta_n\circ\kappa_{n+1}) && \text{by (15.3).}
\end{aligned}$$

Therefore

$$\begin{aligned}
\kappa_n\circ\nu_{n+14} &\in \{\mathrm{Ext}\,(2\iota_n),\ i_*(\eta_n\circ\kappa_{n+1}),\ \eta_{n+15}\}\\
&\supset \{i^*\,\mathrm{Ext}\,(2\iota_n),\ \eta_n\circ\kappa_{n+1},\ \eta_{n+15}\} && \text{by Proposition 1.2}\\
&= \{2\iota_n,\ \eta_n\circ\kappa_{n+1},\eta_{n+15}\} ,
\end{aligned}$$

where i_* and i^* are induced by the injection i. The composition $\{2\iota_n,\ \eta_n\circ\kappa_{n+1},\ \eta_{n+15}\}$ is a coset of $2\pi_{n+17}(S^n)+\pi_{n+16}^n\circ\eta_{n+16}$, where $2\pi_{n+17}(S^n)=0$ by the fact that $\pi_{n+17}(S^n)$ does not have the odd primary components and the 2-primary component of it is $Z_2\oplus Z_2\oplus Z_2\oplus Z_2$ and $\pi_{n+16}^n\circ\eta_{n+16}=\{\eta_n^*\circ\eta_{n+16},\ \eta_n\circ\mu_{n+1}\circ\sigma_{n+10}\}$. $\{\mathrm{Ext}\,(2\iota_n),\ i_*(\eta_n\circ\kappa_{n+1}),\ \eta_{n+15}\}$ is a coset of $\pi_{n+16}^n\circ\eta_{n+16}+\mathrm{Ext}\,(2\iota_n)\circ\pi_{n+17}(K)$. When n is sufficiently large, the elements of $\pi_{n+17}(K)$ are considered as the coextensions of elements of $\pi_{n+16}(S^{n+6})$. So $\mathrm{Ext}\,(2\iota_n)\circ\pi_{n+17}(K)$ is the set of all $\mathrm{Ext}\,(2\iota_n)\circ\mathrm{Coext}\,(\beta)$, where $\beta\in\pi_{n+16}(S^{n+6})\cong Z_6$ and $3\beta=0$ or $3\beta=\eta_{n+6}\circ\mu_{n+7}$.

By Proposition 1.7,

$$\begin{aligned}
\mathrm{Ext}\,(2\iota_n)\circ\mathrm{Coext}\,(\beta) &= \{2\iota_n,\ \nu_n^2,\ \beta\} && \text{by Proposition 1.7}\\
&= \{2\iota_n,\ 3\nu_n^2,\ \beta\}\\
&\supset \{2\iota_n,\ \nu_n^2,\ 3\beta\} && \text{by Proposition 1.2}\\
&= \{2\iota_n,\ \nu_n^2,\ 0\} \text{ or } \{2\iota_n,\ \nu_n^2,\ \eta_{n+6}\circ\mu_{n+7}\} ,
\end{aligned}$$

where $\{2\iota_n, \nu_n^2, 0\}=0$ by $2\pi_{n+17}(S^n)=0$ and

$$\begin{aligned}\{2\iota_n, \nu_n^2, \eta_{n+6}\circ\mu_{n+7}\} &\supset \{2\iota_n, \nu_n^2, \eta_{n+6}\}\circ\mu_{n+8} \\ &\ni \varepsilon_n\circ\mu_{n+8} && \text{by (6.1) and (3.9)} \\ &= (\bar\nu_n+\sigma_n\circ\eta_{n+7})\circ\mu_{n+8} && \text{by Lemma 6.4} \\ &= \sigma_n\circ\eta_{n+7}\circ\mu_{n+8} && \text{by Theorem 14.1.}\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{Ext}(2\iota_n)\circ\mathrm{Coext}(\beta) &\subset 2\pi_{n+17}(S^n)+\pi_{n+7}(S^n)\circ\beta+\{\sigma_n\circ\eta_{n+7}\circ\mu_{n+8}\} \\ &= \{\sigma_n\circ\eta_{n+7}\circ\mu_{n+8}\} \\ &\subset \pi_{n+16}^n\circ\eta_{n+16}\,.\end{aligned}$$

Thus $\{\mathrm{Ext}(2\iota_n), i_*(\eta_n\circ\kappa_{n+1}), \eta_{n+15}\}$ and $\{2\iota_n, \eta_n\circ\kappa_{n+1}, \eta_{n+15}\}$ are the same coset of the subgroup $\pi_{n+16}^n\circ\eta_{n+16}$ of π_{n+17}^n. Whence, for a sufficiently large n,

$$\begin{aligned}\nu_n\circ\kappa_{n+3} = \kappa_n\circ\nu_{n+14} &\in \{2\iota_n, \eta_n\circ\kappa_{n+1}, \eta_{n+15}\} \\ &= \{\eta_n, \eta_{n+1}\circ\kappa_{n+2}, 2\iota_{n+16}\} && \text{by (3.9).}\end{aligned}$$

Now the lemma follows from the fact that the odd primary components of $\pi_{n+17}(S^n)$ are zero for $n\geq 9$, $n\neq 14$ and that the kernel of E^{n-9}: $\pi_{26}^9\to\pi_{n+17}^n$ is generated by $\sigma_9\circ\eta_{16}\circ\mu_{17}+\eta_9\circ\mu_{10}\circ\sigma_{19}$ (*cf.* Theorem 12.7, 12.17 and Ch. XIV). q.e.d.

Proof of Lemma 15.2. Consider the secondary composition $\{\eta_{12}, 2\iota_{13}, \kappa_{13}\}_3\subset\pi_{28}^{12}$ which can be defined since the order of η_{12} and κ_{10} are 2. This composition is a coset of the subgroup

$$\eta_{12}\circ E^3\pi_{25}^{10}+\pi_{14}^{12}\circ\kappa_{14} = \{\eta_{12}\circ E^4\rho', \eta_{12}\circ\sigma_{13}\circ\bar\nu_{20}, \eta_{12}\circ\bar\varepsilon_{13}, \eta_{12}^2\circ\kappa_{14}\},$$

where $\eta_{12}\circ E^4\rho' = \eta_{12}\circ 2\rho_{13} = 2(\eta_{12}\circ\rho_{13}) = 0$ by Lemma 10.9

$\eta_{12}\circ\sigma_{13}\circ\bar\nu_{20} = 0$ by (10.18)

and $\eta_{12}\circ\bar\varepsilon_{13} = \eta_{12}^2\circ\kappa_{14} = 0$ by (15.3) and (15.4).

So it consists of a single element. As π_{28}^{12} is generated by $\sigma_{12}\circ\mu_{19}$,

$$\begin{aligned}\text{so}\quad \{\eta_{12}, 2\iota_{13}, \kappa_{13}\}_3 &= a(\sigma_{12}\circ\mu_{19}) && \text{where } a=0,1 \\ &= a(\eta_{12}\circ\rho_{13}) && \text{by Proposition 12.20.}\end{aligned}$$

Therefore, if $n\geq 12$,

$$\{\eta_n, 2\iota_{n+1}, \kappa_{n+1}\}_{n-9}\supset E^{n-12}\{\eta_{12}, 2\iota_{13}, \kappa_{13}\}_3\ni E^{n-12}(a\eta_{12}\circ\rho_{13}) = a(\eta_n\circ\rho_{n+1})\,.$$

This concludes the lemma. q.e.d.

