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Homotopy groups of symplectic groups

By

Mamoru MIMURA and Hirosi TODA

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§1. Introduction

The present paper is one of our series on the homotopy groups of simple Lie groups, following from the previous paper [7].

We shall consider the homotopy groups $\pi_i(Sp(n))$ of symplectic groups $Sp(n)$.

When $i \leq 4n+1$, the groups are stable and computed by Bott [3]:

$$\pi_i(Sp(n)) \cong \begin{cases} Z & i \equiv 3, 7 \quad (\text{mod } 8), \quad i \leq 4n+1, \\ Z_2 & i \equiv 4, 5 \quad (\text{mod } 8), \quad i \leq 4n+1, \\ 0 & i \equiv 0, 1, 2, 6 \ (\text{mod } 8), \quad i \leq 4n+1. \end{cases}$$

For almost stable cases $i = 4n+2$, $4n+3$, $4n+4$, the following results are obtained (Theorem 2.2).

$$\pi_{4n+2}(Sp(n)) \cong \begin{cases} Z_{2\cdot(2n+1)!} & \text{for odd } n, \\ Z_{(2n+1)!} & \text{for even } n, \end{cases}$$

$$\pi_{4n+3}(Sp(n)) \cong Z_2,$$

$$\pi_{4n+4}(Sp(n)) \cong \begin{cases} Z_2 & \text{for odd } n, \\ Z_2 \oplus Z_2 & \text{for even } n. \end{cases}$$

These results were already computed in [4], except the last one which will be determined in §2 by use of secondary compositions.

For $i \leq 23$, the groups $\pi_i(Sp(1)) = \pi_i(S^3)$ and $\pi_i(Sp(2))$ are determined in [11], [6] and [7]. Then the following table of $\pi_i(Sp(n))$ is established by the computation of the groups $\pi_i(Sp(3))$, $17 \leq i \leq 23$, and $\pi_i(Sp(4))$, $21 \leq i \leq 24$. The computation will be given in §3 by the aid of lemmas in §4 and §5. Generators of the 2-primary components are also given in §4. In the table, the symbols ∞, $+$ and an integer r indicate an infinite cyclic groups, direct sum and a cyclic group Z_r of order r. The notations and terminologies in [7], [6] and [11] will be used in the present paper.

$i \backslash n$	1	2	3	4	5	$n\geq 6$
1	(0)	(0)	(0)	(0)	(0)	(0)
2	(0)	(0)	(0)	(0)	(0)	(0)
3	(∞)	(∞)	(∞)	(∞)	(∞)	(∞)
4	(2)	(2)	(2)	(2)	(2)	(2)
5	(2)	(2)	(2)	(2)	(2)	(2)
6	2·3!	(0)	(0)	(0)	(0)	(0)
7	2	(∞)	(∞)	(∞)	(∞)	(∞)
8	2	(0)	(0)	(0)	(0)	(0)
9	3	(0)	(0)	(0)	(0)	(0)
10	15	5!	(0)	(0)	(0)	(0)
11	2	2	(∞)	(∞)	(∞)	(∞)
12	2+2	2+2	(2)	(2)	(2)	(2)
13	12+3	4+2	(2)	(2)	(2)	(2)
14	84+2+2	7!/3	2·7!	(0)	(0)	(0)
15	2+2	2	2	(∞)	(∞)	(∞)
16	6	2+2	2	(0)	(0)	(0)
17	30	40	0	(0)	(0)	(0)
18	30	7!/2+2	3·7!	9!	(0)	(0)
19	6+2	2+2	2	2	(∞)	(∞)
20	12+2+2	2+2+2	2+2	2+2	(2)	(2)
21	12+2+2	32+2	12+2	6+2	(2)	(2)
22	132+2	44·5!+2+2	11!/120+2	11!/2	2·11!	(0)
23	2+2	2+2+2	2+2	2	2	(∞)
24				2+2	2	(0)

