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Affine transformations of Finsler spaces

Dedicated to Professor Y. Akizuki on his 60th birthday

By

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The notion of Finsler space is based on the Finsler metric, in similar manner with the case of Riemann space. However, we may consider the geometry of spaces with an affine connection without Riemann metric, and have a beautiful theory of connections in principal bundles. Recently, a theory of Finsler spaces has been successfully developed by T. Okada [1] from a modern and global point of view. He introduce first a connection in a certain fibre bundle. The connection is composed of two distributions, and hence is called a pair-connection, which is of the most general type of connections derived from a Finsler metric by several authors. Thus we may say that a theory of connection of the Finsler type is established *without Finsler metric.*

The present paper is the first part of a series dealing with a theory of transformations of Finsler spaces. According to the theory due to T. Okada, affine transformations can be treated for Finsler spaces without use of the metric, parallel to the case for spaces with ordinary affine connection. The base space of the principal bundle in which a pair-connection is defined is the tangent vector bundle B of the manifold M unter consideration, and hence it is natural to start from transformations of B, not of M. We shall introduce linear transformations of B at the beginning of Section 2, which will be more general than that induced from transformations of M. It will be shown that a linear transformation of B is constituted from an induced part and a deviation, that is, a rotation part of elements of support. Sections 6 and 7 are devoted to the thorough investigation of the rotation part. Finally, in Sections 8 and 9 is presented a theory of infinitesimal affine transformations.

§1. Finsler connections

In the first place, we shall sketch the theory of Finsler connections with use of fibre bundles.

Consider a differentiable manifold M of n dimensions, and let $P(M, \pi, G)$ be the bundle of frames of M, where π is the projection $P \to M$ and $G = GL(n, R)$. Let F be an n-dimensional real vector space with a fixed basis (e_a), $a = 1, \cdots, n$, and then $g = (g^a{}_b) \in G$ acts on F by $\rho_g(e_a) = g^b{}_a e_b$. With this standard fibre F, the associated fibre bundle of P is obtained, which is the tangent vector bundle $B(M, \tau, F, G)$ of M. The induced bundle $\tau^{-1}P = Q(B, \bar{\pi}, G)$ is then obtained from P by the projection $\tau : B \to M$. An element $q \in Q$ is a pair $(b, p) \in B \times P$ such that $\tau(b) = \pi(p)$. If $\tau(b)$ has the coordinate (x^i), and

$$b = b^i \frac{\partial}{\partial x^i}, \quad p = (p_a), \quad p_a = p^i{}_a \frac{\partial}{\partial x^i},$$

we call $(x^i, b^i, p^i{}_a)$ the *canonical* coordinate of $q = (b, p)$. From a right translation R_g of P by $g \in G$, the mapping $\bar{R}_g : Q \to Q$ is induced which is defined by $\bar{R}_g(b, p) = (b, R_g(p))$. $\bar{R}_g$ is called the right translation of Q by $g \in G$.

A *Finsler connection* (called a pair-connection by T. Okada) in Q is a pair of distributions $(\Gamma^v, \Gamma^h) : Q \ni q \to (\Gamma^v{}_q, \Gamma^h{}_q)$, which satisfies the following three conditions:

1. The tangent space Q_q at $q \in Q$ is the direct sum $Q^r{}_q + \Gamma^v{}_q + \Gamma^h{}_q$, where $Q^r{}_q$ is the vertical subspace of Q_q.
2. For every $g \in G$ and $q \in Q$, $\bar{R}_g \Gamma^v{}_q = \Gamma^v{}_{qg}$, $\bar{R}_g \Gamma^h{}_q = \Gamma^h{}_{qg}$.
3. For every $q = (b, p) \in Q$, $\bar{\pi}\Gamma^v{}_q = B^r{}_b$, the vertical subspace of the tangent space B_b to B at b.

The direct sum $\Gamma = \Gamma^v + \Gamma^h$ gives an ordinary connection in Q, and hence its *connection form* ω is defined. For a tangent vector $X \in Q_q$ the decomposition of X with respect to the Finsler connection is expressed as follows:

$$X = vX + hX = vX + h^v X + h^h X ,$$
$$vX \in Q^v{}_q, \quad hX \in \Gamma_q, \quad h^v X \in \Gamma^v{}_q, \quad h^h X \in \Gamma^h{}_q .$$

The lift of a given vector $X \in B_b$ to $q=(b, p) \in Q$ is denoted by $l_q X$ which is uniquely determined by $l_q X \in \Gamma_q$ and $\bar{\pi} l_q X = X$. The Finsler connection induces the decomposition of B_b into the direct sum $B_b = B^v{}_b \dot{+} H_b$, where $H_b = \bar{\pi}\Gamma^h{}_q$, $q=(b, p)$. The distribution $H: B \ni b \to H_b$ on B is called the *non-linear connection* induced from the Finsler connection. For a tangent vector $X \in B_b$, the decomposition of X with respect to H is expressed as follows:

$$X = v'X + h'X, \quad v'X \in B^v{}_b, \quad h'X \in H_b .$$

Further, the lift $l'_b X$ of a tangent vector $X \in M_x$, $x = \tau(b)$, to $b \in B$ is defined by $l'_b X \in H_b$ and $\tau l'_b X = X$.

In the case of an ordinary linear connection in a principal bundle, the basic form and the basic vector field are defined. In the case of a Finsler connection, two kinds of basic forms and basic vector fields are obtained as follows. The *h-basic form* $\theta^{(h)}$ and *h-basic vector field* $B^{(h)}(f)$ associated with $f \in F$ are defined by

$$\theta^{(h)}{}_q = p^{-1} \circ \tau \circ \bar{\pi} , \quad B^{(h)}(f)_q = l_q \circ l_b' \circ pf , \quad q = (b, p) ,$$

where $p \in P$ is thought of as an admissible mapping of $B: F \to \tau^{-1} \circ \pi(p)$. It is shown that $\theta^{(h)}(X) = 0$ for $h^h X = 0$, $B^{(h)}(f) \in \Gamma^h$ and $\theta^{(h)}(B^{(h)}(f)) = f$. Moreover, the *v-basic form* $\theta^{(v)}$ and *v-basic vector field* $B^{(v)}(f)$ are obtained as follows:

$$\theta^{(v)}{}_q = j_{\gamma(q)}{}^{-1} \circ p^{-1} \circ v' \circ \bar{\pi} , \quad B^{(v)}(f)_q = l_q \circ p \circ j_{\gamma(q)} f , \quad q = (b, p) ,$$

where p^{-1} means the differential of the inverse of the admissible mapping p, the mapping $\gamma: Q \to F$ is called the *characteristic vector field* defined by $\gamma(b, p) = p^{-1} b \in F$, and $j_f: F \to F_f$ (a tangent space to F at f) is the identification defined by $j_f(f_1) = f_1^a \, (\partial/\partial f^a)_f$, $f_1 = f_1^a e_a$. It is shown that $\theta^{(v)}(X) = 0$ for $h^v X = 0$, $B^{(v)}(f) \in \Gamma^v$, and $\theta^{(v)}(B^{(v)}(f)) = f$.

In terms of a canonical coordinate, basic vector fields $B^{(h)}(f)$ and $B^{(v)}(f)$ are expressed as follows:

$$B^{(h)}(f)=f^a p^i{}_a\left(\frac{\partial}{\partial x^i}-F^j{}_i\frac{\partial}{\partial b^j}-p^j{}_b F_j{}^k{}_i\frac{\partial}{\partial p_b{}^k}\right),$$

$$B^{(v)}(f)=f^a p^i{}_a\left(\frac{\partial}{\partial b^i}-p^j{}_b C_j{}^k{}_i\frac{\partial}{\partial p_b{}^k}\right),$$

where $C_j{}^i{}_k$, $F_j{}^i{}_k$ and $F^i{}_j$ are called *coefficients of the Finsler connection*, which are functions of x^i and b^i.

With respect to the connection $\Gamma=\Gamma^v+\Gamma^h$, we obtain the covariant differential $D\alpha$ of a form α on Q such that $D\alpha=d\alpha\circ h$. Since $h=h^v+h^h$ in the case of a Finsler connection, $D\alpha$ may be written in the decomposed form. For an example, if α is an 1-form,

$$D\alpha=D^v\alpha+2D^{vh}\alpha+D^h\alpha,\quad D^v\alpha=d\alpha(h^v,h^v),$$

$$D^{vh}\alpha=\frac{1}{2}(d\alpha(h^v,h^h)+d\alpha(h^h,h^v)),\quad D^h\alpha=d\alpha(h^h,h^h).$$

Therefore, the covariant differential of the connection form ω gives three kinds of *curvature forms*:

$$D^v\omega=\Omega^v=-\frac{1}{2}S_b{}^a{}_{cd}\theta^{(v)c}\wedge\theta^{(v)d}\hat{g}_a{}^b,$$

(1.1) $$D^{vh}\omega=\Omega^{vh}=\frac{1}{2}P_b{}^a{}_{dc}\theta^{(v)c}\wedge\theta^{(h)d}\hat{g}_a{}^b,$$

$$D^h\omega=\Omega^h=-\frac{1}{2}R_b{}^a{}_{cd}\theta^{(h)c}\wedge\theta^{(h)d}\hat{g}_a{}^b,$$

where $\hat{g}_a{}^b=(\partial/\partial g_b{}^a)_e$ is the basis of the Lie algebra $\hat{G}$, and $\theta^{(v)}=\theta^{(v)a}e_a$, $\theta^{(h)}=\theta^{(h)a}e_a$. From the coefficients $S_b{}^a{}_{cd}$, $P_b{}^a{}_{dc}$ and $R_b{}^a{}_{cd}$ we have three kinds of curvature tensors, for an example, $S_j{}^i{}_{kl}=p_a{}^i p_j^{-1b}p_k^{-1c}p_l^{-1d}S_b{}^a{}_{cd}$.

Next, $D\theta^{(v)}=\Theta^{(v)}$ is called the *v-torsion form* which is written

$$\Theta^{(v)}=\Theta^{(v)v}+2\Theta^{(v)vh}+\Theta^{(v)h},$$

$$\Theta^{(v)v}=-\frac{1}{2}S_b{}^a{}_c\theta^{(v)b}\wedge\theta^{(v)c}e_a,$$

(1.2) $$\Theta^{(v)vh}=\frac{1}{2}P_c{}^a{}_b\theta^{(v)b}\wedge\theta^{(h)c}e_a,$$

$$\Theta^{(v)h}=-\frac{1}{2}R_b{}^a{}_c\Theta^{(h)b}\wedge\theta^{(h)c}e_a.$$

Coefficients $S_b{}^a{}_c$, $P_c{}^a{}_b$ and $R_b{}^a{}_c$ give three kinds of v-torsion tensors. Finally, $D\theta^{(h)}=\Theta^{(h)}$ is the *h-torsion form* which is of a little different form as follows:

$$\Theta^{(h)} = 2\Theta^{(h)vh}+\Theta^{(h)h},$$

$$(1.3)\qquad \Theta^{(h)vh} = \frac{1}{2}C_c{}^a{}_b\theta^{(v)b}\wedge\theta^{(h)c}e_a,$$

$$\Theta^{(h)h} = -\frac{1}{2}T_b{}^a{}_c\theta^{(h)b}\wedge\theta^{(h)c}e_a.$$

Hence there are only two kinds of h-torsion tensors.