Lemma 15. 3. *There exists an element $\bar{\kappa}'$ of $\pi_{26}(S^6)$ such that*

$$2\bar{\kappa}' = \nu_6^2\circ\kappa_{12} \quad \textit{and} \quad 4\bar{\kappa}' = 0\,.$$

Proof. By the relation $\nu_6\circ\eta_9=0$ (*cf.* (5.9)) and (15.4), we can define a secondary composition $\{\nu_6, \eta_9, \eta_{10}\circ\kappa_{11}\}$, from which we choose an element $\bar{\kappa}'$.

$$\begin{aligned} 2\bar{\kappa}' = \bar{\kappa}'\circ 2\iota_{26} &\in \{\nu_6, \eta_9, \eta_{10}\circ\kappa_{11}\}\circ 2\iota_{26} \\ &= -\nu_6\circ\{\eta_9, \eta_{10}\circ\kappa_{11}, 2\iota_{25}\} \qquad \text{by Proposition 1.4.} \end{aligned}$$

On the other hand, by Lemma 15.1 for $n=9$, we have

$$\nu_6\circ\{\eta_9, \eta_{10}\circ\kappa_{11}, 2\iota_{25}\}_4 \equiv \nu_6^2\circ\kappa_{12}$$
$$\bmod \nu_6\circ\eta_9\circ E^4\pi_{22}^6 + \{\nu_6\circ\eta_9\circ\mu_{10}\circ\sigma_{19} + \nu_6\circ\sigma_9\circ\eta_{16}\circ\mu_{17}\}\,,$$

where

$$\begin{aligned} \nu_6\circ\eta_9 &= 0 && \text{by (5.9) and} \\ \nu_6\circ\sigma_9\circ\eta_{16}\circ\mu_{17} &= \nu_6\circ\varepsilon_9\circ\mu_{17} && \text{(see page 152)} \\ &= 2(\bar{\nu}_6\circ\nu_{14})\circ\mu_{17} && \text{by (7.18)} \\ &= 0\,. \end{aligned}$$

That is,

$$\begin{aligned} \nu_6^2\circ\kappa_{12} &= \nu_6\circ\{\eta_9, \eta_{10}\circ\kappa_{11}, 2\iota_{25}\}_4 \\ &\subset \nu_6\circ\{\eta_9, \eta_{10}\circ\kappa_{11}, 2\iota_{25}\} = -\{\nu_6, \eta_9, \eta_{10}\circ\kappa_{11}\}\circ 2\iota_{26}. \end{aligned}$$

The last composition is a coset of

$$\nu_6\circ\pi_{26}(S^9)\circ 2\iota_{26} = \nu_6\circ 2\pi_{26}(S^9) = 0\,, \quad \text{since} \quad \pi_{26}(S^9) \cong Z_2\oplus Z_2\oplus Z_2\oplus Z_2\,.$$

Therefore $\qquad 2\bar{\kappa}' = -\nu_6^2\circ\kappa_{12} = \nu_6^2\circ\kappa_{12}\,.$

This implies that $4\bar{\kappa}'=2\nu_6^2\circ\kappa_{12}=0$. q.e.d.

Consider a cell-complex $L=S^{13}\cup e^{14}$ where e^{14} is attached to S^{13} by a mapping of degree 2. Apply Proposition 1.7 to Lemma 15.2, then we have an extension $\mathrm{Ext}(\eta_{12})\in\pi(L, S^{12})$ of η_{12} and a coextension $\mathrm{Coext}(\kappa_{13})\in\pi_{18}(L)$ of κ_{13}, such that $\mathrm{Ext}(\eta_{12})\circ\mathrm{Coext}(\kappa_{13})=0$. By Proposition 1.9, Proposition 5.9 and Theorem 13.4,

$$\begin{aligned} \nu_9\circ\mathrm{Ext}(\eta_{12}) &\in (E^5p)^*\{\nu_9, \eta_{12}, 2\iota_{13}\}_9 \\ &\subset (E^5p)^*\pi_{14}(S^9) = 0\,, \end{aligned}$$

where $p: S^4 \cup e^5 \to S^5$ is a mapping which shrinks S^4 to a point. Therefore we can define a secondary composition $\{\nu_9, \mathrm{Ext}(\eta_{12}), \mathrm{Coext}(\kappa_{13})\}$. We choose an element $\bar{\kappa}_9$ from this. Then we have

$$\begin{aligned} 2\bar{\kappa}_9 &\in \{\nu_9, \mathrm{Ext}(\eta_{12}), \mathrm{Coext}(\kappa_{13})\} \circ 2\iota_{29} \\ &\subset \{\nu_9, \mathrm{Ext}(\eta_{12}), \mathrm{Coext}(\kappa_{13}) \circ 2\iota_{28}\} && \text{by Proposition 1. 2} \\ &\subset \{\nu_9, \mathrm{Ext}(\eta_{12}), -i_*\{2\iota_{13}, \kappa_{13}, 2\iota_{27}\}_3\} && \text{by Proposition 1. 8.} \end{aligned}$$

As $E^{n-13}: \pi_{28}^{13} \to \pi_{n+15}^{n}$ is an isomorphism onto and $\{2\iota_n, \kappa_n, 2\iota_{n+14}\}_{n-10}$ contains an element $\kappa_n \circ \eta_{n+14} = \eta_n \circ \kappa_{n+1}$ by Corollary 3. 7, so $\{2\iota_{13}, \kappa_{13}, 2\iota_{27}\}_3$ contains the element $\kappa_{13} \circ \eta_{27} = \eta_{13} \circ \kappa_{14}$. Here, the composition $\{2\iota_{13}, \kappa_{13}, 2\iota_{27}\}_3$ is a coset of $2\iota_{13} \circ E^3\pi_{15}(S^{10}) + \pi_{28}(S^{13}) \circ 2\iota_{28} = 2(\pi_{28}(S^{13}))$. We have $2\pi_{28}(S^{13}) = 2\iota_{13} \circ \pi_{28}(S^{13})$, by use of the isomorphism $E: \pi_{28}(S^{13}) \to \pi_{29}(S^{14})$. As $i_*(2\iota_{13}) = 0$, $i_*\{2\iota_{13}, \kappa_{13}, 2\iota_{17}\}_3$ consists of the single element $i_*(\kappa_{13} \circ \eta_{27}) = i_*(\eta_{13} \circ \kappa_{14})$. Therefore we have

$$\begin{aligned} 2\bar{\kappa}_9 &\in \{\nu_9, \mathrm{Ext}(\eta_{12}), -i_*(\eta_{13} \circ \kappa_{14})\} \\ &= \{\nu_9, \mathrm{Ext}(\eta_{12}), i_*(\eta_{13} \circ \kappa_{14})\} \\ &\subset \{\nu_9, i^*\mathrm{Ext}(\eta_{12}), \eta_{13} \circ \kappa_{14}\} && \text{by Proposition 1. 2} \\ &= \{\nu_9, \eta_{12}, \eta_{13} \circ \kappa_{14}\} \\ &\supset \{\nu_9, \eta_{12}, \eta_{13} \circ \kappa_{14}\}_3 \ni E^3\bar{\kappa}' && \text{by Lemma 15. 3.} \end{aligned}$$

Therefore $\quad 2\bar{\kappa}_9 \equiv E^3\kappa' \quad \bmod \nu_9 \circ \pi_{29}(S^{12}) + \pi_{14}^9 \circ \eta_{14} \circ \kappa_{15}$.

So $\quad 4\bar{\kappa}_9 \equiv 2E^3\bar{\kappa}' \quad \bmod \nu_9 \circ 2\pi_{29}(S^{12}) + \pi_{14}^9 \circ 2(\eta_{14} \circ \kappa_{15}) = 0$

$\quad = \nu_9^2 \circ \kappa_{15}$ by Lemma 15. 3.

Lemma 4. 5 and the exactness of $\pi_{27}^7 \to \pi_{28}^8 \to \pi_{28}^{15} = 0$ imply that $E: \pi_{27}^7 \to \pi_{28}^8$ is an isomorphism onto. That $E: \pi_{28}^8 \to \pi_{29}^9$ is an isomorphism onto follows from the exact sequence $0 = \pi_{30}^{17} \xrightarrow{\Delta} \pi_{28}^8 \xrightarrow{E} \pi_{29}^9 \xrightarrow{H} \pi_{29}^{17} = 0$.

By these facts, there exists an element $\kappa_7 \in \pi_{27}^7$ such that $4\bar{\kappa}_7 = 2E\bar{\kappa}' = \nu_7^2 \circ \kappa_{13}$ and $8\bar{\kappa}_7 = 0$. Thus we have obtained

Lemma 15. 4. *There exists an element $\bar{\kappa}_7$ of π_{27}^7 such that*

$$4\bar{\kappa}_7 = 2E\bar{\kappa}' = \nu_7^2 \circ \kappa_{13} \quad \textit{and} \quad 8\bar{\kappa}_7 = 0.$$

Hereafter we denote $\bar{\kappa}_n = E^{n-7}\bar{\kappa}_7$ for $n \geq 7$ and $E^\infty \bar{\kappa}_7 = \bar{\kappa}$.