§2. Almost stable groups

Consider the following exact sequence associated with the fibering $(Sp(n+1),\ p,\ S^{4n+3},\ Sp(n))$:

$$(2.1)_n \quad \cdots \longrightarrow \pi_i(Sp(n)) \xrightarrow{i_*} \pi_i(Sp(n+1)) \xrightarrow{p_*} \pi_i(S^{4n+3}) \xrightarrow{\Delta} \pi_{i-1}(Sp(n)) \cdots,$$

where i_* (resp. p_*) is a homomorphism induced by the injection (resp. the projection) and Δ is a boundary homomorphism. Therefore we have isomorphisms

$$i_*:\ \pi_i(Sp(n)) \cong \pi_i(Sp(n+1)) \qquad \textit{for} \quad i \leq 4n+1,$$

since we have $\pi_{i+1}(S^{4n+3})=0$ for $i \leq 4n+1$.

In this stable range the following results are well-known. (See Bott [3].)

$$(2.2) \qquad \begin{aligned} &\text{i)} \quad \pi_{4n-2}(Sp(n)) = 0, \\ &\text{ii)} \quad \pi_{4n-1}(Sp(n)) \cong Z, \\ &\text{iii)} \quad \pi_{4n}(Sp(n)) \cong \begin{cases} Z_2 & \textit{for odd } n, \\ 0 & \textit{for even } n, \end{cases} \\ &\text{iv)} \quad \pi_{4n+1}(Sp(n)) \cong \begin{cases} Z_2 & \textit{for odd } n, \\ 0 & \textit{for even } n. \end{cases} \end{aligned}$$

The following diagram is evidently commutative and exact:

$$\begin{tikzcd}
 & \pi_{4n+3}(SU(2n+1)) \arrow[d] & \\
 & \pi_{4n+3}(SU(2n+1)/Sp(n)) \arrow[d, "\Delta_2"] & \\
Z \cong \pi_{4n+3}(S^{4n+3}) \arrow[r, "\Delta"] \arrow[d, equal] & \pi_{4n+2}(Sp(n)) \arrow[r] \arrow[d, "i_*"] & \pi_{4n+2}(Sp(n+1)) = 0 \arrow[d] \\
\pi_{4n+3}(S^{4n+3}) \arrow[r, "\Delta_1"] & \pi_{4n+2}(SU(2n+1)) \arrow[r] \arrow[d, "p_*"] & \pi_{4n+2}(SU(2n+2)) \\
 & \pi_{4n+2}(SU(2n+1)/Sp(n)) \arrow[d, "\Delta_3"] & \\
 & \pi_{4n+1}(Sp(n)) \arrow[d] & \\
 & \pi_{4n+1}(SU(2n+1)), &
\end{tikzcd}$$

where $\pi_{4n+3}(SU(2n+1))=0$ by [10], $\pi_{4n+3}(SU(2n+1)/Sp(n))\cong\pi_{4n+2}(SU(2n+1)/Sp(n))\cong\pi_{4n+1}(Sp(n))\cong Z_2$ for odd n and $=0$ for even n, $\pi_{4n+2}(SU(2n+2))=0$ and $\pi_{4n+1}(SU(2n+1))\cong Z$ by [3], $\pi_{4n+2}(SU(2n+1))\cong Z_{(2n+1)!}$ by [2], [10].

First we have that Δ_3 is an epimorphism and thus an isomorphism. It follows that Δ, Δ_1, and i_* are epimorphisms and Δ_2 is a monomorphism. Thus we have that $\pi_{4n+2}(Sp(n))$ is cyclic, and

$$\pi_{4n+2}(Sp(n))\cong\begin{cases} Z_{2\cdot(2n+1)!} & \text{for odd } n\,,\\ Z_{(2n+1)!} & \text{for oven } n\,.\end{cases}$$

We denote by θ_{4n-1} a generator of $\pi_{4n-1}(Sp(n))\cong Z$. Then we have the following

Theorem 2.1. *When n is odd, the generators of $\pi_{4n}(Sp(n))\cong Z_2$ and $\pi_{4n+1}(Sp(n))\cong Z_2$ are $\theta_{4n-1}\circ\eta_{4n-1}$ and $\theta_{4n-1}\circ\eta^2_{4n-1}$ respectively, where η_{4n-1} is the generator of $\pi_{4n}(S^{4n-1})\cong Z_2$.*

The proof is similar to that of Lemma 2 of [5], by use of Bott's periodicity $Sp(\infty)\simeq\Omega^8 Sp(\infty)$.