We obtain two kinds of covariant derivatives $\Delta^v T$ and $\Delta^h T$ of a tensor T by the equations

$$\Delta^v T = dT(B_a{}^{(v)})\otimes e^a, \quad \Delta^h T = dT(B_a{}^{(h)})\otimes e^a,$$

where $B_a{}^{(v)}=B^{(v)}(e_a)$ and $B_a{}^{(h)}=B^{(h)}(e_a)$ form the basis of Γ^v and Γ^h respectively, and (e^a) is the dual basis of F^* to (e_a). If, for an example, T is of (1, 1)-type such that $T=T_b{}^a e_a\otimes e^b$, then we have

$$\Delta^v T = T_b{}^a|_c e_a\otimes e^b\otimes e^c, \quad \Delta^h T = T_b{}^a{}_{|c} e_a\otimes e^b\otimes e^c.$$

The *structure equation* for ω is of the same form with the well-known one for an ordinary connection, namely, $\Omega=d\omega+[\omega,\omega]$. On the other hand, since there are two kinds of basic forms, we have the *additional structure equations*

$$\Theta^{(v)}=d\theta^{(v)}+\omega\wedge\theta^{(v)}, \quad \Theta^{(h)}=d\theta^{(h)}+\omega\wedge\theta^{(h)}.$$

Let $\bar{F}(A)$ be a fundamental vector field on Q induced from $A\in\hat{G}$. It is well known that $[\bar{F}(A),\bar{F}(A')]=\bar{F}([A,A'])$. Further brackets of $\bar{F}(A)$, $B^{(v)}(f)$ and $B^{(h)}(f)$ are written in the following forms:

$$[\bar{F}(A), B^{(v)}(f)] = B^{(v)}(Af), \quad [\bar{F}(A), B^{(h)}(f)] = B^{(h)}(Af),$$

$$[B^{(v)}(f), B^{(v)}(f')] = S_b{}^a{}_{cd}f^c f'^d\bar{F}(\hat{g}_a{}^b)+S_b{}^a{}_c f^b f'^c B_a{}^{(v)},$$

$$(1.4)\quad [B^{(v)}(f), B^{(h)}(f')] = -P_b{}^a{}_{dc}f^c f'^d\bar{F}(\hat{g}_a{}^b) - P_c{}^a{}_b f^b f'^c B_a{}^{(v)} - C_c{}^a{}_b f^b f'^c B_a{}^{(h)},$$

$$[B^{(h)}(f), B^{(h)}(f')] = R_b{}^a{}_{cd}f^c f'^d\bar{F}(\hat{g}_a{}^b) + R_b{}^a{}_c f^b f'^c B_a{}^{(v)} + T_b{}^a{}_c f^b f'^c B_a{}^{(h)}.$$

Bianchi identities are necessary in the final two sections. These are given by $D\Omega=0$ for the curvature tensors and $D\Theta^{(v)}=\Omega\wedge\theta^{(v)}$, $D\Theta^{(h)}=\Omega\wedge\theta^{(h)}$ for the torsion tensors.

We have noted in the above only what will be used in the following. Refer to the paper [1] for the detail, if necessary, in particular, expressions of curvature and torsion tensors in terms of coefficients of connection, the condition of integrability of covariant derivatives and further Bianchi identities.

§ 2. Quasi-connections in P

We first notice that the bundle space Q may be identified with the product $F\times P$ by the mapping

$$i:Q\rightarrow F\times P,\quad q\rightarrow(\gamma(q),\eta(q)),$$

where γ is the characteristic vector field and η is the induced mapping $Q\rightarrow P$, $(b,p)\rightarrow p$. By means of the identification i, various notions in Q are transfered into $F\times P$, and, in particular, we obtain the characteristic vector field γ^* on $F\times P$, right translations $R_g{}^*$ of $F\times P$, and fundamental vector fields $F^*(A)$ on $F\times P$.

Proposition 1. (1) *Let γ be the characteristic vector field on Q, and then the mapping*

$$\gamma^*=\gamma\circ i^{-1}:F\times P\rightarrow F$$

is canonical, that is, $F\times P\ni(f,p)\rightarrow f$.

(2) *Let $\bar{R}_g$ be a right translation of Q by $g\in G$, and then the mapping*

$$R^*_g=i\circ R_g\circ i^{-1}:F\times P\rightarrow F\times P$$

is given by the equation $R^*_g(f,p)=(g^{-1}f,pg)$.

(3) *Let $\bar{F}(A)$ be a fundamental vector field on Q corresponding to $A\in\dot{G}$, and put $F^*(A)=i\bar{F}(A)$. According to the decomposition $(F\times P)_{(f,p)}=F_f+P_p$, $F^*(A)$ is written in the form*

$$F^*(A)_{(f,p)}=-\sigma_f A+F(A)_p,$$

in which $F(A)$ is the fundamental vector field on P and the mapping $\sigma_f:G\rightarrow F$ is defined by $\sigma_f(g)=gf$, $f\in F$, $g\in G$.

Proof. (1) and (2) are direct results from definitions γ^* and R^*_g respectively. We shall verify (3). $\bar{F}(A)$ on Q is defined by the mapping $L_q: G \to Q,\ g \to qg,\ q \in Q$, and we have $\bar{F}(A)_q = L_q(A)$. If $q=(b, p)$, then

$$i \circ L_q(g) = i \circ (b, pg) = (g^{-1}p^{-1}b, pg) = R^*_g \circ i(q),$$

from which it follows that, if we take the mapping $L^*_{(f, p)}: G \to F \times P,\ g \to R^*_g(f, p)$, then $F^*(A)_{(f, p)} = L^*_{(f, p)}A$. Denote by π_2 the canonical mapping $F \times P \to P$, and the decomposition of $F^*(A)$ is written

$$F^*(A)_{(f, p)} = \gamma^* \circ L^*_{(f, p)}A + \pi_2 L^*_{(f, p)}A .$$

Further, making use of mappings $(\ )^{-1}: G \to G,\ g \to g^{-1}$, and $L_p: G \to P,\ g \to pg,\ p \in P$, the above equation is rewritten

$$F^*(A)_{(f, p)} = \sigma_f \circ (\ )^{-1}(A) + L_p(A),$$

and thus (3) is established.

Now, we shall introduce a *quasi-connection* in P induced from the Finsler connection in Q and a fixed element $f \in F$. Take the mapping $\bar{K}_f: P \to Q,\ p \to (pf, p)$, and define $h_f(X)$ for $X \in P_p$ as follows:

$$h_f(X) = \eta \circ h^h \bar{K}_f(X) \in P_p .$$

The tangent vector $h_f(X)$ will be called the *f-horizontal component* of X. It is easily verified that the vector $v_f(X) = X - h_f(X)$ is vertical. Then the f-horizontal subspace $\Gamma_{(f)p}$ of P_p is defined by $\{X \in P_p,\ v_f(X) = 0\}$, and the distribution $\Gamma_{(f)p}: P \ni p \to \Gamma_{(f)p}$ will be called the quasi-connection in P with respect to $f \in F$. It is noticed that $\Gamma_{(f)}$ is not a notion of an ordinary connection in P, because h_f does not commute with right translations R_g, but we get

$$R_g \circ h_f = h_{g^{-1}f} \circ R_g, \tag{2.1}$$

which will be at once verified by means of the relation $\bar{R}_g \circ \bar{K}_f = \bar{K}_{g^{-1}f} R_g$.

When an ordinary connection is given in P, we shall have the connection form which is defined by the well-known characteristic properties. In like manner, for a quasi-connection in P, a $\hat{G}$-valued

1-form $\omega^*_{(f)}$ will be defined and satisfy

$$\omega^*_{(f)}(F(A)) = A, \quad \omega^*_{(f)}(h_f X) = 0, \quad X \in P_p,$$

which are satisfied by an ordinary connection form. In the following we shall introduce such a form $\omega^*_{(f)}$.

Let ω and $\theta^{(v)}$ be the connection form and the v-basic form of the Finsler connection in Q under consideration. If we take mappings $\chi_p : F \rightarrow F \times P$, $f \rightarrow (f, p)$, and $\chi_f : P \rightarrow F \times P$, $p = (f, p)$, then the following forms are defined:

$$\omega_{(p)} = \omega \circ i^{-1} \circ \chi_p, \quad \omega_{(f)} = \omega \circ i^{-1} \circ \chi_f .$$

The former is the $\hat{G}$-valued 1-form on F which depends upon $p \in P$, and the latter is the $\hat{G}$-valued 1-form on P which depends upon $f \in F$. In like manner F-valued 1-forms $\theta^{(v)}_{(p)}$ and $\theta^{(v)}_{(f)}$ will be obtained from $\theta^{(v)}$.

Lemma 1. *Let σ_f be the mapping defined in Proposition* 1, *and j_f the identification $F \rightarrow F_f$. Then $\omega_{(p)}$, $\omega_{(f)}$, $\theta^{(v)}_{(p)}$ and $\theta^{(v)}_{(f)}$ satisfy the following equations*:

(2.2) $$\omega_{(f)}(F(A)_p) = A + \omega_{(p)}(\sigma_f A),$$

(2.3) $$\theta^{(v)}_{(p)}(\dot{f}) = f,$$

(2.4) $$\theta^{(v)}_{(f)}(F(A)) = Af,$$

where we putted $j_{f_1}(f) = \dot{f} \in F_{f_1}$.

Proof. According to (3) of Proposition 1, we have

$$A = \omega(\bar{F}(A)) = \omega \circ i^{-1}(F^*(A)) = \omega_{(p)}(-\sigma_f A) + \omega_{(f)}(F(A)),$$

and thus (2.2) is obtained.

Next, we consider (2.3). We first see that

$$\bar{\pi} \circ i^{-1} \circ \chi_p(\dot{f}) = p\dot{f} \in B_b^{(v)}, \quad b = pf_1,$$

and hence the v-basic vector field $B^{(v)}(f)$ is such that

$$B^{(v)}(f)_q = l_q \circ p\dot{f} = l_q \circ \bar{\pi} \circ i^{-1} \chi_p(\dot{f}) = h^v \circ i^{-1} \circ \chi_p(\dot{f}).$$

Therefore we have

$$\theta^{(v)}_{(p)}(\hat{f}) = \theta^{(v)} \circ i^{-1} \circ X_p(\hat{f}) = \theta^{(v)} \circ h^v \circ i^{-1} \circ X_p(\hat{f}) = \theta^{(v)}(B^{(v)}(f)) = f ,$$

and hence (2.3) is proved.

Finally we consider (2.4). According to (3) of Proposition 1, we obtain

$$0 = \theta^{(v)}(\bar{F}(A)) = \theta^{(v)}_{(p)}(-\sigma_f A) + \theta^{(v)}_{(f)}(F(A)) .$$

Then (2.4) is the direct result from $j_f(Af) = \sigma_f A$ and (2.3).

Theorem 1. *Let* $\omega^*_{(f)}$ *be the* $\hat{G}$*-valued* 1*-form on* P *defined by*

$$\omega^*_{(f)p} = \omega_{(f)p} - \omega_{(p)f} \circ j_f \circ \theta^{(v)}_{(f)} . \tag{2.5}$$

Then $\omega^*_{(f)}$ *satisfies the following equations*:

$$\omega^*_{(f)}(F(A)) = A , \quad \omega^*_{(f)}(X) = 0 , \quad X \in \Gamma_{(f)} .$$

Proof. According to Lemma 1, we have

$$\begin{aligned} \omega^*_{(f)}(F(A)_p) &= \omega_{(f)}(F(A)_p) - \omega_{(p)f} \circ j_f \circ \theta^{(v)}_{(f)}(F(A)_p) \\ &= A + \omega_{(p)}(\sigma_f A) - \omega_{(p)f} \circ j_f(Af) = A , \end{aligned}$$

and hence the first equation is proved. Next, take a mapping $K_f = \bar{\pi} \circ i^{-1} \circ X_f : P \to B$, $p \to pf$, $f \in F$, and then

$$j_f \circ \theta^{(v)}_{(f)}(h_f X) = p^{-1} \circ v' \circ K_f(h_f X) , \quad X \in P_p .$$

Both of the vector $K_f(h_f X)$ contained in the right-hand side of the above and the vector $\bar{\pi} \circ h^h \circ \bar{K}_f X$ are tangent to B at pf. Put $Y = K_f(h_f X) - \bar{\pi} \circ h^h \circ \bar{K}_f X$. It is clear that $\bar{\pi} \circ h^h \circ \bar{K}_f X$ is horizontal, while Y is vertical. Therefore we have $Y = v'Y = v' \circ K_f(h_f X)$, and hence

$$j_f \theta^{(v)}_{(f)}(h_f X) = p^{-1} Y = p^{-1}(K_f \circ \eta \circ i^{-1} - \bar{\pi} \circ i^{-1}) i \circ h^h(\bar{K}_f X) .$$

Since $i \circ h^h(\bar{K}_f X) \in (F \times P)_{(f,p)} = F_f + P_p$, we may put $i \circ h^h(\bar{K}_f X) = X'_f + X''_p$, where $X'_f \in F_f$ and $X''_p \in P_p$. Then we obtain

$$j_f \circ \theta^{(v)}_{(f)}(h_f X) = p^{-1}(-pX'_f) = -X'_f = -\gamma^* \circ i \circ h^h(\bar{K}_f X) .$$

Therefore we see

$$\begin{aligned} 0 = \omega(h^h \bar{K}_f X) &= \omega_{(p)}(\gamma^* \circ i \circ h^h \bar{K}_f X) + \omega_{(f)}(\pi_2 \circ i \circ h^h \bar{K}_f X) \\ &= -\omega_{(p)}(j_f \theta^{(v)}_{(f)}(h_f X)) + \omega_{(f)}(h_f X) . \end{aligned}$$

This completes the proof of the second equation.