§ XVI. Computation of $\pi_{n+20}(S^n)$.

(i) *2-primary components of* $\pi_{n+20}(S^n)$.

We shall compute the 2-primary components of $\pi_{n+20}(S^n)$. First we have

$$\pi_{22}^2 = \{\eta_2\circ\bar{\mu}', \eta_2\circ\nu'\circ\mu_6\circ\sigma_{15}\} \approx Z_4\oplus Z_2$$

by (5.2) and Theorem 12.9.

Consider the exact sequence

$$\cdots \longrightarrow \pi_{24}^5 \xrightarrow{\Delta} \pi_{22}^2 \xrightarrow{E} \pi_{23}^3 \xrightarrow{H} \pi_{23}^5 \xrightarrow{\Delta} \pi_{21}^2 \longrightarrow \cdots,$$

where $\pi_{23}^5 = \{\zeta_5\circ\sigma_{16}, \nu_5\circ\bar{\varepsilon}_8, \eta_5\circ\bar{\mu}_6\} \approx Z_8\oplus Z_2\oplus Z_2$.

We have in the page 149 of [3]

$$\Delta(\zeta_5\circ\sigma_{16}) \equiv \pm(\eta_2\circ\mu'\circ\sigma_{14}) \bmod \{\eta_2\circ\nu'\circ\bar{\varepsilon}_6\} + \{\eta_2^2\circ\bar{\mu}_4\},$$
$$\Delta(\nu_5\circ\bar{\varepsilon}_8) = \eta_2\circ\nu'\circ\bar{\varepsilon}_6,$$
$$\Delta(\eta_5\circ\bar{\mu}_6) = 0.$$

Then we have $H\pi_{23}^3 = \mathrm{Ker}\,\Delta = \{\eta_5\circ\bar{\mu}_6, 4\zeta_5\circ\sigma_{16}\} \approx Z_2\oplus Z_2$.

And $E(\eta_2\circ\nu'\circ\mu_6\circ\sigma_{15}) = 0$ by Lemma 5.7

$E^2(\eta_2\circ\bar{\mu}') = \eta_4\circ E^2(\bar{\mu}')$

$= \eta_4\circ(2\bar{\zeta}_5)$ by Lemma 12.4

$= \eta_4\circ 2\iota_5\circ\bar{\zeta}_5$ by use of the monomorphism $E: \pi_{24}^5 \to \pi_{25}^6$

$= 0.$

So, by the fact that $E: \pi_{23}^3 \to \pi_{24}^4$ is a monomorphism, we have

(16.1) $$E(\eta_2\circ\bar{\mu}') = 0.$$

Thus

(16.2) $$H: \pi_{23}^3 \approx H\pi_{23}^3 = \mathrm{Ker}\,\Delta.$$

Since $H(\nu'\circ\bar{\mu}_6) = \eta_5\circ\bar{\mu}_6$ by (5.3)

and $H(\nu'\circ\eta_6\circ\mu_7\circ\sigma_{16}) = \eta_5^2\circ\mu_7\circ\sigma_{16}$ by (5.3)

$= 4\zeta_5\circ\sigma_{16}$ by (7.14),

we have that

$$\pi_{23}^3 = \{\nu'\circ\bar{\mu}_6, \nu'\circ\eta_6\circ\mu_7\circ\sigma_{16}\} \approx Z_2\oplus Z_2.$$

By (5.6), Theorem 12.7 and by the above result,

$$\pi_{24}^4 = \{E\nu'\circ\bar{\mu}_7,\ E\nu'\circ\eta_7\circ\mu_8\circ\sigma_{17},\ \nu_4\circ\sigma'\circ\eta_{14}\circ\mu_{15},\ \nu_4^2\circ\kappa_{10},\ \nu_4\circ\bar{\mu}_7,\ \nu_4\circ\eta_7\circ\mu_8\circ\sigma_{17}\}$$
$$\cong Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2.$$

Next consider the exact sequence

$$\cdots \longrightarrow \pi_{26}^9 \xrightarrow{\Delta} \pi_{24}^4 \xrightarrow{E} \pi_{25}^5 \xrightarrow{H} \pi_{25}^9 \xrightarrow{\Delta} \pi_{23}^4 \longrightarrow \cdots,$$

where $\pi_{26}^9 = \{\sigma_9\circ\eta_{16}\circ\mu_{17},\ \nu_9\circ\kappa_{12},\ \bar{\mu}_9,\ \eta_9\circ\mu_{10}\circ\sigma_{19}\} \cong Z_2\oplus Z_2\oplus Z_2\oplus Z_2$
and $\pi_{25}^9 = \{\sigma_9\circ\nu_{16}^3,\ \sigma_9\circ\mu_{16},\ \sigma_9\circ\eta_{16}\circ\varepsilon_{17},\ \mu_9\circ\sigma_{18}\} \cong Z_2\oplus Z_2\oplus Z_2\oplus Z_2.$

We have, by use of Proposition 2.5,

$$\begin{aligned}
\Delta(\sigma_9\circ\eta_{16}\circ\mu_{17}) &= \Delta(\sigma_9\circ\eta_{16})\circ\mu_{15} \\
&= (\nu_4\circ\sigma'\circ\eta_{14}+E\nu'\circ\varepsilon_7)\circ\mu_{15} && \text{by (7.16)} \\
&= \nu_4\circ\sigma'\circ\eta_{14}\circ\mu_{15}+E\nu'\circ\varepsilon_7\circ\mu_{15}, \\
\Delta(\nu_9\circ\kappa_{12}) &= \Delta(\nu_9)\circ\kappa_{10} \\
&= (\pm 2\nu_4^2)\circ\kappa_{10} && \text{by (5.13)} \\
&= \nu_4\circ(\pm 2\nu_7\circ\kappa_{10}) \\
&= 0, && \text{by Theorem 12.7} \\
\Delta(\bar{\mu}_9) &= \Delta(\iota_9)\circ\bar{\mu}_7 \\
&= \pm(2\nu_4-E\nu')\circ\bar{\mu}_7 && \text{by (5.8)} \\
&= \nu_4\circ(\pm 2\bar{\mu}_7)+E\nu'\circ(\pm\bar{\mu}_7) \\
&= E\nu'\circ\bar{\mu}_7, \\
\Delta(\eta_9\circ\mu_{10}\circ\sigma_{19}) &= \Delta(\eta_9)\circ\mu_8\circ\sigma_{17} \\
&= E\nu'\circ\eta_7\circ\mu_8\circ\sigma_{17}, && \text{by (5.11).}
\end{aligned}$$

And in the page 150 we have the following results:

$$\begin{aligned}
\Delta(\sigma_9\circ\nu_{16}^3) &= \nu_4\circ\eta_7\circ\bar{\varepsilon}_8+E(\varepsilon'\circ\nu_{13}^3), \\
\Delta(\sigma_9\circ\mu_{16}) &= \nu_4\circ\sigma'\circ\mu_{14}+E(\varepsilon'\circ\mu_{13}), \\
\Delta(\sigma_9\circ\eta_{16}\circ\varepsilon_{17}) &= \nu_4\circ E\zeta'+E(\varepsilon'\circ\eta_{13}\circ\varepsilon_{14}), \\
\Delta(\mu_9\circ\sigma_{18}) &= E\nu'\circ\mu_7\circ\sigma_{16}.
\end{aligned}$$

Since $\sigma_9\circ\nu_{16}^3$, $\sigma_9\circ\mu_{16}$, $\sigma_9\circ\eta_{16}\circ\varepsilon_{17}$ and $\mu_9\circ\sigma_{18}$ span π_{25}^9 and their images under Δ are independent, the kernel of $\Delta: \pi_{25}^9\to\pi_{23}^4$ is zero. From these facts we have

$$\pi_{25}^5 = \{\nu_5^2\circ\kappa_{11},\ \nu_5\circ\bar{\mu}_8,\ \nu_5\circ\eta_8\circ\mu_9\circ\sigma_{18}\} \cong Z_2\oplus Z_2\oplus Z_2.$$

We have seen also

(16.3) $$\mathrm{Ker}\,\Delta(:\pi_{26}^{9}\to\pi_{24}^{4}) = \{\nu_9\circ\kappa_{12}\} \approx Z_2\,.$$