Consider the exact sequence $(2.1)_n$ for $i=4n+3$:

$$\pi_{4n+3}(Sp(n))\xrightarrow{i_*}\pi_{4n+3}(Sp(n+1))\xrightarrow{p_*}\pi_{4n+3}(S^{4n+3})\xrightarrow{\Delta}\pi_{4n+2}(Sp(n))\longrightarrow$$
$$\longrightarrow\pi_{4n+2}(Sp(n+1))\longrightarrow\cdots.$$

We have that $\pi_{4n+2}(Sp(n+1))=0$ by i) of (2.2), $\pi_{4n+2}(Sp(n))\cong Z_{2\cdot(2n+1)!}$ for odd n and $\cong Z_{(2n+1)!}$ for even n and $\pi_{4n+3}(Sp(n+1))\cong Z$. Therefore the degree of the homomorphism p_* is $2\cdot(2n+1)!$ for odd n, $(2n+1)!$ for even n and i_* is trivial. That is,

$$(2.3)\qquad p_*(\theta_{4n+3})=\begin{cases} 2\cdot(2n+1)!\,\iota_{4n+3} & \textit{for odd } n\,,\\ (2n+1)!\,\iota_{4n+3} & \textit{for even } n\,.\end{cases}$$

Whence we have (for n: even)

$$p_*(\theta_{4n+3}\circ\eta_{4n+3})=p_*(\theta_{4n+3})\circ\eta_{4n+3}=(2n+1)!\,\eta_{4n+3}=0$$

and $$p_*(\theta_{4n+3}\circ\eta^2_{4n+3})=p_*(\theta_{4n+3})\circ\eta^2_{4n+3}=(2n+1)!\,\eta^2_{4n+3}=0\,,$$

for the generators $\theta_{4n+3}\circ\eta_{4n+3}$ of $\pi_{4n+4}(Sp(n+1))$ and $\theta_{4n+3}\circ\eta^2_{4n+3}$ of $\pi_{4n+5}(Sp(n+1))$. Thus we have the following two exact sequences:

$$0 \longrightarrow \pi_{4n+4}(S^{4n+3}) \xrightarrow{\Delta} \pi_{4n+3}(Sp(n)) \longrightarrow 0\,,$$

$$0 \longrightarrow \pi_{4n+5}(S^{4n+3}) \xrightarrow{\Delta} \pi_{4n+4}(Sp(n)) \xrightarrow{i_*} \pi_{4n+4}(Sp(n+1)) \longrightarrow 0\,,$$

where $\pi_{4n+4}(S^{4n+3}) \cong \pi_{4n+5}(S^{4n+3}) \cong Z_2$, $\pi_{4n+4}(Sp(n+1)) = \{\theta_{4n+3} \circ \eta_{4n+3}\} \cong Z_2$ for even n and $=0$ for odd n. For odd n, obviously,

$$\pi_{4n+4}(Sp(n)) \cong Z_2\,.$$

But for even n, we must determine the following extension:

$$0 \longrightarrow Z_2 \longrightarrow \pi_{4n+4}(Sp(n)) \longrightarrow Z_2 \longrightarrow 0\,.$$

Let n be even and $\lambda = \Delta\iota_{4n+3}$, then

$$\begin{aligned} \lambda \circ ((2n+1)!\,\iota_{4n+2}) &= (2n+1)!\,\Delta\iota_{4n+3} \\ &= \Delta p_*(\theta_{4n+3}) \\ &= 0\,. \end{aligned}$$

$Sp(n+1)$ contains a subspace $Sp(n) \bigcup_\lambda e^{4n+3}$ such that $p | Sp(n) \bigcup_\lambda e^{4n+3}$: $(Sp(n) \bigcup_\lambda e^{4n+3},\ Sp(n)) \rightarrow (S^{4n+3}, *)$ is a relative homeomorphism preserving orientations.