The result of this theorem is that $\omega^*_{(f)}$ satisfies two of the characteristic equations for an ordinary connection form. In this point of view, the form $\omega^*_{(f)}$ is to be called the *quasi-connection form* of the quasi-connection $\Gamma_{(f)}$. However, $\omega^*_{(f)}$ does not satisfy the another characteristic equation: $\omega^*_{(f)}\circ R_g = ad(g^{-1})\omega^*_{(f)}$, but, corresponding to (2.1), we obtain the following equation:

$$\omega^*_{(f)}\circ R_g = ad(g^{-1})\omega^*_{(gf)}\,. \tag{2.6}$$

To prove (2.6), we shall find first a relation between $\omega_{(f)}$ and a right translation R_g. We have

$$\begin{aligned}\omega_{(f)pg}\circ R_g &= \omega\circ i^{-1}\circ X_f\circ R_g = \omega\circ i^{-1}\circ R_g^*\circ X_{gf}\\ &= ad(g^{-1})\omega\circ i^{-1}\circ X_{gf}\,.\end{aligned}$$

Applying the similar process to $\theta^{(v)}_{(f)}$, we obtain

$$\omega_{(f)}\circ R_g = ad(g^{-1})\omega_{(gf)}, \quad \theta^{(v)}_{(f)}\circ R_g = g^{-1}\theta^{(v)}_{(gf)}\,. \tag{2.7}$$

Next, it is easily verified that the operation ρ_g of $g\in G$ on F and the identification $j_f: F\to F_f$ satisfy $\rho_g\circ j_f = j_{gf}\circ\rho_g$. From this relation it follows that

$$\omega_{(p)}\circ\rho_g = ad(g)\omega_{(pg)}\,. \tag{2.8}$$

According to (2.7) and (2.8), we have (2.6) immediately.

It will be interesting to know the local expression of the quasi-connection form $\omega^*_{(f)}$ in terms of canonical coordinate $(x^i, p_a{}^i)$. From expressions of forms ω and $\theta^{(v)}$, following equations are derived at once:

$$\begin{aligned}&\omega_{(f)}{}^a{}_b = p_i^{-1a}(dp_b{}^i + p_b{}^j\Gamma_j{}^i{}_k(pf)\,dx^k + p_b{}^jC_j{}^i{}_k(pf)f^c dp_c{}^k)\,,\\ &\omega_{(p)}{}^a{}_b = p_i^{-1a}p_b{}^jp_c{}^kC_j{}^i{}_k(pf)\,df^c\,,\\ &\theta^{(v)a}_{(f)} = p_i^{-1a}(f^b db_b{}^i + F_j{}^i(pf)\,dx^j)\,,\end{aligned} \tag{2.9}$$

from which it follows immediately that

$$\omega^*_{(f)}{}^a{}_b = p_i^{-1a}(dp_b{}^i + p_b{}^jF_j{}^i{}_k(pf)\,dx^k)\,. \tag{2.10}$$

If the canonical expression of $X\in P_p$ is $X^i(\partial/\partial x^i) + X_a{}^i(\partial/\partial p_a{}^i)$, the f-horizontal component h_fX of X is given by

$$h_f X = X^i \frac{\partial}{\partial x^i} - p_a{}^j F_j{}^i{}_k(pf) X^k \frac{\partial}{\partial p_a^i} . \tag{2.11}$$

In the case of an ordinary connection, the horizontal component hX of X is given by

$$hX = X^i \frac{\partial}{\partial x^i} - p_a{}^j \left\{ {i \atop j\ k} \right\}(x) X^k \frac{\partial}{\partial p_a^i} ,$$

where $\left\{ {i \atop j\ k} \right\}(x)$ are coefficients of connection and depend upon x^i alone. While $F_j{}^i{}_k$ are functions not only of x^i but also of b^i. For $F_j{}^i{}_k$ in (2.10) and (2.11), $p_a{}^i f^a$ are inserted for b^i, which means that $h_f X$ depends upon the choice of an element $f \in F$ really.

§ 3. Linear transformations of B

A Finsler connection is defined in the bundle space Q, and the base space of Q is the tangent vector bundle B of M. Further, elements of support, that is, elements of B play an essential rôle in the Finsler geometry, and are treated as independent of the base point. But, if our attention will be confined to a transformation induced from the one of M, the transformation of elements of support is determined by the one of the base point. In this point of view, it seems natural to investigate a behaviour of a Finsler connection under a transformation of Q induced from the one of B. However, we may not consider a general transformation of B without any restriction, and then we are led to define a linear transformation as follows.

A linear transformation φ of the tangent bundle space B is a transformation which satisfies the following two conditions:

1. φ is fibre-preserving, i. e. $\tau \circ \varphi(b) = \tau \circ \varphi(b')$ for any b, b' in $\tau^{-1}(x)$, $x \in M$.

2. φ is linear on each fibre, i. e. $\varphi(ub + u'b') = u\varphi(b) + u'\varphi(b')$ for any b, b' in $\tau^{-1}(x)$ and real numbers u, u'.

If a transformation ψ of the base manifold M is given, the transformation of B induced from ψ, that is, the differential of ψ is linear in the above sense clearly.

Let φ be a linear transformation of B. Since φ is fibre-preserving, the transformation $\underline{\varphi}$ of M is derived which satisfies the equation $\tau\circ\varphi=\underline{\varphi}\circ\tau$. We say that $\underline{\varphi}$ is the *projection* of φ. Then, from the projection $\underline{\varphi}$, the linear transformation φ_0 of B is induced. On the other hand, the linear transformation φ gives the transformation φ^* of P naturally, which is defined by

$$\varphi^*(p)=(\varphi(p_1),\cdots,\varphi(p_n)),\quad p=(p_1,\cdots,p_n).$$

It is easily observed that

1. φ^* commutes with every right translation R_g of P,

2. we interprete an element of P as an admissible mapping $F\to B$, and then

$$\varphi^*(p)f=\varphi(pf),\quad p\in P,\quad f\in F. \tag{3.1}$$

φ^* is called to *be associated with* φ. In like manner we obtain the transformation φ_0^* of P associated with the induced φ_0.

Next, a linear transformation of P is by definition a transformation which is fibre-preserving and commutes with every right translations R_g. The above φ^* and φ_0^* are linear in this sense obviously. If a linear transformation φ^* of P is given firstly, there exists uniquely a linear transformation φ of B such that φ^* is associated with φ. In fact, if φ is defined by

$$\varphi(b)=\varphi^*(p)p^{-1}b,\quad b\in B,\quad p\in\pi^{-1}\circ\tau(b), \tag{3.2}$$

it is shown that the above φ is linear and does not depend on the choice of $p\in\pi^{-1}\circ\tau(b)$. Comparing (3.1) and (3.2), it follows that the original φ^* is associated with φ as given by (3.2).

Thus we obtained as above the linear transformations φ^* and φ_0^* of P from the same φ. Since $\varphi^*(p)$ and $\varphi_0^*(p)$, $p\in P$, are on the same fibre of P, there exists the element $g\in G$ such that $\varphi^*(p)=\varphi_0^*(p)g$. Therefore we have the mapping $\lambda: P\to G$ which is such that

$$\varphi^*(p)=\varphi_0^*(p)\lambda(p). \tag{3.3}$$

The mapping λ is called the *deviation* of φ. From the commutability of φ^* and φ_0^* with R_g, it is known that λ is of adjoint type, i.e.

(3.4) $$\lambda(pg) = g^{-1}\lambda(p)g, \quad p \in P, \quad g \in G.$$

The following theorem makes clear the property of linear transformations.

Theorem 2. *Suppose that a transformation $\underline{\varphi}$ of the base manifold M is given. Then there is the one-to-one correspondence between the set of linear transformations of B having $\underline{\varphi}$ as the projection and the set of mappings λ of P into G satisfying* (3.4).

Proof. As have already been shown, to a linear transformation φ corresponds its deviation λ. Conversely, if a mapping $\lambda : P \to G$ is given such that (3.4) holds, then we take first the induced transformation φ_0^* of P from the given $\underline{\varphi}$, and then (3.3) gives the linear transformation φ^*, and (3.2) does the linear transformation φ of B, which has the given $\underline{\varphi}$ as its projection and λ as its deviation.

The above theorem shows that the notion of linear transformations is obtained by synthesis of notions of induced transformations and deviations.

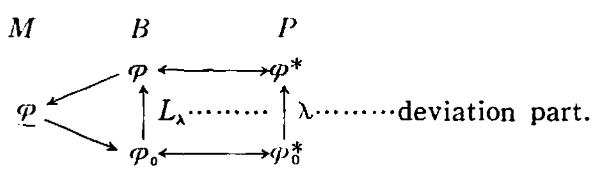
Consider linear transformations φ and φ_0 of B, the latter being the induced one from the projection of the former. If we take the mapping

$$L_\lambda : B \to B, \quad b \to p(\lambda(p)p^{-1}b), \quad p \in \pi^{-1}\circ\tau(b),$$

it is verified in virtue of (3.4) that $L_\lambda(b)$ does not depend upon the choice of $p \in \pi^{-1}\circ\tau(b)$. By means of (3.2) and (3.3) we have

(3.5) $$\varphi = \varphi_0 \circ L_\lambda .$$

The mapping L_λ may be said the *deviation part* of φ.



The element $\lambda(p) \in G$ is expressed by a matrix $(a_b{}^a)$, $a, b = 1, \cdots, n$. The equation (3.4) means that quantities $\lambda_j{}^i = p_a{}^i a_b{}^a p_j^{-1b}$ ($p = (x^i, p_a{}^i)$) are functions of x^i only. In terms of $\lambda_j{}^i$, deviation parts of φ^* and φ are expressible respectively by

$$(3.6)\qquad \begin{aligned} p\lambda(p) &= (x^i, \lambda_j{}^i(x)p_a{}^j), & p &= (x^i, p_a{}^i), \\ L_\lambda(b) &= (x^i, \lambda_j{}^i(x)b^j), & b &= (x^i, b^i). \end{aligned}$$

§4. Induced transformations of Q

We consider a linear transformation φ of B and the associated φ^* with φ. Then a transformation $\bar{\varphi}$ of the bundle space Q over B is naturally obtained as follows:

$$\bar{\varphi}: Q \to Q, \quad (b, p) \to (\varphi(b), \varphi^*(p)),$$

which will be called the *transformation induced from* φ. It is to be noted that the induced $\bar{\varphi}$ does not contain differentials of φ and φ^*.

Proposition 2. *The characteristic vector field* $\gamma: Q \to F$ *and fundamental vector fields on* Q *are invariant by the induced transformation* $\bar{\varphi}$.

Proof. We shall recall the definition of $\gamma: Q \to F$, $(b, p) \to p^{-1}b$. Then its invariance $\gamma \circ \bar{\varphi} = \gamma$ is expressed as

$$(4.1)\qquad \varphi^*(p)^{-1}(\varphi(b)) = p^{-1}b.$$

This follows from (3.2) immediately. Next, we take a fundamental vector field $\bar{F}(A)$ on Q which is defined from $A \in \dot{G}$ by $\bar{F}(A)_q = L_q(A)$, where $L_q(g) = \bar{R}_g(q)$, $g \in G$. Since $\bar{\varphi}$ commutes with $\bar{R}_g$, $\bar{\varphi} \circ L_q = L_{\bar{\varphi}(q)}$ is obvious. Hence we have

$$\bar{\varphi}\bar{F}(A)_q = \bar{\varphi} \circ L_q(A) = L_{\bar{\varphi}(q)}(A) = \bar{F}(A)_{\bar{\varphi}(q)}.$$

This completes the proof.

In an entirely similar way, we deduce that any fundamental vector field on P is as well invariant by φ^*.