By (5.9), $E(\nu_5\circ\eta_8\circ\mu_9\circ\sigma_{18})=0$. Thus $0\neq\nu_5\circ\eta_8\circ\mu_9\circ\sigma_{18}\in\mathrm{Ker}\,E=\Delta\pi_{27}^{11}$. Since $\pi_{27}^{11}\approx Z_2$ is generated by $\sigma_{11}\circ\mu_{18}$, we have

(16.4) $$\Delta\pi_{27}^{11} = \{\Delta(\sigma_{11}\circ\mu_{18})\} = \{\nu_5\circ\eta_8\circ\mu_9\circ\sigma_{18}\}\,,$$
$$E\pi_{25}^{5} = \{\nu_6^2\circ\kappa_{12},\ \nu_6\circ\bar{\mu}_9\} \approx Z_2\oplus Z_2\,.$$

We have that the Δ-image of $\pi_{26}^{11}=\{E^2\rho',\ \bar{\varepsilon}_{11}\}\approx Z_{16}\oplus Z_2$ is trivial by (12.15). So we obtain an exact sequence

$$0\longrightarrow E\pi_{25}^{5}\longrightarrow\pi_{26}^{6}\xrightarrow{\ H\ }\pi_{26}^{11}\longrightarrow 0\,.$$

Since
$$\begin{aligned}\Delta(\bar{\varepsilon}_{13}) &= \Delta(\eta_{13}\circ\kappa_{14}) && \text{by (15.3)}\\ &= \Delta(\eta_{13})\circ\kappa_{12} && \text{by Proposition 2.5}\\ &= \Delta(H(\sigma'))\circ\kappa_{12} && \text{by Lemma 5.14}\\ &= 0\,,\end{aligned}$$

the image of $\Delta:\ \pi_{28}^{13}\to\pi_{26}^{6}$ is generated by $\Delta\rho_{13}$.

As
$$\begin{aligned}H(\Delta(2\rho_{13})) &= H(\Delta(E^4\rho')) && \text{by Lemma 10.9}\\ &= \pm 2E^2\rho' && \text{by Proposition 2.7,}\end{aligned}$$

(16.5) $$H(\Delta\rho_{13})\equiv\pm E^2\rho' \mod \{8E^2\rho',\ \bar{\varepsilon}_{11}\}\,.$$

So $H(\Delta(8\rho_{13})) = \pm 8E^2\rho'\neq 0$ and $H(\Delta(16\rho_{13})) = \pm 16E^2\rho' = 0$.

i.e., $$\Delta(16\rho_{13})\in\text{kernel of } H=E\pi_{25}^{5}\,.$$

Therefore $\mathrm{Im}\,\Delta\cap\mathrm{Ker}\,H=\mathrm{Im}\,\Delta\cap\mathrm{Im}\,E$ is generated by $\Delta(16\rho_{13})$. Obviously $2\Delta(16\rho_{13})=\Delta(32\rho_{13})=\Delta(0)=0$. By the definition of $\bar{\mu}_n$,

$$\begin{aligned}E(\nu_6\circ\bar{\mu}_9) &= \nu_7\circ\bar{\mu}_{10} = \nu_7\circ E^7\{\mu_3,\ 2\iota_{12},\ 8\sigma_{12}\}\\ &= \{\nu_7,\ \mu_{10},\ 2\iota_{19}\}_7\circ 8\sigma_{20} && \text{by Proposition 1.4}\\ &\subset \pi_{20}^{7}\circ 8\sigma_{20} = 0 && \text{by Theorem 7.7.}\end{aligned}$$

Thus $\nu_6\circ\bar{\mu}_9\in\Delta\pi_{27}^{11}\cap E\pi_{25}^{5}$. As $\nu_6\circ\bar{\mu}_9\neq 0$, we have

(16.6) $$\nu_6\circ\bar{\mu}_9 = 16\Delta\rho_{13}\,.$$

That is, the order of $\Delta\rho_{13}$ is 32. In Lemma 15.5 we have chosen such an essential element $\bar{\kappa}'$ of π_{26}^{6} that $2\bar{\kappa}'=\nu_6^2\circ\kappa_{12}$. Then we have

$$\pi_{26}^{6} = \{\Delta\rho_{13},\ \bar{\kappa}'\} \approx Z_{32}\oplus Z_4$$

and

$$E\pi_{26}^{6} = \{E\bar{\kappa}'\} \approx Z_4\,.$$

Consider the exact sequence

$$0 \longrightarrow E\pi_{26}^{6} \longrightarrow \pi_{27}^{7} \xrightarrow{H} \pi_{27}^{13} \xrightarrow{\Delta} \pi_{25}^{6} \xrightarrow{E} \pi_{26}^{7} \longrightarrow \cdots,$$

where $\pi_{27}^{13}=\{\sigma_{13}^{2}, \kappa_{13}\}\approx Z_{16}\oplus Z_{2}$. By Theorem 12.9, $\Delta\pi_{27}^{13}=\text{Ker}\,E=\{2\bar{\sigma}_{6}\}\approx Z_{16}$. Thus $H\pi_{27}^{7}=\text{Ker}\,\Delta\approx Z_{2}$, and we obtain the exact sequence

$$0 \longrightarrow E\pi_{26}^{6} \longrightarrow \pi_{27}^{7} \longrightarrow Z_{2} \longrightarrow 0,$$

where $E\pi_{26}^{6}\approx Z_{4}$.

Here

$$\begin{aligned} 4\bar{\kappa}_{7} &= \nu_{7}^{2}\circ\kappa_{13} && \text{by Lemma 15.4} \\ &= 2E\bar{\kappa}' && \text{by Lemma 15.3} \\ &\neq 0. \end{aligned}$$

Therefore the order of $\bar{\kappa}_{7}$ is 8 and

$$\pi_{27}^{7} = \{\bar{\kappa}_{7}\} \approx Z_{8}.$$

Remark that

(16.7) $$2\bar{\kappa}_{7} = \pm E\bar{\kappa}'.$$

We obtain immediately

$$\pi_{28}^{8} = \{\bar{\kappa}_{8}\} \approx Z_{8} \qquad \text{by (5.15) and } \pi_{28}^{15}=0.$$

Since $\pi_{29}^{17}=\pi_{30}^{17}=0$, E: $\pi_{28}^{8}\rightarrow\pi_{29}^{9}$ is an isomorphism onto. Namely,

$$\pi_{29}^{9} = \{\bar{\kappa}_{9}\} \approx Z_{8}.$$

Lemma 16.1. *There exists an element β' of π_{30}^{10} such that*

$$E\beta' = \theta'\circ\varepsilon_{23}, \quad H(\beta') = \zeta_{19}, \quad \pm 2\beta' = \Delta(\zeta_{21})$$

and the order of β' is 8.

Proof. By (i) of Proposition 11.11 in the case that $i=29$, $n=9$ and $\alpha=\eta_{18}\circ\varepsilon_{19}\in\pi_{27}^{18}$, it follows the existence of β such that

$$E^{2}(\beta) = \Delta(\eta_{25}\circ\varepsilon_{26}) \quad \text{and} \quad H(\beta)\in\{2\iota_{19}, \eta_{19}, \eta_{20}\circ\varepsilon_{21}\}_{2}.$$

On the other hand

$$\begin{aligned} E(\theta'\circ\varepsilon_{23}) &= E\theta'\circ\varepsilon_{24} \\ &= \Delta\eta_{25}\circ\varepsilon_{24} && \text{by (7.30)} \\ &= \Delta(\eta_{25}\circ\varepsilon_{26}) = E^{2}\beta && \text{by Proposition 2.5.} \end{aligned}$$

As $$\Delta\pi_{33}^{23} = \{\Delta(\eta_{23}\circ\mu_{24})\} \quad \text{by Theorem 7.3}$$
$$= \{\Delta(H(\theta\circ\mu_{24}))\} \quad \text{by Lemma 7.5}$$
$$= 0,$$

(16.8) $E: \pi_{31}^{11}\to\pi_{32}^{12}$ *is a monomorphism.*

Thus $$\theta'\circ\varepsilon_{23} = E\beta.$$

Next we consider the secondary composition $\{2\iota_{19}, \eta_{19}, \eta_{20}\circ\varepsilon_{21}\}_2$.