Let f: $S^{4n+3} \rightarrow Sp(n) \bigcup_\lambda e^{4n+3} \subset Sp(n+1)$ be a coextension of $(2n+1)!\,\iota_{4n+2}$. Then f represents θ_{4n+3}, since $p \circ f$ represents $p_* \theta_{4n+3} = (2n+1)!\,\iota_{4n+3}$. It follows from Proposition 1.8 of [11]

$$\theta_{4n+3} \circ \eta_{4n+3} = i_*(\alpha)\,, \quad \alpha \in -\{\Delta\iota_{4n+3},\ (2n+1)!\,\iota_{4n+2},\ \eta_{4n+2}\}\,,$$

where i is the inclusion of $Sp(n)$ into $Sp(n+1)$.
We have

$$\begin{aligned} &-\{\Delta\iota_{4n+3},\ (2n+1)!\,\iota_{4n+2},\ \eta_{4n+2}\} \circ 2\iota_{4n+4} \\ &= \Delta\iota_{4n+3} \circ \{(2n+1)!\,\iota_{4n+2},\ \eta_{4n+2},\ 2\iota_{4n+3}\} \end{aligned}$$

by Proposition 1.4 of [11]

$$\begin{aligned} &= \Delta\iota_{4n+3} \circ ((2n+1)!/2)\eta^2_{4n+2} \\ &= 0\,, \end{aligned}$$

since $(2n+1)!$ is divisible by 4. Therefore, $2\alpha = \alpha \circ 2\iota_{4n+4} = 0$, for even n, and the above sequence splits.

Summarizing the above results, we have the following

Theorem 2. 2.

$$\pi_{4n+2}(Sp(n)) \cong \begin{cases} Z_{2\cdot(2n+1)!} & \textit{for odd } n, \\ Z_{(2n+1)!} & \textit{for even } n, \end{cases}$$

$$\pi_{4n+3}(Sp(n)) \cong Z_2,$$

$$\pi_{4n+4}(Sp(n)) \cong \begin{cases} Z_2 & \textit{for odd } n, \\ Z_2 \oplus Z_2 & \textit{for even } n. \end{cases}$$

§ 3. Computation of $\pi_i(Sp(n))$, $i \leq 23$

The groups $\pi_i(Sp(1)) \cong \pi_i(S^3)$, $i \leq 23$, are determined in [11] and [6]. The groups $\pi_i(Sp(2))$, $i \leq 23$, are determined in [7].

The groups $\pi_i(Sp(n))$ are given in § 2 for $i \leq 4n+4$. So, it is sufficient to compute the groups $\pi_i(Sp(3))$, $17 \leq i \leq 23$, and $\pi_i(Sp(4))$, $21 \leq i \leq 24$. The computations will be done algebraically by use of the exact sequences $(2.1)_n$ and by the aid of Lemmas 3.1 to 3.6, which will be proved in the next section.

Lemma 3. 1. *The homomorphism* $\Delta : \pi_{18}(S^{11}) \rightarrow \pi_{17}(Sp(2))$ *is an epimorphism.*