The notion of an induced transformation of Q is transfered to $F \times P$ by the identification i used in Section 2. We see from (4.1)

$$\begin{aligned} i \circ \bar{\varphi} \circ i^{-1}(p^{-1}b, p) &= i \circ \bar{\varphi}(b, p) = i(\varphi(b), \varphi^*(p)) \\ &= (p^{-1}b, \varphi^*(p)). \end{aligned}$$

If we define a transformation $(1, \varphi^*)$ of $F\times P$ by $(1, \varphi^*)(f, p) = (f, \varphi^*(p))$, then it follows that $\bar{\varphi}=i^{-1}\circ(1, \varphi^*)\circ i$.

Let $\bar{\varphi}_0$ be the transformation of Q induced from the induced φ_0 of B and λ the deviation of φ. The

$$\begin{aligned}\bar{\varphi}(b, p) &= i^{-1}\circ(1, \varphi_0^*\circ\lambda)\circ i(b, p)\\ &= ((\varphi_0^*(p)\lambda(p))p^{-1}b, \varphi_0^*(p)\lambda(p))\\ &= (\varphi_0(p\lambda(p)p^{-1}b), \varphi_0^*(p\lambda(p)) = (\varphi_0\circ L_\lambda(b), \varphi_0^*(p\lambda(p))\,.\end{aligned}$$

If we define $\bar{L}_\lambda : Q\rightarrow Q$ by the equation $\bar{L}_\lambda(b, p)=(L_\lambda(b), p\lambda(p))$, then we obtain the expression $\bar{\varphi}=\bar{\varphi}_0\circ\bar{L}_\lambda$, which corresponds to (3.5). Thus $\bar{L}_\lambda$ may be said to be the *deviation part* of $\bar{\varphi}$.

We now turn to a consideration of behaviour of the h-basic form $\theta^{(h)}$ under the induced transformation $\bar{\varphi}$. Neglecting first the deviation part of $\bar{\varphi}$, we have the following lemma.

Lemma 2. *The h-basic form $\theta^{(h)}$ on Q is invariant by the transformation $\bar{\varphi}_0$ induced from an induced φ_0 of B.*

Proof. From the definition of $\theta^{(h)}$ it follows that

$$\theta^{(h)}_{\bar{\varphi}_0(q)}\circ\bar{\varphi}_0 = \varphi_0^*(p)^{-1}\circ\tau\circ\bar{\pi}\circ\bar{\varphi}_0 = \varphi_0^*(p)^{-1}\circ\underline{\varphi}\circ\tau\circ\bar{\pi}\ ,$$

where $q=(b, p)$, and $\underline{\varphi}$, $\bar{\pi}$, π mean differentials of these mappings written by the same symbols. The differential of $\underline{\varphi}$ is the induced transformation φ_0. Then the lemma will be derived at once from (4.1) replacing φ and φ^* by φ_0 and φ_0^* respectively.

It is noticed that the above lemma corresponds to the similar one for the case of the basic form on P and induced transformation from the one of the base manifold. However, in our case, we must take attention to deviations, and the result is as follows.

Proposition 3. *By the transformation $\bar{\varphi}$ induced from a linear transformation φ of B the h-basic form $\theta^{(h)}$ on Q is generally not invariant, but $\theta^{(h)}\circ\bar{\varphi}=\lambda^{-1}\theta^{(h)}$, where λ is the deviation.*

Proof. It was shown above that $\bar{\varphi}$ was expressed as $\bar{\varphi}=\bar{\varphi}_0\circ\bar{L}_\lambda$, and hence it is sufficient to prove that $\theta^{(h)}\circ\bar{L}_\lambda=\lambda^{-1}\theta^{(h)}$, because of Lemma 2.

$$\theta^{(h)}_{L_\lambda(q)} \circ \bar{L}_\lambda = \lambda(p)^{-1} p^{-1} \circ \tau \circ \bar{\pi} \circ \bar{L}_\lambda = \lambda(p)^{-1} p^{-1} \circ \tau \circ L_\lambda \circ \bar{\pi}$$
$$= \lambda(p)^{-1} p^{-1} \circ \tau \circ \bar{\pi}, \quad q = (b, p).$$

This completes the proof.

We note here that the h-basic form $\theta^{(h)}$ as above treated is defined without a Finsler connection, while the v-basic form $\theta^{(v)}$ is not so. Therefore we can not discuss the behaviour of $\theta^{(v)}$ under general induced transformation $\bar{\varphi}$.

§ 5. Affine transformations of Finsler connections

Suppose that a connection is defined in the frame bundle P. It is well known that the basic form is always invariant by induced transformations. Thus an affine transformation $\underline{\varphi}$ of M is defined such that the connection form is invariant by the induced transformation φ_0^*, which is equivalent to the fact that the transformation preserves horizontal subspaces.

In the case of a Finsler connection in Q, however, we have generalized a transformation to a linear transformation and proved Proposition 3. In this point of view, it is natural to define an affine transformation of B as follows: *an affine transformation of B is a linear transformation which preserves both of h- and v-horizontal subspaces.* We can state the definition of an affine transformation in terms of the associated ordinary connection and the non-linear connection in B as follows.

Proposition 4. *A necessary and sufficient condition for a linear transformation φ of B to be affine with respect to the given Finsler connection in Q is that $\varphi H = H$ and $\bar{\varphi}\Gamma = \Gamma$, where Γ is the associated ordinary connection and H the non-linear connection.*

Proposition 5. *The above condition is given by the equations*

(5.1) $$\omega \circ \bar{\varphi} = \omega, \quad \theta^{(v)} \circ \bar{\varphi} = \theta^{(v)},$$

where ω is the connection form and $\theta^{(v)}$ the v-basic form.

Proof. Suppose that φ is affine, and then $\omega \circ \bar{\varphi}(\Gamma) = 0$ and further $\omega \circ \bar{\varphi}(\bar{F}(A)) = A$ in virtue of Proposition 2. From the commutability of $\bar{\varphi}$ with right translations it follows that $\omega \circ \bar{\varphi} \circ \bar{R}_g$

$=ad(g^{-1})\omega\circ\bar{\varphi}$. Thus the form $\omega\circ\bar{\varphi}$ satisfies the characteristic properties of the connection form, and hence $\omega\circ\bar{\varphi}=\omega$. Next we see

$$\theta^{(v)}_{\bar{\varphi}(q)}\circ\bar{\varphi}=\varphi^*(p)^{-1}\circ\varphi\circ v'\circ\bar{\pi}=p^{-1}\circ v'\circ\bar{\pi}\ ,$$

and hence the second of (5.1) is obtained.

Conversely, suppose that (5.1) holds. It is easy to show that the first of (5.1) gives the invariance of horizontal subspace Γ of Q. Now, if $X\in H_b$, we have $\bar{\varphi}\circ l_q(X)\in\Gamma^h$ by means of (5.1), and so $\bar{\pi}\circ\bar{\varphi}\circ l_q(X)\in H_{\varphi(b)}$ from the definition of H. Making use of $\bar{\pi}\circ\bar{\varphi}=\varphi\circ\bar{\pi}$, we see that $\bar{\pi}\circ\bar{\varphi}\circ l_q(X)=\varphi(X)\in H_{\varphi(b)}$, and thus we obtain $\varphi H=H$.

Proposition 6. *The above condition is also given by*

$$\bar{\varphi}B^{(v)}(f)=B^{(v)}(f),\quad \bar{\varphi}B^{(h)}(f)=B^{(h)}(\lambda^{-1}f)\ , \tag{5.2}$$

where $B^{(v)}(f)$ and $B^{(h)}(f)$ are v- and h-basic vector fields respectively.

This proposition will be easily verified by means of (5.1) and Proposition 3.

§6. Rotations of B

Theorem 2 shows that, if we take the identity transformation of M, a mapping λ of P into G corresponds to a special linear transformation φ of B. For a Finsler connection, an element b of B is thought of as an element of support, and the base point of b is invariant by the above φ. Therefore we may say that the φ is a *rotation* of B. The associated φ^* of P and the induced $\bar{\varphi}$ of Q are called rotations as well. The discussions of this and next sections will be confined to rotations.

From (3.4) it follows that for a rotation φ^*

$$\lambda\circ\varphi^*=\lambda\ . \tag{6.1}$$

Define a $\hat{G}$-valued 1-form Λ on P by $\Lambda_p=\lambda(p)^{-1}\lambda$, where λ is the differential, that is, for $X\in P_p$ we have a tangent vector $\lambda(X)$ to G at $\lambda(p)$. The form Λ will be denoted by the λ-*form* of the rotation.

Proposition 7. *The λ-form Λ of a rotation φ has the following two properties:*

1. *Λ is of adjoint type.*
2. *Λ is invariant by the associated rotation φ^*.*

Proof. From (3.4) it follows that

$$\Lambda_{pg}\circ R_g = \lambda(pg)^{-1}\lambda\circ R_g = (g^{-1}\lambda(p)^{-1}g)(g^{-1}\lambda g) = g^{-1}\Lambda_p g ,$$

which is the first property. Next, in virtue of (6.1) we obtain

$$\Lambda_{\varphi^*(p)}\varphi^* = \lambda(\varphi^*(p))^{-1}\lambda\circ\varphi^* = \lambda(p)^{-1}\lambda = \Lambda_p .$$

Thus the second is proved.

Proposition 8. *If a connection form ω on P is given, we get*

$$\Lambda = \omega\varphi^* - ad(\lambda^{-1})\omega , \tag{6.2}$$

where φ^* is the associated rotation corresponding to λ.

Proof. Since $\varphi^*(p) = p\lambda(p)$, we have

$$\varphi^*(X) = X\lambda(p) + p\lambda(X), \quad X\in P_p . \tag{6.3}$$

Let ψ_g be the left translation of G by $g\in G$, and then

$$L_{\varphi^*(p)} = L_p\circ\psi_{\lambda(p)} , \tag{6.4}$$

where L_p is the mapping $G\to P$ as has already been used in the proof of Proposition 2. From (6.3) and (6.4) it follows that

$$\begin{aligned}\omega_{\varphi^*(p)}\varphi^*(X) &= \omega_{\varphi^*(p)}(R_{\lambda(p)}X) + \omega_{\varphi^*(p)}(L_p\lambda(X))\\ &= ad(\lambda(p)^{-1})\omega_p(X) + \omega_{\varphi^*(p)}(L_{\varphi^*(p)}\psi_{\lambda(p)^{-1}}\lambda(X))\\ &= ad(\lambda(p)^{-1})\omega_p(X) + \lambda(p)^{-1}\lambda(X) .\end{aligned}$$

The last term is the $\Lambda_p(X)$ and hence (6.2) is established.

The form Λ is originally defined by the deviation λ and hence the value of Λ for a fundamental vector field $F(A)$ does not depend upon connections. But the equation (6.2) gives the value directly as follows.

Corollary 1. *Let $F(A)$ be a fundamental vector field on P corresponding to a $A\in\hat{G}$, and then*

(6.5) $$\Lambda(F(A)) = A - ad(\lambda^{-1})A\,.$$

This follows from (6.2) and the observation that $F(A)$ is invariant by φ^*.

Now, if a connection is given in P, the following problem arises: whether an associated rotation φ^* remains invariant the connection or not. The following theorem is the clear answer of this problem.

Theorem 3. *The necessary and sufficient condition for a connection form ω on P to be invariant by an associated rotation φ^* is that the λ-form Λ of the rotation is equal to zero for any horizontal vector.*

Proof. If $\omega\varphi^* = \omega$, then we have $\Lambda = \omega - ad(\lambda^{-1})\omega$ in virute of (6.2), and hence $\Lambda(X) = 0$ for any horizontal vector X. Conversely, if $\Lambda(X) = 0$ for $X \in \Gamma_p$ (horizontal subspace), we have $\omega\varphi^*(X) = 0$ from (6.2). It is easily verified that the form $\omega\varphi^*$ statisfies the characteristic properties of a connection form, and hence $\omega\varphi^* = \omega$.

The contents of Theorem 3 can be expressed in terms of a certain tensor. The λ-*tensor* T is by definition the $F \otimes F^*$-valued tensor which is given by $T(p)(f) = \lambda(p)f$, $p \in P$, $f \in F$. The tensor T is of adjoint type as shown from (3.4), and $dT(X)$ is given by $dT(X)f = d\lambda(X)f$, $X \in P_p$, $f \in F$.