$$H(\beta)\in\{2\iota_{19}, \eta_{19}, \eta_{20}\circ\varepsilon_{21}\}\equiv\zeta_{19}, \quad \bmod 2\pi_{30}(S^{19}) \quad \text{by Lemma 9.1.}$$

So $H(\beta)\equiv\zeta_{19}$ $\quad \bmod 2\pi_{30}(S^{19})+\pi_{21}^{19}\circ\eta_{21}\circ\varepsilon_{22}$
$$= 2\pi_{30}(S^{19})+\{\eta_{19}^3\circ\varepsilon_{22}\}$$
$$= 2\pi_{30}(S^{19})+\{4\nu_{19}\circ\varepsilon_{22}\} \quad \text{by (5.5)}$$
$$= 2\pi_{30}(S^{19}) \quad \text{by Lemma 6.1.}$$

For some odd integer r, $H(r\beta)=\zeta_{19}$. Set $\beta'=r\beta$, then $E\beta'=rE\beta$ $=r(\theta'\circ\varepsilon_{23})=\theta'\circ\varepsilon_{23}$, because $2\varepsilon_{23}=0$. We have $E(2\beta')=2E\beta'=2(\theta'\circ\varepsilon_{23})$ $=\theta'\circ(2\varepsilon_{23})=0$. That is, $2\beta'\in\operatorname{Ker} E=\Delta\pi_{32}^{21}=\{\Delta\zeta_{21}\}$. Therefore $2\beta'$ $=x\Delta\zeta_{21}$ for some integer x.

As $$8\beta' = 4x\Delta\zeta_{21}$$
$$= x\Delta(4\zeta_{21})$$
$$= x\Delta(\eta_{21}^2\circ\mu_{23}) \quad \text{by (7.14)}$$
$$= x\Delta(H(\theta'\circ\mu_{23})) = 0 \quad \text{Lemma 7.5}$$

and the order of $H(\beta')=\zeta_{19}$ is 8, so the order of β' is 8.

Since $$2\zeta_{19} = H(2\beta')$$
$$= H(x\Delta\zeta_{21})$$
$$= \pm 2x\zeta_{19} \quad \text{by Proposition 2.7,}$$

we have $$2\equiv\pm 2x \quad \bmod 8$$
$$1\equiv\pm x \quad \bmod 4.$$

Therefore $\Delta\zeta_{21}\equiv\pm 2\beta' \bmod 4(2\beta')$, and $8\beta'=0$. We have

(16.9) $$\Delta\zeta_{21} = \pm 2\beta'.$$ q.e.d.

Since $\Delta\pi_{30}^{19}=\{\Delta\zeta_{19}\}=0$ by (12.22) and $\pi_{31}^{19}=0$, we obtain the exact sequence

$$0\longrightarrow\pi_{29}^{9}\xrightarrow{E}\pi_{30}^{10}\xrightarrow{H}\pi_{30}^{19}\longrightarrow 0,$$

where $\pi_{29}^{9}=\{\bar{\kappa}_9\}\approx Z_8$ and $\pi_{30}^{19}=\{\zeta_{19}\}\approx Z_8$ by Theorem 7.4.

By the above lemma, we conclude that this sequence splits. Therefore we have

$$\pi_{30}^{10}=\{\bar{\kappa}_{10},\beta'\}\approx Z_8\oplus Z_8 .$$

Lemma 16.2. *There exists an element β'' of π_{31}^{11} such that*

$$\Delta(\mu_{25})=E\beta'',\quad H(\beta'')=\eta_{21}\circ\mu_{22}\quad \textit{and}\quad 2\beta''=0 .$$

Proof. Let $i=30$, $n=10$ and $\alpha=\mu_{20}$ in (i) of Proposition 11.10. Then the existence of β'' such that $E\beta''=\Delta(\mu_{25})$ and $H(\beta'')=\eta_{21}\circ\mu_{22}$ follows immediately. Since the order of μ_{25} is 2, we have

$$2E\beta''=2\Delta\mu_{25}=\Delta(2\mu_{25})=0 .$$

$E:\pi_{31}^{11}\rightarrow\pi_{32}^{12}$ is a monomorphism by (16.8). Therefore $2\beta''=0$. q.e.d.

In the exact sequence

$$\cdots\longrightarrow\pi_{32}^{21}\xrightarrow{\ \Delta\ }\pi_{30}^{10}\xrightarrow{\ E\ }\pi_{31}^{11}\xrightarrow{\ H\ }\pi_{31}^{21}\longrightarrow\cdots,$$

$\pi_{32}^{21}=\{\zeta_{21}\}\approx Z_8$ and $\pi_{31}^{21}=\{\eta_{21}\circ\mu_{22}\}\approx Z_2$. We have that $\Delta\pi_{32}^{21}=\{2\beta'\}\approx Z_4$ by Lemma 16.1, H is an epimorphism, and we have the following result by Lemma 16.2:

$$\pi_{31}^{11}=\{\bar{\kappa}_{11},\theta'\circ\varepsilon_{23},\beta''\}\approx Z_8\oplus Z_2\oplus Z_2 .$$

Lemma 16.3. *There exists an element $\beta'''\in\pi_{32}^{12}$ such that*

$$E^2\beta'''=8\Delta\sigma_{29},\quad H(\beta''')=\mu_{23}\quad \textit{and}\quad 2\beta'''=0 .$$

Proof. Consider (ii) of Proposition 11.11 in the case that $n=11$, $i=31$ and $\alpha=8\sigma_{22}\in\pi_{29}^{22}$, then we easily obtain an element β of π_{32}^{12} such that $E^2\beta=8\Delta\sigma_{29}$ and $H(\beta)\in\{\eta_{23},2\iota_{24},8\sigma_{24}\}_2$.

Here $\{\eta_{23},2\iota_{24},8\sigma_{24}\}_2\equiv\mu_{23}\mod G$ by Lemma 6.5,

where $G=\eta_{23}\circ E^2\pi_{30}^{22}+\pi_{25}^{23}\circ 8\sigma_{25}$

$=\{\eta_{23}\circ\varepsilon_{24},\eta_{23}\circ\bar{\nu}_{24}\}$ by Theorem 7.1 aad 7.2

$=\{\eta_{23}\circ\varepsilon_{24},\nu_{23}^{3}\}$ by Lemma 6.3.

That is, $H(\beta)\equiv\mu_{23}\mod\{\eta_{23}\circ\varepsilon_{24},\nu_{23}^{3}\}$.

As $H(\theta\circ\varepsilon_{24})=\eta_{23}\circ\varepsilon_{24}$ by Lemma 7.5 and $H(\theta\circ\bar{\nu}_{24})=\eta_{23}\circ\bar{\nu}_{24}=\nu_{23}^{3}$ by Lemma 7.5 and Lemma 6.3, there exists $\beta'''\in\pi_{32}^{12}$ such that $\beta'''\equiv\beta$ mod $\{\theta\circ\varepsilon_{24},\theta\circ\bar{\nu}_{24}\}$ and $H(\beta''')=\mu_{23}$. Also

$$E^2\beta''' \equiv E^2\beta \qquad \bmod \{E^2(\theta\circ\varepsilon_{24}),\ E^2(\theta\circ\bar\nu_{24})\} = 0 \qquad \text{by (7.30)},$$
$$E^2\beta''' = 8\Delta\sigma_{29}.$$

Next consider the secondary composition $\{\sigma_{12}, \nu_{19}, \mu_{22}\}_1$ which can be defined by (7.20) and (7.25), and choose an element γ from this secondary composition. By Proposition 2.6,

$$H(\gamma) \in H\{\sigma_{12}, \nu_{19}, \mu_{22}\}_1 = -\Delta^{-1}(\sigma_{11}\circ\nu_{18})\circ\mu_{23}$$
$$\ni -\iota_{23}\circ\mu_{23} = \mu_{23} \qquad \text{by (7.21)},$$

i.e., $\qquad H(\gamma) \equiv \mu_{23} \qquad \bmod H(\sigma_{12}\circ\pi_{32}^{19} + \pi_{23}^{12}\circ\mu_{23}) = 0.$

As $H(\gamma-\beta''') = \mu_{23}-\mu_{23} = 0$, $\gamma-\beta''' \in E\pi_{31}^{11} = \{\bar\kappa_{12}, E\theta'\circ\varepsilon_{24}, E\beta''\}$.