Corollary 2. *The condition of Theorem* 3 *is equivalent to* $DT = 0$, *where D is the covariant differential operator with respect to the connection under consideration.*

Proof. Since T is of adjoint type, it is well known that

$$DT(X) = dT(X) + \omega(X)T - T\omega(X)\,, \quad X \in P_p\,,$$

which is rewritten

$$= d\lambda(X) + \omega(X)\lambda(p) - \lambda(p)\omega(X)$$
$$= \lambda(p)\Lambda(X) + \omega(X)\lambda(p) - \lambda(p)\omega(X)\,,$$

and, in virtue of (6.2), we obtain

(6.6) $$DT = \lambda\omega\varphi^* - \lambda\omega\,.$$

It follows that, if $\omega\varphi^*=\omega$, then $DT=0$. Conversely, if $DT=0$, we have $\omega\varphi^*(X)=0$ for $X\in\Gamma_p$ and $\Lambda(X)=0$ from (6.2).

We conclude this section by a remark that $DT=0$ is expressed by a very simple form. In fact, it has already been noted in Section 3 that $\lambda(p)^a{}_b=p_i^{-1a}\lambda_j{}^i p_b{}^j$, $p=(x^i, p_a{}^i)$, where $\lambda_j{}^i$ are functions of x^i. $DT=0$ is expressed by $\lambda_j{}^i{}_{;k}=0$, where semicolon denotes the covariant differentiation.

§ 7. Affine rotations

A linear transformation φ of B is called an *affine rotation* if φ is a rotation and an affine transformation with respect to the Finsler connection in Q. We shall find a necessary and sufficient condition for a rotation φ to be affine. In Section 2, we have deduced a notion of a quasi-connection in P with respect to a fixed $f\in F$, which will be used throughout this section. And we have introduced the forms $\omega_{(f)}$ and $\omega_{(p)}$ from the connection form ω. In like manner, if a form α on Q is given, we obtain the form $\alpha_{(p)}$ on F defined by $\alpha_{(p)}=\alpha\circ i^{-1}\circ\chi_p$, which will be called the *p-induced form* from α. Similarly we shall obtain the *f-induced form* $\alpha_{(f)}$ on P.

Lemma 3. *The necessary and sufficient coondition for a form α on Q to be invariant by the induced rotation from a rotation φ is that the p-induced form $\alpha_{(p)}$ on F and the f-induced form $\alpha_{(f)}$ on P satisfy the following equations:*

$$\alpha_{(f)}\varphi^*=\alpha_{(f)}, \quad \alpha_{(\varphi^*(p))}=\alpha_{(p)},$$

where φ^* is the associated rotation of P.

Proof. If we take $\alpha^*=\alpha\circ i^{-1}$ which is a form on $F\times P$, then it is obvious from $\bar{\varphi}=i^{-1}\circ(1,\varphi^*)\circ i$ that the invariance of α by $\bar{\varphi}$ is equivalent to that of α^* by $(1,\varphi^*)$. Thus we see that

$$\alpha_{(f)}\varphi^*=\alpha^*\chi_f\circ\varphi^*=\alpha^*(1,\varphi^*)\chi_f=\alpha^*\chi_f=\alpha_{(f)},$$
$$\alpha_{(\varphi^*(p))}=\alpha^*\chi_{\varphi^*(p)}=\alpha^*(1,\varphi^*)\chi_p=\alpha^*\chi_p=\alpha_{(p)},$$

and hence the necessary is obtained. Conversely

$$\alpha^*_{(f,\varphi^*(p))}(1,\varphi^*)=(\alpha_{(\varphi^*(p))}\gamma^*+\alpha_{(f)}\pi_2)(1,\varphi^*)$$
$$=\alpha_{(p)}\gamma^*+\alpha_{(f)}\varphi^*\pi_2=\alpha_{(p)}\gamma^*+\alpha_{(f)}\pi_2=\alpha^*_{(f,p)}.$$

This shows the sufficiency.

According to Lemma 3, the necessary and sufficient condition for a rotation φ to be affine is given by

$$\text{(7.1)}\qquad \begin{array}{ll}(1)\ \omega_{(f)}\varphi^*=\omega_{(f)}, & (3)\ \theta^{(v)}_{(f)}\varphi^*=\theta^{(v)}_{(f)},\\ (2)\ \omega_{(\varphi^*(p))}=\omega_{(p)}, & (4)\ \theta^{(v)}_{(\varphi^*(p))}=\theta^{(v)}_{(p)}.\end{array}$$

In the following we shall discuss these equations in detail and change them to the forms given in the following theorem 4.

It is first observed that the fourth of (7.1) is automatically satisfied by virtue of (2.3), and may be removed from the condition. Next, the process used in Section 6 to obtain (6.2) is applicable to any form on P, and then we deduce

$$\text{(7.2)}\qquad \alpha_{(f)}\varphi^*=\alpha_{(f)}R_\lambda+\alpha_{(f)}(F(\Lambda)).$$

Applying (7.2) to $\omega_{(f)}$ and making use of (2.2) and (2.7), we obtain

$$\omega_{(f)}\varphi^*=\omega_{(f)}R_\lambda+\omega_{(f)}(F(\Lambda))$$
$$=ad(\lambda(p)^{-1})\omega_{(\lambda f)}+\Lambda+\omega_{(p)}(\sigma_f\Lambda).$$

Therefore (1) of (7.1) is rewritten in the following form:

$$\text{(7.3)}\qquad \omega_{(f)}=ad(\lambda^{-1})\omega_{(\lambda f)}+\Lambda+\omega_{(p)}(\sigma_f\Lambda).$$

In like manner, (3) of (7.1) is changed into

$$\text{(7.4)}\qquad \theta^{(v)}_{(f)}=\lambda^{-1}\theta^{(v)}_{(\lambda f)}+\Lambda f.$$

Next, we shall show that (7.3) is further rewritten in a simpler form in terms of the quasi-connection form $\omega^*_{(f)}$ as defined in Section 2. In fact, we have from (7.4) that $\sigma_f\Lambda=j_f\theta^{(v)}_{(f)}-j_f\lambda^{-1}\theta^{(v)}_{(\lambda f)}$, where $j_f\colon F\to F_f$ is the identification. Substitution of this into (7.3) gives

$$\omega_{(f)p}=ad(\lambda^{-1})\omega_{(\lambda f)p}+\Lambda+\omega_{(p)f}(j_f\theta^{(v)}_{(f)})$$
$$-\omega_{(p)f}(j_f\lambda^{-1}\theta^{(v)}_{(\lambda f)}),$$

the last term of which is

$$-\omega_{(p)f}(\rho_{\lambda^{-1}}j_{\lambda f}\theta^{(v)}_{(\lambda f)})=-ad(\lambda^{-1})\omega_{(p\lambda^{-1})\lambda f}(j_{\lambda f}\theta^{(v)}_{(\lambda f)}),$$

in consequence of (2. 8). Making use of (2) of (7. 1), this term is moreover changed to $-ad(\lambda^{-1})\omega_{(p)\lambda f}(j_{\lambda f}\theta^{(v)}_{(\lambda f)})$. Therefore (7. 3) is written in the following form:

$$\omega_{(f)p}-\omega_{(p)f}(j_f\theta^{(v)}_{(f)}) = \Lambda + ad(\lambda^{-1})(\omega_{(\lambda f)p} \\ -\omega_{(p)\lambda f}(j_{\lambda f}\theta^{(v)}_{(\lambda f)}))\,.$$

We shall recall the definition of the quasi-connection form $\omega^*_{(f)}$, and the above equation is written

$$\omega^*_{(f)} = \Lambda + ad(\lambda^{-1})\omega^*_{(\lambda f)}\,. \tag{7.5}$$

Consequently, at the present stage, the condition (7. 1) is equivalent to (2) of (7. 1), (7. 4) and (7. 5).

We now observe that values of both sides of (7. 4) for a fundamental vector $F(A)$ are equal to Af in consequence of (2. 4) and (6. 5). Hence (7. 4) may be exchanged for

$$\theta^{(v)}_{(f)}h_f = \lambda^{-1}\theta^{(v)}_{(\lambda f)}h_f + \Lambda(h_f)f\,. \tag{7.6}$$

By virtue of Theorem 1, the similar result is seen with respect to (7. 5), and further $\omega^*_{(f)}h_f=0$, and hence (7. 5) may be exchanged for the following:

$$\Lambda h_f + ad(\lambda^{-1})\omega^*_{(\lambda f)}h_f = 0\,. \tag{7.7}$$

Finally we shall show that (7. 6) is written in the simpler form

$$\theta^{(v)}_{(f)}h_f = \lambda^{-1}\theta^{(v)}_{(\lambda f)}h_{\lambda f}\,. \tag{7.8}$$

In fact, if we take $X\in P_p$, then $h_{\lambda f}(X)-h_f(X)$ is clearly vertical and hence there exists $A\in\hat{G}$ such that $h_{\lambda f}(X)-h_f(X)=F(A)_p$. According to Theorem 1, we have

$$\omega^*_{(\lambda f)}(h_{\lambda f}(X)-h_f(X)) = -\omega^*_{(\lambda f)}h_f(X) = A\,.$$

We see from (2. 4) that $\theta^{(v)}_{(\lambda f)}(F(A)_p)=A\lambda f$. Therefore

$$\theta^{(v)}_{(\lambda f)}(h_{\lambda f}(X)-h_f(X)) = -\omega^*_{(\lambda f)}(h_f(X))\lambda f\,.$$

Substituting (7. 7) in the right hand side of the above, we see that (7. 6) is written in the form (7. 8).

Consequently we establish

Theorem 4. *The necessary and sufficient condition for a rotation φ to be affine is that the deviation λ of φ satisfies the following three equations*:

$$\Lambda h_f + ad(\lambda^{-1})\omega^*_{(\lambda f)}h_f = 0\,, \tag{1}$$

$$\lambda\theta^{(v)}_{(f)}h_f = \theta^{(v)}_{(\lambda f)}h_{\lambda f}\,, \tag{2}$$

$$\omega_{(p\lambda)} = \omega_{(p)}\,, \tag{3}$$

for every $f \in F$ and $p \in P$.

At the end of this section, we shall express the above equations in terms of a canonical coordinate. In the final paragraph of Section 2, some canonical expressions of forms and vectors have been obtained. The equation (3) is immediately expressed in the form

$$C_j{}^i{}_k(\varphi(b)) = \lambda_h{}^i\lambda^{-1}{}_j{}^l\lambda^{-1}{}_k{}^m C_l{}^h{}_m(b)\,. \tag{3'}$$

Next, the following is easily derived.

$$(\theta^{(v)}_{(f)}h_f)^a = p_i^{-1a}D^i{}_k(pf)\,dx^k\,,\quad D^i{}_k(b) = F^i{}_k(b) - b^jF_j{}^i{}_k(b)\,.$$

It is to be noted that *Condition F* due to T. Okada is expressed by $D_k{}^i = 0$. The equation (2) is then written

$$D^i{}_k(\varphi(b)) = \lambda_h{}^iD^h{}_k(b)\,. \tag{2'}$$

We consider (1) finally. By means of the definition of the λ-form Λ, we have

$$\Lambda^a{}_b = p_i^{-1a}(dp_b{}^i - \lambda^{-1}{}_h{}^i\lambda_k{}^jp_j^{-1c}p_b{}^kdp_c{}^h + p_b{}^j\lambda_k^{-1i}d\lambda_j{}^k)\,,$$
$$(\Lambda h_f)^a{}_b = p_i^{-1a}p_b{}^j\lambda_h^{-1i}\lambda_j{}^h{}_{|k}dx^k\,,$$

where the symbol $(|)$ denotes h-covariant differentiation. Besides, we have from (2.10)

$$(ad(\lambda^{-1})\omega^*_{(\lambda f)}h_f)^a{}_b = p_i^{-1a}p_b{}^j\lambda_h^{-1i}\lambda_j{}^l(F_l{}^h{}_k(p\lambda f) - F_l{}^h{}_k(pf))\,dx^k\,,$$

and hence the equation (1) is expressible in the form

$$\lambda_j{}^i{}_{|k} + (F_l{}^i{}_k(\varphi(b)) - F_l{}^i{}_k(b))\lambda_j{}^l = 0\,. \tag{1'}$$

As has already been stated in Theorem 3, the condition for a rotation φ to be affine with respect to an ordinary connection in P is expressed by $\Lambda h=0$, namely $\lambda_j{}^i{}_{;k}=0$. In this case, coefficients $\left\{ {i \atop j\,k} \right\}$ of the connection depend upon x^i only, and hence the second term of (1'), $\left\{ {i \atop l\,k} \right\}(\varphi(b))-\left\{ {i \atop l\,k} \right\}(b)$, vanishes. Thus the equation (1) is looked upon as the direct generalization of $\Lambda h=0$ in the case of a connection in P.

§8. Infinitesimal affine transformations

We consider a 1-parameter quasi-group of linear transformation φ_t. Let λ_t be the deviation of φ_t, $A_\lambda \in \hat{G}$ the tangent vector at $e\in G$ of the curve $\lambda_t(p)$, $p\in P$, and X the infinitesimal transformation of φ_t. Then we have from Proposition 3

$$\mathfrak{L}_X \theta^{(h)} = -A_\lambda \theta^{(h)}, \tag{8.1}$$

where $\mathfrak{L}_X$ denotes the operator of Lie derivative with respact to X.

Now, in consequance of Proposition 6, we know that the necessary and sufficient condition for φ_t to be a 1-parameter quasi-group of affine transformation is that

$$\mathfrak{L}_X B^{(v)}(f)=0, \quad \mathfrak{L}_X B^{(h)}(f)=B^{(h)}(A_\lambda f). \tag{8.2}$$

We first treat the first of (8.2), which is equivalent to

$$\omega(\mathfrak{L}_X B^{(v)}(f))=0, \quad \theta^{(v)}(\mathfrak{L}_X B^{(v)}(f))=0, \quad \theta^{(h)}(\mathfrak{L}_X B^{(v)}(f))=0. \tag{8.3}$$

According to the definition of Lie derivative, we obtain from a 1-form α on Q

$$\begin{aligned}\mathfrak{L}_X\alpha(Y) &= d\alpha(X, Y)+Y(\alpha(X)), \qquad Y\in Q_q,\\ &= X(\alpha(Y))-Y(\alpha(X))-\alpha([X, Y])+Y(\alpha(X)).\end{aligned}$$

Therefore we have

$$\mathfrak{L}_X\alpha(Y)=X(\alpha(Y))-\alpha(\mathfrak{L}_X Y). \tag{8.4}$$

We apply (8.4) to ω and have

$$\begin{aligned}\omega(\mathfrak{L}_X B^{(v)}(f)) &= X(\omega(B^{(v)}(f))-\mathfrak{L}_X\omega(B^{(v)}(f))\\ &= -d\omega(X, B^{(v)}(f))-B^{(v)}(f)(\omega(X)).\end{aligned}$$

Referring to the structure equation, we obtain

$$= -\Omega(X, B^{(v)}(f)) - B^{(v)}(f)(\omega(X)).$$

Thus the first of (8.3) is written

$$B^{(v)}(f)\omega(X) = \Omega(B^{(v)}(f), X). \tag{8.5}$$

In like manner the second of (8.3) is written

$$B^{(v)}(f)(\theta^{(v)}(X)) = \Theta^{(v)}(B^{(v)}(f), X) + \omega(X)f, \tag{8.6}$$

where we made use of the additional structure equation. On the other hand, the third of (8.3) is automatically satisfied. In fact, applying (8.4) to $\alpha = \theta^{(h)}$, $Y = B^{(v)}(f)$, we obtain

$$\theta^{(h)}(\mathfrak{L}_X B^{(v)}(f)) = -\mathfrak{L}_X \theta^{(h)}(B^{(v)}(f)).$$

The right-hand side of the above vanishes, because of (8.1).

Next, the second of (8.2) is written

$$\omega(\mathfrak{L}_X B^{(h)}(f)) = 0, \quad \theta^{(v)}(\mathfrak{L}_X B^{(h)}(f)) = 0,$$
$$\theta^{(h)}(\mathfrak{L}_X B^{(h)}(f)) = A_\lambda f.$$

In this case also, making use of (8.4), we shall deduce from above equations the following two equations:

$$B^{(h)}(f)(\omega(X)) = \Omega(B^{(h)}(f), X), \tag{8.7}$$

$$B^{(h)}(f)(\theta^{(v)}(X)) = \Theta^{(v)}(B^{(h)}(f), X). \tag{8.8}$$

Therefore we have deduced the following proposition.

Proposition 9. *The necessary and sufficient condition for φ_t to be a 1-parameter quasi-group of affine transformations is that the infinitesimal transformation X of φ_t satisfies the system of differential equations* (8.5)-(8.8).

We shall express left hand sides of these equations in terms of covariant differentiations Δ^v and Δ^h. From the definition of Δ^v we see

$$\Delta^v\omega(X)(f) = (d(\omega(X))B_a^{(v)} \otimes e^a)(f^b e_b)$$
$$= (B_a^{(v)}(\omega(X)) \otimes e^a)(f^b a_b) = B^{(v)}(f)(\omega(X)).$$

In the similar manner we obtain $\Delta^h\omega(X)(f) = B^{(h)}(f)(\omega(X))$, and

further the similar equations with respect to $\theta^{(v)}(X)$.
Therefore the system (8.5)-(8.8) is written in the forms

$$\begin{aligned}\Delta^{v}\omega(X)(f) &= \Omega(B^{(v)}(f), X),\\ \Delta^{v}\theta^{(v)}(X)(f) &= \Theta^{(v)}(B^{(v)}(f), X)+\omega(X)f,\\ \Delta^{h}\omega(X)(f) &= \Omega(B^{(h)}(f), X),\\ \Delta^{h}\theta^{(v)}(X)(f) &= \Theta^{(v)}(B^{(h)}(f), X).\end{aligned}$$

Further we shall find explicit forms of the system in regard to the expression

$$X = X^{a}{}_{b}F^{b}{}_{a} - X^{(a)}B^{(v)}_{a} + X^{a}B^{(h)}_{a}.$$

It is easily seen that the system is written in the following:

$$(8.9)\qquad X^{a}{}_{b}|_{c} = -S_{b\cdot cd}^{\ a}X^{(d)} + P_{b\cdot dc}^{\ a}X^{d},$$

$$(8.10)\qquad X^{(a)}|_{b} = -S_{b\ c}^{\ a}X^{(c)} + P_{c\ b}^{\ a}X^{c} + X^{a}{}_{b},$$

$$(8.11)\qquad X^{a}{}_{b|c} = -P_{b\cdot cd}^{\ a}X^{(d)} - R_{b\cdot cd}^{\ a}X^{d},$$

$$(8.12)\qquad X^{(a)}{}_{|b} = -P_{b\ c}^{\ a}X^{(c)} - R_{b\ c}^{\ a}X^{c}.$$

We observe that covariant derivatives of X^{a}, h-horizontal components of X, do not appear in the system (8.9)-(8.12). However, for the latter use, we shall find expessions of $X^{a}|_{b}$ and $X^{a}{}_{|b}$ in terms of $X^{a}{}_{b}$, $X^{(a)}$ and X^{a}. It follows from (8.1) that

$$\begin{aligned}\mathfrak{L}_{X}\theta^{(h)}(B^{(v)}(f)) &= d\theta^{(h)}(X, B^{(v)}(f)) + B^{(v)}(f)(\theta^{(h)}(X))\\ &= \Theta^{(h)}(X, B^{(v)}(f)) + B^{(v)}(f)(\theta^{(h)}(X))\\ &= -A_{\lambda}\theta^{(h)}(B^{(v)}(f)) = 0,\\ \mathfrak{L}_{X}\theta^{(h)}(B^{(h)}(f)) &= \Theta^{(h)}(X, B^{(h)}(f)) - \omega(X)f\\ &\quad + B^{(h)}(f)(\theta^{(h)}(X)) = -A_{\lambda}f.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}B^{(v)}(f)(\theta^{(h)}(X)) &= \Theta^{(h)}(B^{(v)}(f), X),\\ B^{(h)}(f)(\theta^{(h)}(X)) &= \Theta^{(h)}(B^{(h)}(f), X) + \omega(X)f - A_{\lambda}f,\end{aligned}$$

which are written in the following explicit forms:

$$(8.13)\qquad \begin{aligned}X^{a}|_{b} &= C_{c\ b}^{\ a}X^{c},\\ X^{a}{}_{|b} &= -C_{b\ c}^{\ a}X^{(c)} + T_{c\ b}^{\ a}X^{c} + X^{a}{}_{b} - \eta^{a}{}_{b},\end{aligned}$$

in which $\eta^a{}_b$ are components of the tangent vector A_λ to G at e.

We now shall find the condition of integrability of the system (8.5)-(8.8). These conditions will be obtained from the explicit form (8.9)-(8.12) by direct calculation, making use of conditions of integrability of covariant differentiations and substituting (8.13). However, we shall proceed in the following by the other way, for the purpose of avoiding tensor calculus.

It we put $S_1{}^1{}_2 = S_b{}^a{}_c f_1{}^b f_2{}^c e_a \in F$ and $S_1{}^2{}_2 = S_b{}^a{}_{cd} f_1{}^c f_2{}^d \hat{g}_a{}^b \in \dot{G}$, it follows from (1.1) and (1.4) that

$$\begin{aligned}(8.14)\qquad &\Omega(B^{(v)}(f_1), B^{(v)}(f_2)) = -S_1{}^2{}_2\,,\\ &[B^{(v)}(f_1), B^{(v)}(f_2)] = \bar{F}(S_1{}^2{}_2) + B^{(v)}(S_1{}^1{}_2)\,.\end{aligned}$$

It follows from (8.2) that

$$\begin{aligned}0 = [X, B^{(v)}(f)] &= [vX, B^{(v)}(f)] + [hX, B^{(v)}(f)]\\ &= [\bar{F}(\omega(X)), B^{(v)}(f)] + [hX, B^{(v)}(f)]\,,\end{aligned}$$

and thus we obtain in virtue of (1.4)

$$(8.15)\qquad [hX, B^{(v)}(f)] = -B^{(v)}(\omega(X)f)\,.$$

According to the Bianchi identity $D\Omega = 0$, it is derived that

$$\begin{aligned}0 &= d\Omega(B^{(v)}(f_1), B^{(v)}(f_2), hX)\\ &= B^{(v)}(f_1)\Omega(B^{(v)}(f_2), hX) - B^{(v)}(f_2)\Omega(B^{(v)}(f_1), hX)\\ &\quad + hX\Omega(B^{(v)}(f_1), B^{(v)}(f_2)) - \Omega([B^{(v)}(f_1), B^{(v)}(f_2)], hX)\\ &\quad + \Omega([B^{(v)}(f_1), hX], B^{(v)}(f_2)) - \Omega([B^{(v)}(f_2), hX], B^{(v)}(f_1))\,.\end{aligned}$$

In consequence of (8.14) and (8.15), the above equation gives

$$\begin{aligned}(8.16)\qquad &B^{(v)}(f_1)\Omega(B^{(v)}(f_2), hX) - B^{(v)}(f_2)\Omega(B^{(v)}(f_1), hX)\\ &= hX(S_1{}^2{}_2) + \Omega(B^{(v)}(S_1{}^1{}_2), hX) - \Omega(B^{(v)}(\omega(X)f_1), B^{(v)}(f_2))\\ &\quad + \Omega(B^{(v)}(\omega(X)f_2), B^{(v)}(f_1))\,.\end{aligned}$$

Next, we see that

$$\begin{aligned}d\omega(X, \bar{F}(A)) &= X\omega(\bar{F}(A)) - \bar{F}(A)\omega(X) - \omega([X, \bar{F}(A)])\\ &= -\bar{F}(A)\omega(X)\,,\end{aligned}$$

where we used $[X, \bar{F}(A)] = \mathfrak{L}_X \bar{F}(A) = 0$ from Proposition 2. On the other hand we have from the structure equation

$$= \Omega(X, \bar{F}(A)) - \omega(X)\omega(\bar{F}(A)) + \omega(\bar{F}(A))\omega(X)$$
$$= -\omega(X)A + A\omega(X) .$$