Here $\qquad 2\gamma \in \{\sigma_{12}, \nu_{19}, \mu_{22}\}_1\circ 2\iota_{32} = \sigma_{12}\circ E\{\nu_{18}, \mu_{21}, 2\iota_{30}\}$

$$\subset \sigma_{12}\circ\pi_{32}^{19} = 0 \qquad \text{by Theorem 6.7},$$
$$2(\gamma-\beta''') = -2\beta''' \in 2E\pi_{31}^{11} = \{2\bar\kappa_{12}, 2(E\theta'\circ\varepsilon_{24}), 2E\beta''\}$$
$$= \{2\bar\kappa_{12}, E\theta'\circ(2\varepsilon_{24}), \Delta(2\mu_{25})\}$$
$$= \{2\bar\kappa_{12}\} \qquad \text{by Lemmas 6.1, 6.5.}$$

Therefore $2\beta''' = 2x\bar\kappa_{12}$ for some integer x.

The kernel of $E^2: \pi_{32}^{12} \to \pi_{34}^{14}$ is generated by $E\theta'\circ\varepsilon_{24}$, $\Delta\mu_{25}$, $\theta\circ\bar\nu_{24}$ and $\theta\circ\varepsilon_{24}$, since $\pi_{34}^{25} = \{\eta_{25}\circ\varepsilon_{26}, \nu_{25}^3, \mu_{25}\}$, $\pi_{35}^{27} = \{\bar\nu_{27}, \varepsilon_{27}\}$ and the following relations hold:

(16.10) $\quad \Delta(\bar\nu_{27}) = \Delta(\iota_{27})\circ\bar\nu_{25} = E\theta\circ\bar\nu_{25} \qquad$ by (7.30),

$\qquad \Delta(\varepsilon_{27}) = \Delta(\iota_{27})\circ\varepsilon_{25} = E\theta\circ\varepsilon_{25} \qquad$ by (7.30).

(16.11) $\quad \Delta(\eta_{25}\circ\varepsilon_{26}) = \Delta(\eta_{25})\circ\varepsilon_{24} = E\theta'\circ\varepsilon_{24} \qquad$ by (7.30),

$\qquad \Delta(\nu_{25}^3) = \Delta(\nu_{25}^2)\circ\nu_{29} = \Delta(H(\lambda))\circ\nu_{29} = 0 \qquad$ by Lemma 12.18.

As $\qquad 2x\bar\kappa_{14} = E^2(2x\bar\kappa_{12}) = E^2(2\beta''') = 16\Delta\sigma_{29} = \Delta(16\sigma_{29}) = 0$,

so $\qquad 2x\bar\kappa_{12} = y(E\theta'\circ\varepsilon_{24}) + z(\Delta\mu_{25}) + u(\theta\circ\bar\nu_{24}) + v(\theta\circ\varepsilon_{24})$

by the above facts, where $y, z, u, v = 0, 1$. Operating H we know $u = v = 0$. Then

$$2x\bar\kappa_{12} = y(E\theta'\circ\varepsilon_{24}) + z(\Delta\mu_{25})$$
$$= yE(\theta'\circ\varepsilon_{23}) + zE\beta''.$$

By (16.8) $\qquad 2x\bar\kappa_{11} = y(\theta'\circ\varepsilon_{23}) + z\beta''.$

Since $\bar\kappa_{11}$, $\theta'\circ\varepsilon_{23}$ and β'' are independent generators of π_{31}^{11}, we have $y = z = 0$. Thus $2\beta''' = 2x\bar\kappa_{12} = 0$. q.e.d.

In the exact sequence $\cdots\to\pi_{33}^{23}\xrightarrow{\Delta}\pi_{31}^{11}\xrightarrow{E}\pi_{32}^{12}\xrightarrow{H}\pi_{32}^{23}\to\cdots$, E is a monomorphism by (16.8) and H is an epimorphism, since π_{32}^{23} is generated by ν_{23}^3, μ_{23} and $\eta_{23}\circ\varepsilon_{24}$, and

$$\text{(16.12)}\qquad \begin{aligned} H(\theta\circ\bar\nu_{24}) &= \eta_{23}\circ\bar\nu_{24} = \nu_{23}^3 && \text{by Lemma 7.5,}\\ H(\theta\circ\varepsilon_{24}) &= \eta_{23}\circ\varepsilon_{24} && \text{by Lemma 7.5,}\\ H(\beta''') &= \mu_{23} && \text{by Lemma 16.3.}\end{aligned}$$

Obviously the order of $\theta\circ\bar\nu_{24}$, $\theta\circ\varepsilon_{24}$ and β''' is 2. So the following exact sequence splits:

$$0\longrightarrow\pi_{31}^{11}\xrightarrow{E}\pi_{32}^{12}\xrightarrow{H}\pi_{32}^{23}\longrightarrow 0.$$

Therefore $\quad\pi_{32}^{12}=\{\bar\kappa_{12}, E\theta'\circ\varepsilon_{24}, \Delta\mu_{25}, \theta\circ\varepsilon_{24}, \theta\circ\bar\nu_{24}, \beta'''\}$
$$\approx Z_8\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2.$$

Consider the exact sequence

$$\cdots\longrightarrow\pi_{34}^{25}\xrightarrow{\Delta}\pi_{32}^{12}\xrightarrow{E}\pi_{33}^{13}\xrightarrow{H}\pi_{33}^{25}\xrightarrow{\Delta}\pi_{31}^{12}\longrightarrow\cdots,$$

where $\pi_{34}^{25}=\{\nu_{25}^3, \mu_{25}, \eta_{25}\circ\varepsilon_{26}\}$ and $\pi_{33}^{25}=\{\bar\nu_{25}, \varepsilon_{25}\}$.

By (12.26), $\Delta:\pi_{33}^{25}\to\pi_{31}^{12}$ is a monomorphism, so H is trivial. Therefore we obtain

$$\pi_{33}^{13}=E\pi_{32}^{12}=\{\bar\kappa_{13}, E\theta\circ\varepsilon_{25}, E\theta\circ\bar\nu_{25}, E\beta'''\}\approx Z_8\oplus Z_2\oplus Z_2\oplus Z_2,$$

by (16.11).

As $\Delta\pi_{34}^{27}=\{\Delta\sigma_{27}\}=\{\xi_{13}\circ\eta_{31}\}$ by (12.26), we have $H\pi_{34}^{14}=\{2\sigma_{27}\}\approx Z_8$ in the exact sequence

$$\cdots\longrightarrow\pi_{35}^{27}\xrightarrow{\Delta}\pi_{33}^{13}\xrightarrow{E}\pi_{34}^{14}\xrightarrow{H}\pi_{34}^{27}\xrightarrow{\Delta}\pi_{32}^{13}\longrightarrow\cdots,$$

where $\pi_{35}^{27}=\{\bar\nu_{27}, \varepsilon_{27}\}$ and $\pi_{34}^{27}=\{\sigma_{27}\}\approx Z_{16}$.

By the use of (16.10), $E\pi_{33}^{13}=\{\bar\kappa_{14}, E^2\beta'''\}\approx Z_8\oplus Z_2$. Then, we have obtained the exact sequence

$$0\longrightarrow\{\bar\kappa_{13}, E\beta'''\}\xrightarrow{E}\pi_{34}^{14}\xrightarrow{H}\{2\sigma_{27}\}\longrightarrow 0.$$

Since $8\Delta\sigma_{29}=E^2\beta'''\neq 0$ by Lemma 16.3, the order of $\Delta\sigma_{29}$ is 16. We have also $H\Delta\sigma_{29}=\pm 2\sigma_{27}$. Therefore

$$\pi_{34}^{14} = \{\Delta\sigma_{29}, \bar{\kappa}_{14}\} = Z_{16}\oplus Z_8 .$$

Consider the exact sequence

$$\cdots \longrightarrow \pi_{36}^{29} \xrightarrow{\Delta} \pi_{34}^{14} \xrightarrow{E} \pi_{35}^{15} \xrightarrow{H} \pi_{35}^{29} \xrightarrow{\Delta} \pi_{33}^{14} \longrightarrow \cdots ,$$

where $\pi_{36}^{29}=\{\sigma_{29}\}$, $\pi_{35}^{29}=\{\nu_{29}^2\}$ and $\pi_{33}^{14}=\{\omega_{14}\circ\nu_{30}, \bar{\sigma}_{14}, \bar{\zeta}_{14}\}$.