Therefore we obtain

$$\bar{F}(A)\omega(X) = \omega(X)A - A\omega(X) . \tag{8.17}$$

Now we consider first the value $[B^{(v)}(f_1), B^{(v)}(f_2)]\omega(X)$. From the definition of the bracket it follows that

$$[B^{(v)}(f_1), B^{(v)}(f_2)]\omega(X)$$
$$= B^{(v)}(f_1)(B^{(v)}(f_2)\omega(X)) - B^{(v)}(f_2)(B^{(v)}(f_1)\omega(X))$$

and, by means of the differential equation (8.5), we get

$$= B^{(v)}(f_1)\Omega(B^{(v)}(f_2), X) - B^{(v)}(f_2)\Omega(B^{(v)}(f_1), X) ,$$

which is the left-hand side of (8.16). Besides, it follows from (8.14) that

$$= \bar{F}(S_1{}^2{}_2)\omega(X) + B^{(v)}(S_1{}^1{}_2)\omega(X) ,$$

which is written by virtue of (8.5) and (8.17)

$$= \omega(X)S_1{}^2{}_2 - S_1{}^2{}_2\omega(X) + \Omega(B^{(v)}(S_1{}^1{}_2), hX) .$$

Consequently we establish

$$hX(S_1{}^2{}_2) + \Omega^v(B^{(v)}(\omega(X)f_2), B^{(v)}(f_1))$$
$$- \Omega^v(B^{(v)}(\omega(X)f_1), B^{(v)}(f_2)) - \omega(X)S_1{}^2{}_2 + S_1{}^2{}_2\omega(X) = 0 . \tag{8.18}$$

In like manner we shall consider values of brackets $[B^{(v)}(f_1), B^{(h)}(f_2)]\omega(X)$ and $[B^{(h)}(f_1), B^{(h)}(f_2)]\omega(X)$. Furthermore, values of those three kinds of brackets for $\theta^{(v)}(X)$ will be examined by applying the similar considerations to those applied to $\omega(X)$, in which, instead of (8.15) and (8.17), we shall use $[hX, B^{(h)}(f)] = -B^{(h)}((\omega(X) - A_\lambda)f)$ and $\bar{F}(A)\theta^{(v)}(X) = -A\theta^{(v)}(X)$ as will be easily obtained, and further the Bianchi identity $D\Theta^{(v)} = \Omega \wedge \theta^{(v)}$ will play a important rôle. Thus we obtain the following five equations.

$$hX(R_1{}^2{}_2) + \Omega^h(B^{(h)}((\omega(X) - A_\lambda)f_2), B^{(h)}(f_1))$$
$$- \Omega^h(B^{(h)}((\omega(X) - A_\lambda)f_1), B^{(h)}(f_2))$$
$$- \omega(X)R_1{}^2{}_2 + R_1{}^2{}_2\omega(X) = 0 , \tag{8.19}$$

(8.20) $$hX(P_2{}^2{}_1)+2\Omega^{vh}(B^{(v)}(\omega(X)f_1)),\ B^{(h)}(f_2)) \\ -2\Omega^{vh}(B^{(h)}((\omega(X)-A_\lambda)f_2),\ B^{(v)}(f_1)) \\ -\omega(X)P_2{}^2{}_1+P_2{}^2{}_1\omega(X)=0\,,$$

(8.21) $$hX(S_1{}^1{}_2)+\Theta^{(v)v}(B^{(v)}(\omega(X)f_2),\ B^{(v)}(f_1)) \\ -\Theta^{(v)v}(B^{(v)}(\omega(X)f_1),\ B^{(v)}(f_2))-\omega(X)S_1{}^1{}_2=0\,,$$

(8.22) $$hX(R_1{}^1{}_2)+\Theta^{(v)h}(B^{(h)}((\omega(X)-A_\lambda)f_2),\ B^{(h)}(f_1)) \\ -\Theta^{(v)h}(B^{(h)}((\omega(X)-A_\lambda)f_1),\ B^{(h)}(f_2))-\omega(X)R_1{}^1{}_2=0\,,$$

(8.23) $$hX(P_2{}^1{}_1)+2\Theta^{(v)vh}(B^{(v)}(\omega(X)f_1),\ B^{(h)}(f_2)) \\ -2\Theta^{(v)vh}(B^{(h)}((\omega(X)-A_\lambda)f_2),\ B^{(v)}(f_1))-\omega(X)P_2{}^1{}_1=0\,.$$

In these equations quantities $R_1{}^1{}_2$ and $R_1{}^2{}_2$ are defined in a similar way as for $S_1{}^1{}_2$ and $S_1{}^2{}_2$, and further $P_2{}^1{}_1=P_c{}^a{}_b f_2{}^c f_1{}^b e_a$, $P_2{}^2{}_1=P_b{}^a{}_{\cdot dc} f_2{}^d f_1{}^c \hat{g}_a{}^b$.

The system of six equations (8.18)-(8.23) is the condition of integrability of the system (8.5)-(8.8). For the latter use, we shall write down the system (8.18)-(8.23) in the following explicit forms.

(8.18′) $$S_b{}^a{}_{\cdot cd}|_e X^{(e)}+S_b{}^a{}_{\cdot cd|e}X^e-S_b{}^e{}_{\cdot cd}X_e{}^a+S_e{}^a{}_{\cdot cd}X_b{}^e \\ +S_b{}^a{}_{\cdot ed}X_c{}^e-S_b{}^a{}_{\cdot ec}X_d{}^e=0\,,$$

(8.19′) $$R_b{}^a{}_{\cdot cd}|_e X^{(e)}+R_b{}^a{}_{\cdot cd|e}X^e-R_b{}^e{}_{\cdot cd}X_e{}^a+R_e{}^a{}_{\cdot cd}X_b{}^e \\ +R_b{}^a{}_{\cdot ed}(X_c{}^e-\eta_c{}^e)-R_b{}^a{}_{\cdot ec}(X_d{}^e-\eta_d{}^e)=0\,,$$

(8.20′) $$P_b{}^a{}_{\cdot cd}|_e X^{(e)}+P_b{}^a{}_{\cdot cd|e}X^e-P_b{}^e{}_{\cdot cd}X_e{}^a+P_e{}^a{}_{\cdot cd}X_b{}^e \\ +P_b{}^a{}_{\cdot ed}(X_c{}^e-\eta_c{}^e)+P_b{}^a{}_{\cdot ce}X_d{}^e=0\,,$$

(8.21′) $$S_b{}^a{}_c|_d X^{(d)}+S_b{}^a{}_{c|d}X^d-S_b{}^d{}_c X_d{}^a+S_d{}^a{}_c X_b{}^d-S_d{}^a{}_b X_c{}^d=0\,,$$

(8.22′) $$R_b{}^a{}_c|_d X^{(d)}+R_b{}^a{}_{c|d}X^d-R_b{}^d{}_c X_d{}^a+R_d{}^a{}_c(X_b{}^d-\eta_b{}^d) \\ -R_d{}^a{}_b(X_c{}^d-\eta_c{}^d)=0\,,$$

(8.23′) $$P_b{}^a{}_c|_d X^{(d)}+P_b{}^a{}_{c|d}X^d-P_b{}^d{}_c X_d{}^a+P_d{}^a{}_c(X_b{}^d-\eta_b{}^d) \\ +P_b{}^a{}_d X_c{}^d=0\,.$$

§ 9. Complete integrability of infinitesimal affine transformations

In the last section, we derived the conditions of integrability

(8.18)-(8.23) of the system of differential equations (8.9)-(8.12) which were satisfied by an infinitesimal affine transformation X. These conditions were written with use of covariant v- and h-differentiations. It will be noted that the v-covariant one in these equations may be replaced, roughly speaking, by the $\mathring{v}$-covariant one which will be defined as follows. If we take the mapping $\bar{\pi}_p^{-1}: \tau^{-1}\circ\pi(p)\to Q$, $p\in P$, defined by $\bar{\pi}_p^{-1}(b)=(b, p)$, then we have a tangent vector field $\mathring{B}^{(v)}(f)$ on Q, $f\in F$, which is defined by $\mathring{B}^{(v)}(f)_q=\bar{\pi}_p^{-1}(pj_{\gamma(q)}f)$, $q=(b, p)$. $\mathring{B}^{(v)}(f)$ will be called the $\mathring{v}$-*basic vector field*. It will be easily seen that $\mathring{B}^{(v)}(f)$ is expressed by

$$\mathring{B}^{(v)}(f)=C_b{}^a{}_c f^c F_a{}^b+B^{(v)}(f),\quad f=f^a e_a,$$

where $F_a{}^b=\bar{F}(\hat{g}_a{}^b)$. Therefore, if $X\in Q_q$ is expressed

$$X=X_b{}^a F_a{}^b+X^{(a)}B_a^{(v)}+X^a B_a^{(v)}=\mathring{X}_b{}^a F_a{}^b+\mathring{X}^{(a)}\mathring{B}_a^{(v)}+\mathring{X}^a B_a^{(h)},$$

then we obtain

$$(9.1)\qquad X^a=\mathring{X}^a,\quad X^{(a)}=\mathring{X}^{(a)},\quad X_b{}^a=\mathring{X}_b{}^a+\mathring{X}^{(c)}C_b{}^a{}_c.$$

Referring to $\mathring{B}^{(v)}(f)$, we shall define the $\mathring{v}$-*covariant derivative* $\mathring{\Delta}^v$ in similar manner with the v-covariant one Δ^v by means of $B^{(v)}(f)$. Thus, for a (1, 1)-type tensor $K=K_b{}^a e_a\otimes e^b$, we shall write $\mathring{\Delta}^v K=K_b{}^a{}_{\|c}e_a\otimes e^b\otimes e^c$. The following equation will be easily verified.

$$K_b{}^a|_c X^{(c)}-K_b{}^d X_d{}^a+K_d{}^a X_b{}^d=K_b{}^a{}_{\|c}\mathring{X}^{(c)}-K_b{}^d\mathring{X}_d{}^a+K_d{}^a\mathring{X}_b{}^d,$$

both hand sides of the above being of the same type. Consequently we may replace $X_b{}^a$, $X^{(a)}$, X^a and the v-covariant differentiation (|) in (8.18')-(8.23') by $\mathring{X}_b{}^a$, $\mathring{X}^{(a)}$, $\mathring{X}^a$ and the $\mathring{v}$-covariant one (‖) respectively.

We now return to a consideration of an infinitesimal linear transformation X, which may be written in the form of sum $Y+Z$, where Y is the induced part from a transformation of the base manifold M, and Z is the rotation part arising from the deviation λ. In the following we shall first treat the induced part Y and then the rotation part Z.

If we denote by $\xi=\xi^i\,(\partial/\partial x^i)$ the infinitesimal transformation of M, the induced Y is expressed by

$$Y=\xi^i\frac{\partial}{\partial x^i}+\frac{\partial\xi^i}{\partial x^j}b^j\frac{\partial}{\partial b^i}+\frac{\partial\xi^i}{\partial x^j}p_a{}^j\frac{\partial}{\partial p_a^i},$$

at $q=(x^i, b^i, p_a{}^i)$. Hence, if we put

$$\xi_j{}^i=\xi^i{}_{;j}+T_j{}^i{}_k\xi^k,\quad \xi_0{}^i=\xi_j{}^ib^j,$$

then vertical, v- and h-horizontal components $Y_b{}^a$, $Y^{(a)}$, Y^a of Y are

$$\text{(9.2)}\qquad \begin{aligned} Y_b{}^a&=p_i^{-1a}p_b{}^j(\xi_j{}^i+C_j{}^i{}_k\xi_0{}^k+C_j{}^i{}_kD_l{}^k\xi^l),\\ Y^{(a)}&=p_i^{-1a}(\xi_o{}^i+D_j{}^i\xi^j),\quad Y^a=p_i^{-1a}\xi^i, \end{aligned}$$

or, from (9.1), we have the following simpler equations:

$$\text{(9.2')}\qquad \mathring{Y}_b{}^a=p_i^{-1a}p_b{}^j\xi_j{}^i,\quad \mathring{Y}^{(a)}=Y^{(a)},\quad \mathring{Y}^a=Y^a.$$

Since the condition of complete integrability will be somewhat complicated for the most general Finsler connection, in subsequent discussion attention is confined to such a Finsler connection that *Conditions F and I* due to T. Okada are satisfied. Condition F is that $D_j{}^i=0$, while Condition I means the *homogeneity* of coefficients of the connection, and it does not seem to us that these conditions impose upon the connection some strong restriction.

According to the note as given at the beginning of this section, we substitute (9.2′) into (8.18′), and then we have

$$\text{(9.3)}\quad S_j{}^i{}_{kl;h}+\frac{\partial S_j{}^i{}_{kl}}{\partial b^r}b^sT_s{}^r{}_h+S_r{}^i{}_{kl}T_j{}^r{}_h+S_j{}^i{}_{rl}T_k{}^r{}_h-S_j{}^i{}_{rk}T_l{}^r{}_h=0,$$

$$\text{(9.4)}\quad \frac{\partial S_j{}^i{}_{kl}}{\partial b^h}b^m-S_j{}^m{}_{kl}\delta_h{}^i+S_h{}^i{}_{kl}\delta_j{}^m+S_j{}^i{}_{hl}\delta_k{}^m-S_j{}^i{}_{hk}\delta_l{}^m=0,$$

the first being a coefficient of ξ^h and the second being of $\xi^h{}_{;m}$. Since $S_j{}^i{}_{kl}$ are homogeneous of degree -2 with respect to b^i, contraction of (9.4) by b^h gives

$$\text{(9.5)}\qquad -2b^mS_j{}^i{}_{kl}+\delta_k{}^mS_j{}^i{}_{ol}-\delta_l{}^mS_j{}^i{}_{ok}-\delta_j{}^mS_o{}^i{}_{kl}-b^iS_j{}^m{}_{kl}=0,$$

where the index (o) means contraction by b^i. Take quantities b_i such that $b_o=b^ib_i\neq 0$, and contract (9.5) by b_i, and we have

$$\text{(9.6)}\qquad b_oS_j{}^m{}_{kl}=-2b^mS_j{}^o{}_{kl}+\delta_k{}^mS_j{}^o{}_{ol}-\delta_l{}^mS_j{}^o{}_{ok}+\delta_j{}^mS_o{}^o{}_{kl}.$$

Further, contraction by b_m gives

$$3b_0 S_{j\cdot kl}^{\ o} = b_k S_{j\cdot ol}^{\ o} - b_l S_{j\cdot ok}^{\ o} + b_j S_{o\cdot kl}^{\ o} ,$$

from which we obtain $S_{o\cdot kl}^{\ o}=S_{j\cdot ol}^{\ o}=0$ immediately, and hence $S_{j\cdot kl}^{\ o}=0$. Consequently (9.6) gives $S_{j\cdot kl}^{\ m}=0$, and then the another equation (9.3) is reduced to be trivial.

We next consider (8.19′), where it is to be noted that $\eta_b{}^a=0$ for the induced Y. In this case we obtain (9.4) replaced S by R. Since $R_{j\cdot kl}^{\ i}$ are homogeneous of degree 0, we get $R_{j\cdot kl}^{\ i}=0$ immediately by putting $m=h$ and summing. By the same process (8.20′) gives $P_{j\cdot kl}^{\ i}=0$. Remaining three equations (8.21′)-(8.23′) give us easily that torsion tensors $S_j{}^i{}_k$, $R_j{}^i{}_k$ and $P_j{}^i{}_k$ vanish as well.

Consequently we conclude that

Theorem 5. *Let* (Γ^v, Γ^h) *be a Finsler connection in* $Q(B)$ *satisfying the condition of homogeneity and Condition* F: $F_j{}^i=b^kF_k{}^i{}_j$. *If the connection admits a group of induced affine transformations of the maximum order* n^2+n (n=*dimension of the base manifold* M), *then all kinds of curvature tensors vanish and three kinds of torsion tensors, namely,* $S_j{}^i{}_k$, $R_j{}^i{}_k$ *and* $P_j{}^i{}_k$ *vanish also.*

According to this theorem, we have not any conclusion about the torsion tensor $T_j{}^i{}_k$, while the torsion tensor $C_j{}^i{}_k$ is to be symmetric with respect to subscripts, because $S_j{}^i{}_k=C_j{}^i{}_k-C_k{}^i{}_j$. It is to be noted that the following equations have to be satisfied:

$$C_j{}^i{}_{(k}|_{l)} - C_j{}^h{}_{(k}C_h{}^i{}_{l)} = 0 , \quad T_{(j}{}^i{}_{k;l)} - T_{(j}{}^h{}_k T_h{}^i{}_{l)} = 0 ,$$
$$C_{(k}{}^i{}_{j|l)} - T_k{}^i{}_l|_j - C_{(k}{}^h{}_j T_h{}^i{}_{l)} + T_k{}^h{}_l C_h{}^i{}_j = 0 ,$$

which are given from Bianchi identities with respect to torsion tensors.

The remainder of this section will be devoted to the study of the rotation part Z. Since we putted $\eta_j{}^i=(d\lambda_j{}^i/dt)_o$, Z is given by

$$Z = \eta_o{}^i \frac{\partial}{\partial b^i} + \eta_j{}^i p_a{}^j \frac{\partial}{\partial p_a^i} ,$$

from which it follows that

$$(9.7) \qquad Z_b{}^a = p_i^{-1a} p_b{}^j(\eta_j{}^i + C_j{}^i{}_k \eta_o{}^k), \quad Z^{(a)} = p_i^{-1a}\eta_o{}^i, \quad Z^a = 0 .$$

Comparing (9.7) with (9.2) for the case of Y, we know that Z is of the simpler form than Y, because terms containing $D_j{}^i$ do not appear, and hence we assume in this case that the connection is imposed the condition of homogeneity only.

We are concerned with the condition of complete integrability (8.18′)-(8.23′). The first (8.18′) gives

$$\frac{\partial R_j{}^i{}_{kl}}{\partial b^h} b^m + \delta_j{}^m R_h{}^i{}_{kl} - \delta_h{}^i R_j{}^m{}_{kl} = 0 \,. \tag{9.8}$$

Contraction of (9.8) by b^h gives

$$\delta_j{}^m R_o{}^i{}_{kl} - b^i R_j{}^m{}_{kl} = 0 \,.$$

Take b_i as used in the case of Y and contract the above by b_i. We then obtain $R_j{}^i{}_{kl} = \frac{1}{n} \delta_j{}^i R_h{}^h{}_{kl}$ immediately. Substitution of this expression into (9.8) shows that $R_h{}^h{}_{kl}$ are functions of x^i only. Thus there exist quantities $R_{kl}(x)$ such that $R_j{}^i{}_{kl} = \delta_j{}^i R_{kl}(x)$.

Next, we have from (8.20′)

$$\frac{\partial P_j{}^i{}_{kl}}{\partial b^h} b^m + \delta_l{}^m P_j{}^i{}_{kh} - \delta_h{}^i P_j{}^m{}_{kl} + \delta_j{}^m P_h{}^i{}_{kl} = 0 \,.$$

Contraction by b^h gives

$$-b^m P_j{}^i{}_{kl} + \delta_l{}^m P_j{}^i{}_{ko} - b^i P_j{}^m{}_{kl} + \delta_j{}^m P_o{}^i{}_{kl} = 0 \,, \tag{9.9}$$

because $P_j{}^i{}_{kl}$ are homogeneous of degree -1. The process by means of which from (9.5) we got $S_j{}^i{}_{kl} = 0$ is applied to (9.9), and then we obtain

$$b_o{}^2 P_j{}^m{}_{kl} = \left(-\frac{b^m}{b_o} b_j b_l + \delta_j{}^m b_l + \delta_l{}^m b_j \right) P_o{}^o{}_{ko} \,.$$

Substitution of this expression into (9.9) gives the equation of the form $(\cdots)\ P_o{}^o{}_{ko} = 0$, the coefficient $(\cdots)$ being

$$\frac{2}{b_o^2} b^i b^m b_j b_l - \frac{b^m}{b_o} (\delta_l{}^i b_j + \delta_j{}^i b_l)$$
$$- \frac{b^i}{b_o} (\delta_l{}^m b_j + \delta_j{}^m b_l) + \delta_l{}^m \delta_j{}^i + \delta_j{}^m \delta_l{}^i \,.$$

which does not vanish as will be easily seen. Therefore we obtain $P_{j\,:kl}^{\,i}=0$.

Finally we treat (8.21′), (8.22′) and (8.23′), which give

$$\frac{\partial S_j{}^i{}_k}{\partial b^l}b^h+\delta_j{}^h S_l{}^i{}_k-\delta_k{}^h S_l{}^i{}_j-\delta_l{}^i S_j{}^h{}_k=0\,, \tag{9.10}$$

$$\frac{\partial R_j{}^i{}_k}{\partial b^l}b^h-\delta_l{}^i R_j{}^h{}_k=0\,, \tag{9.11}$$

$$\frac{\partial P_j{}^i{}_k}{\partial b^l}b^h+\delta_k{}^h P_j{}^i{}_l-\delta_l{}^i P_j{}^h{}_k=0\,, \tag{9.12}$$

respectively. Contracting these equations by b^l and noting that $S_j{}^i{}_k$, $R_j{}^i{}_k$ and $P_j{}^i{}_k$ are homogeneous of degree -1, 1 and 0 respectively, we get

$$-b^h S_j{}^i{}_k+\delta_j{}^h S_o{}^i{}_k-\delta_k{}^h S_o{}^i{}_j-b^i S_j{}^h{}_k=0\,, \tag{9.13}$$

$$b^h R_j{}^i{}_k-b^i R_j{}^h{}_k=0, \tag{9.14}$$

$$\delta_k{}^h P_j{}^i{}_o-b^i P_j{}^h{}_k=0\,, \tag{9.15}$$

respectively. It is easily deduced from (9.13) that $S_j{}^i{}_k=0$, and (9.14) and (9.15) show that there exist quantities $\bar{R}_{jk}$ and P_j such that $R_j{}^i{}_k=b^i\bar{R}_{jk}$ and $P_j{}^i{}_k=\delta_k{}^i P_j$. Substitution of these into (9.11) and (9.12) respectively gives that both of $\bar{R}_{jk}$ and P_j are functions of x^i only. Consequently we establish

Theorem 6. *Let (Γ^v, Γ^h) be a Finsler connection in $Q(B)$ satisfying the condition of homogeneity. If the connection admits a group of affine rotations of the maximum order n^2 (n=dimension of the base manifold M), then the following equations are satisfied*:

$$S_{j\,:kl}^{\,i}=P_{j\,:kl}^{\,i}=S_j{}^i{}_k=0\,,\quad R_{j\,:kl}^{\,i}=\delta_j{}^i R_{kl}(x)\,,$$
$$R_j{}^i{}_k=b^i\bar{R}_{jk}(x)\,,\quad P_j{}^i{}_k=\delta_k{}^i P_j(x)\,.$$

The theorem was derived under the condition of homogeneity only. If certain conditions are further imposed on the connection, it will be expected that some of above functions $R_{jk}(x)$, $\bar{R}_{jk}(x)$ and $P_j(x)$ or all of them vanish, especially for a connection induced from a Finsler metric under certain hypothesis. It is to be remarked that Bianchi identities for curvature and torsion give following equations:

$$\begin{aligned}
&\delta_j{}^i(P_{(k|l)}+P_h T_k{}^h{}_l+R_{kl})-(\delta_j{}^i+C_o{}^i{}_j)\bar{R}_{kl}=0\,,\\
&\bar{R}_{(jk|l)}+T_{(jk}^{h}\bar{R}_{l)h}+\bar{R}_{(jk}P_{l)}=0\,,\\
&C_{(k}{}^i{}_{j|l)}-T_k{}^i{}_{l||j}+P_{(k}C_{l)j}^{i}=0\,,\\
&T_{(j}{}^i{}_{k|l)}+T_{(j}{}^h{}_k T_{l)}{}^i{}_h+\bar{R}_{(jk}C_{l)}{}^i{}_o-\delta_{(j}^{i}R_{kl)}=0\,.
\end{aligned}$$

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Note

In equations under Theorems 5 and 6, branckets and parenthesisses of indices mean that, for an example,

$$\begin{aligned}
&C_{(k}{}^i{}_{j|l)}=C_k{}^i{}_{j|l}-C_l{}^i{}_{j|k}\,,\\
&T_{(j}{}^h{}_k T_h{}^i{}_{l)}=T_j{}^h{}_k T_h{}^i{}_l+T_k{}^h{}_l T_h{}^i{}_j+T_l{}^h{}_j T_h{}^i{}_k\,.
\end{aligned}$$