Since $\Delta(\nu_{29}^2)=E^2\omega'=2(\omega_{14}\circ\nu_{30})\neq 0$ by (12.27), the above H is trivial. It is obvious that the kernel of E: $\pi_{34}^{14}\rightarrow\pi_{35}^{15}$ is generated by $\Delta\sigma_{29}$. Thus we obtain $\pi_{35}^{15}=\{\bar{\kappa}_{15}\}\cong Z_8$.

$\Delta(\nu_{31}^2)=\Delta(H(\nu_{16}^*\circ\nu_{34}))=0$ by Lemma 12.16, where ν_{31}^2 is a generator of π_{37}^{31}. We have also $\pi_{36}^{31}=0$. Thus E: $\pi_{35}^{15}\rightarrow\pi_{36}^{16}$ is an isomorphism onto. Then

$$\pi_{36}^{16} = \{\bar{\kappa}_{16}\} \cong Z_8 .$$

Next $\pi_{37}^{17}=\{\bar{\kappa}_{17}\}\cong Z_8$ follows immediately from $\pi_{38}^{33}=\pi_{37}^{33}=0$.

Lemma 16.4. *There exists an element $\bar{\beta}$ of π_{39}^{19} such that*

$$E\bar{\beta} = \Delta\eta_{41} \quad \text{and} \quad H(\bar{\beta}) = \eta_{37}^2 .$$

Proof. This follows immediately if we take $n=18$, $i=38$ and $\alpha=\eta_{36}\in\pi_{37}^{36}$ in (ii) of Proposition 11.10. q.e.d.

Now consider the exact sequence

$$\cdots \longrightarrow \pi_{39}^{35} \xrightarrow{\Delta} \pi_{37}^{17} \xrightarrow{E} \pi_{38}^{18} \xrightarrow{H} \pi_{38}^{35} \xrightarrow{\Delta} \pi_{36}^{17} \longrightarrow \cdots ,$$

where $\pi_{39}^{35}=0$, $\pi_{38}^{35}=\{\nu_{35}\}$ and $\pi_{36}^{17}=\{\omega_{17}\circ\nu_{33}, \bar{\sigma}_{17}, \bar{\zeta}_{17}\}$.

$H\pi_{38}^{18}=\{2\nu_{35}\}\cong Z_4$, since $\Delta\nu_{35}=\omega_{17}\circ\nu_{33}$ and $\Delta(2\nu_{35})=2(\omega_{17}\circ\nu_{33})=0$ (see page 170).

$H(\Delta\nu_{37})=\pm 2\nu_{15}$ by Proposition 2.7. Thus $2\Delta\nu_{37}\neq 0$ in π_{38}^{18}.

By the above Lemma 16.4 and (5.5),

$$4\Delta\nu_{37} = \Delta(\eta_{37}^3) = \Delta(H(\bar{\beta}\circ\eta_{39})) = 0 .$$

i.e., the order of $\Delta\nu_{37}$ is 4. Therefore the following exact sequence splits:

$$0 \longrightarrow \pi_{37}^{17} \xrightarrow{E} \pi_{38}^{18} \xrightarrow{H} \{2\nu_{35}\} \longrightarrow 0 ,$$

and

$$\pi_{38}^{18} = \{\bar{\kappa}_{18}, \Delta\nu_{37}\} \cong Z_8\oplus Z_4 .$$

Lemma 16.5. *There exists an element $\bar{\bar{\beta}}$ of π_{40}^{20} such that*

$$E\bar{\bar{\beta}} = \Delta\iota_{43} \quad and \quad H(\bar{\bar{\beta}}) = \eta_{39}.$$

Proof. This lemma immediately follows if we take $n=19$, $i=39$ and $\alpha=\iota_{38}$ in (ii) of Proposition 11.10. q.e.d.

Now we note that $E: \pi_{39}^{19} \to \pi_{40}^{20}$ is a monomorphism, for, in the exact sequence $\pi_{41}^{39} \xrightarrow{\Delta} \pi_{39}^{19} \xrightarrow{E} \pi_{40}^{20}$, the generator η_{39}^2 of π_{41}^{39} is an H-image of $\bar{\bar{\beta}}\circ\eta_{40}$. Since $2E\bar{\beta}=2\Delta\eta_{41}=\Delta(2\eta_{41})=0$ and $E: \pi_{39}^{19}\to\pi_{40}^{20}$ is a monomorphism, the order of $\bar{\beta}$ in Lemma 16.4 is 2. Therefore the following exact sequence splits:

$$0 \longrightarrow E\pi_{38}^{18} \longrightarrow \pi_{39}^{19} \xrightarrow{H} \pi_{39}^{37} \longrightarrow 0,$$

where $E\pi_{38}^{18}=\{\bar{\kappa}_{19}\}\approx Z_8$ and $\pi_{39}^{37}=\{\eta_{37}^2\}$.

By the use of Lemma 16.4,

$$\pi_{39}^{19} = \{\bar{\kappa}_{19}, \bar{\beta}\} \approx Z_8 \oplus Z_2.$$

Since $\Delta(\eta_{39})=0$ and $\Delta(\eta_{39}^2)=0$ by Lemma 16.5, we have an exact sequence

$$0 \longrightarrow \pi_{39}^{19} \xrightarrow{E} \pi_{40}^{20} \xrightarrow{H} \pi_{40}^{39} \longrightarrow 0.$$

Lemma 16.6. *The above exact sequence splits, that is, the order of $\bar{\bar{\beta}}$ in Lemma* 16.5 *is* 2.

Proof. It follows from the homotopy exact sequence associated with the pair $(\Omega^2(S^{20}), S^{18})$ that the following sequence is exact:

$$(16.13) \qquad \cdots \to \pi_{38}^{18} \to \pi_{40}^{20} \to \pi_{38}((\Omega^2(S^{20}), S^{18}):2) \to \pi_{37}^{18} \to \cdots,$$

where $\pi_{38}^{18}=\{\bar{\kappa}_{18}, \Delta\nu_{37}\}\approx Z_8\oplus Z_4$, $\pi_{37}^{18}=\{\zeta_{18}, \bar{\sigma}_{18}\}\approx Z_8\oplus Z_2$ and $\pi_{38}((\Omega^2(S^{20}), S^{18}):2)$ denotes the 2-primary component of $\pi_{38}(\Omega^2(S^{20}), S^{18})$.

By Theorem 11.7 in the case that $i=37$, $n=18$ and $k=2$,

$$\pi_{38}((\Omega^2(S^{20}), S^{18}):2)\approx\pi_{37}(E^{17}P_{18}^{19}:2).$$

Here $P_{18}^{19}=S^{18}\cup e^{19}$ which is homotopy equivalent to $S^{18}\vee S^{19}$. For, the attaching map of e^{19} is trivial since 18 is even.

Therefore

$$\begin{aligned}\pi_{18}((\Omega^2(S^{20}), S^{18}):2) &\cong \pi_{37}(S^{35}\vee S^{36}:2)\\ &\cong \pi_{37}^{35}\oplus\pi_{37}^{36} \qquad \text{by (4.18) of [4]}\\ &\cong Z_2\oplus Z_2 .\end{aligned}$$

The element $2\bar{\bar{\beta}}\in\pi_{40}^{20}$ is a multiple of $E(\bar{\beta})=\Delta\eta_{41}$, since $E(2\bar{\bar{\beta}})=2(\Delta\iota_{43})=0$ and since $\Delta\eta_{41}$ generates the kernel of $E:\pi_{40}^{20}\to\pi_{41}^{21}$. So, there are two possibilities: $\pi_{40}^{20}\cong Z_8\oplus Z_4$ or $\pi_{40}^{20}\cong Z_8\oplus Z_2\oplus Z_2$.

Since $\pi_{38}^{18}=\{\bar{\kappa}_{18}, \Delta\nu_{37}\}\cong Z_8\oplus Z_4$ and $E:\pi_{39}^{19}\to\pi_{40}^{20}$ is a monomorphism, $E^2\pi_{38}^{18}=\{\bar{\kappa}_{20}\}\cong Z_8$. Therefore by considering the exact sequence (16.13) the lemma follows. q.e.d.

By Lemma 16.6,

$$\pi_{40}^{20}=\{\bar{\kappa}_{20}, \Delta\eta_{41}, \bar{\bar{\beta}}\}\cong Z_8\oplus Z_2\oplus Z_2 .$$

Since $\Delta:\pi_{41}^{41}\to\pi_{39}^{20}$ is a monomorphism, we have an exact sequence

$$0\longrightarrow E\pi_{40}^{20}\longrightarrow\pi_{41}^{21}\longrightarrow 0,$$

where $E\pi_{40}^{20}=\{\bar{\kappa}_{21}, E\beta\}\cong Z_8\oplus Z_2$. Therefore

$$\pi_{41}^{21}=\{\bar{\kappa}_{21}, \Delta\iota_{43}\}\cong Z_8\oplus Z_2, \text{ where } \Delta\iota_{43}=E\bar{\bar{\beta}} \quad \text{by Lemma 16.5.}$$

As it is obvious that the kernel of $E:\pi_{41}^{21}\to\pi_{42}^{22}$ is generated by $\Delta\iota_{43}$ and $\pi_{42}^{43}=0$, we obtain $\pi_{42}^{22}=\{\bar{\kappa}_{22}\}\cong Z_8$.

The result for the stable group $(G_{20};2)=\{\bar{\kappa}\}\cong Z_8$ follows immediately.

(ii) *Odd primary components.*

We shall compute p-primary components $\pi_{n+20}(S^n:p)$ for odd prime p in order to complete our calculation of $\pi_{n+20}(S^n)$.

By Serre's isomorphism [1]

(16.14) $$\pi_{i-1}(S^{2m-1}:p)\oplus\pi_i(S^{4m-1}:p)\cong\pi_i(S^{2m}:p)$$

which is given by the correspondence $(\alpha,\beta)\to E\alpha+[\iota_{2m},\iota_{2m}]\circ\beta$, we compute only the groups $\pi_{n+20}(S^n:p)$ for the case that n is odd.

First, there are no odd prime p satisfying

$$2i(p-1)-2=20 \qquad \text{for } i=2,\cdots,p-1,$$

or

$$2i(p-1)-1=20 \qquad \text{for } i=1,2,\cdots,p-1.$$

Whence we have the next result by Theorem 13.4.

(16.15) $\pi_{2m+1+20}(S^{2m+1}:p) = 0$ *for* $m\geq 1$ *and an odd prime* $p\geq 5$.

Next we shall compute the 3-primary component of $\pi_{2m+1+20}(S^{2m+1})$ for $m\geq 1$. We shall use the following results of [2]:

(16.16) 3-*primary component of* $G_{20} = \{\beta_1\circ\beta_2\} \approx Z_3$.

For the simplicity we shall use the following notations.

$$\pi^n_{n+k} = \pi_{n+k}(S^n:3) \quad \text{and} \quad Q^n_i = \pi_i((\Omega^2(S^n), S^{n-2}):3)\,.$$

The next two exact sequences are mainly used in this last section.

(16.17) $\cdots \longrightarrow \pi^n_i \xrightarrow{E^2} \pi^{n+2}_{i+2} \longrightarrow Q^{n+2}_i \xrightarrow{\partial} \pi^n_{i-1} \longrightarrow \cdots$ *for* n: *odd*

and

(16.18) $\cdots \longrightarrow \pi^{3(n+1)+1}_{i+2} \xrightarrow{\Delta} \pi^{(3n+1)-1}_i \longrightarrow Q^{n+2}_{i-1} \longrightarrow \pi^{3(n+1)+1}_{i+1} \longrightarrow \cdots$
for $i>3(n+1)-1$ *and* n: *odd*,

where $\Delta\circ E^2=f_{3*}$ for a mapping f_3: $S^{3(n+1)-1}\rightarrow S^{3(n+1)-1}$ of degree 3.

For the case $n=1$, we have the following exact sequence (see Proposition 13.3)

(16.19) $\cdots \longrightarrow \pi^7_{i+2} \xrightarrow{\Delta} \pi^5_i \longrightarrow \pi^3_{i+1} \longrightarrow \pi^7_{i+1} \xrightarrow{\Delta} \cdots$ *for* $i>5$,

such that $\Delta\circ E^2=f_{3*}$.

$\pi^5_{22}=\pi^7_{23}=0$ by Theorem 13.10, hence we have $\pi^3_{23}=0$.

By (13.6)

$$Q^{2m+1}_{2m+1+18} = 0 \qquad \text{for } m\geq 3\,,$$
$$Q^{2m+1}_{2m+1+19} = 0 \qquad \text{for } m\geq 4\,.$$

By inserting these values in the exact sequence (16.17), we obtain the isomorphism onto for $m\geq 3$:

$$E^2:\ \pi^{2m+1}_{2m+1+20} \approx \pi^{2m+3}_{2m+3+20}\,.$$

So by (16.16),

$$\pi^{2m+1}_{2m+1+20} \approx Z_3 \qquad \textit{for } m\geq 3\,.$$

Consider the exact sequence of (16.17) in the case that $n=3$ and $i=23$, that is

$$\cdots \longrightarrow \pi^3_{23} \xrightarrow{E^2} \pi^5_{25} \longrightarrow Q^5_{23} \xrightarrow{\partial} \pi^3_{22} \xrightarrow{E^2} \pi^5_{24} \longrightarrow \cdots,$$

where $\pi_{23}^3=0$ and $E^2: \pi_{22}^3 \to \pi_{24}^5$ is an isomorphism onto by (vii) of Theorem 13.10. Therefore we have

(16.20) $$\pi_{25}^5 \cong Q_{23}^5 .$$

Next consider the exact sequence (16.18) in the case that $n=3$, $i=21$, that is

$$\cdots \longrightarrow \pi_{26}^{13} \xrightarrow{\Delta} \pi_{24}^{11} \longrightarrow Q_{23}^5 \longrightarrow \pi_{25}^{13} \longrightarrow \cdots ,$$

where $\pi_{26}^{13}=\{\alpha_1(13)\circ\beta_1(16)\} \cong Z_3$, $\pi_{24}^{11}=\{\alpha_1(11)\circ\beta_1(14)\} \cong Z_3$ and $\pi_{25}^{13}=0$ by (i) of Theorem 13.10 and Theorem 13.9. As $\Delta\circ E^2=f_{3*}$,

$$\Delta(\alpha_1(13)\circ\beta_1(16)) = 3(\alpha_1(11)\circ\beta_1(14)) = 0 .$$

Thus $Q_{23}^5 \cong \pi_{24}^{11} \cong Z_3$. Therefore by (16.20)

$$\pi_{25}^5 \cong Z_3 .$$

Summarizing these results, we have the following

Theorem 16.7.

$\pi_{2m+1+20}(S^{2m+1}:p) = 0$ *for* $m \geq 1$ *and odd prime* $p \geq 5$, *and*

$$\pi_{2m+1+20}(S^{2m+1}:3) \cong \begin{cases} 0 & m=1 \\ Z_3 & m \geq 2 \end{cases}$$

For even dimensional spheres, their homotopy groups are easily calculated by use of Serre's isomorphism (16.14) and the known results for $\pi_{2m+1+k}(S^{2m+1}:p)$ where $k \leq 20$, and the results are the following.

Theorem 16.8. *The odd primary components of* $\pi_{2n+20}(S^{2n})$ *are isomorphic to*

Z_3	*for* $n=6, 8$ *and* $n \geq 10$,
$Z_3 \oplus Z_3$	*for* $n=4, 9$,
$Z_{15} \oplus Z_3$	*for* $n=3, 7$,
Z_{33}	*for* $n=1$,
$Z_{63} \oplus Z_3$	*for* $n=5$,
0	*for* $n=2$.

KYOTO UNIVERSITY

BIBLIOGRAPHY

[1] J-P. Serre : *Groupes d'homotopie et classes des groupes abéliens*, Ann. of Math., 58 (1953), 258-294.

[2] H. Toda : *p-primary components of homotopy groups* IV, *Compositions and toric constructions*, Memoirs Univ. of Kyoto, 32 (1959), 297-332.

[3] H. Toda : *Composition methods in homotopy groups of spheres*, Ann. of Math. Studies. No. 49. Princeton (1962).

[4] G. W. Whitehead : *A generalization of the Hopf invariant*, Ann. of Math., 51 (1950), 192-237.