It follows from Lemma 3.1 and the exactness of $(2.1)_2$ that the sequence

$$0 \longrightarrow \pi_{17}(Sp(3)) \longrightarrow \pi_{17}(S^{11}) \longrightarrow \pi_{16}(Sp(2)) \longrightarrow \pi_{16}(Sp(3))$$

is exact. $\pi_{16}(Sp(3)) \cong Z_2$ by Theorem 2.2, $\pi_{16}(Sp(2)) \cong Z_2 \oplus Z_2$ by Theorem 5.1 of [7] and $\pi_{17}(S^{11}) \cong Z_2$. It follows

$$\pi_{17}(Sp(3)) = 0.$$

From $(2.1)_3$, we have an exact sequence

$$\pi_{19}(S^{15}) \longrightarrow \pi_{18}(Sp(3)) \longrightarrow \pi_{18}(Sp(4)) \longrightarrow \pi_{18}(S^{15}) \longrightarrow \pi_{17}(Sp(3)) = 0.$$

We have $\pi_{18}(Sp(4)) \cong Z_{9!}$ by Theorem 2.2, $\pi_{19}(S^{15}) = 0$ and $\pi_{18}(S^{15}) \cong Z_{24}$. It follows

$$\pi_{18}(Sp(3)) \cong Z_{9!/24} = Z_{3\cdot 7!}.$$

It follows from $(2.1)_3$ that $i_* : \pi_{19}(Sp(3)) \rightarrow \pi_{19}(Sp(4))$ is an isomorphism, since $\pi_{20}(S^{15}) = \pi_{19}(S^{15}) = 0$. Thus

$$\pi_{19}(Sp(3)) \cong \pi_{19}(Sp(4)) \cong Z_2 \qquad \text{(Theorem 2.2)}.$$

From the last sequence, we have that $p_*: \pi_{18}(Sp(4)) \to \pi_{18}(S^{15})$ is an epimorphism. Let $[\nu_{15}]$ be an element of $\pi_{18}(Sp(4))$ such that $p_*[\nu_{15}] = \nu_{15} \in \pi_{18}(S^{15}:2) \subset \pi_{18}(S^{15})$. Obviously $p_*([\nu_{15}]\circ\nu_{18}) = \nu_{15}\circ\nu_{18} = \nu_{15}^2$. ν_{15}^2 generates $\pi_{21}(S^{15}) \cong Z_2$. It follows that the homomorphism p_* in the following sequence is an epimorphism:

$$\pi_{21}(Sp(4)) \xrightarrow{p_*} \pi_{21}(S^{15}) \longrightarrow \pi_{20}(Sp(3)) \longrightarrow \pi_{20}(Sp(4)) \longrightarrow \pi_{20}(S^{15}) = 0.$$

Thus we have

$$\pi_{20}(Sp(3)) \cong \pi_{20}(Sp(4)) \cong Z_2 \oplus Z_2 \qquad \text{(Theorem 2.2)}.$$

Lemma 3.2. *The image of* $\Delta: \pi_{21}(S^{11}) \to \pi_{20}(Sp(2))$ *is isomorphic to* $Z_2 \oplus Z_2$.

Lemma 3.3. *The image of* $\Delta: \pi_{22}(S^{11}) \to \pi_{21}(Sp(2))$ *is generated by* $4[\sigma'\sigma_{14}]$ *and isomorphic to* Z_8.

In [7], we have obtained the results:

$$\pi_{20}(Sp(2)) \cong Z_2 \oplus Z_2 \oplus Z_2,$$
$$\pi_{21}(Sp(2)) \cong Z_{32} \oplus Z_2 = \{[\sigma'\sigma_{14}],\ i_*\eta_3\bar{\mu}_4\}.$$

Then it follows from the exactness of $(2.1)_2$ that the sequence $0 \to Z_4 \oplus Z_2 \to \pi_{21}(Sp(3)) \to Z_6 \oplus Z_2 \to Z_2 \oplus Z_2 \to 0$ is exact, where $Z_6 \oplus Z_2 \cong \pi_{21}(S^{11})$ by [11]. Thus we have easily

$$\pi_{21}(Sp(3)) \cong Z_{12} \oplus Z_2.$$

Lemma 3..4. *The image of* $\Delta: \pi_{22}(S^{15}) \to \pi_{21}(Sp(3))$ *is isomorphic to* Z_4.

We have seen that $p_*: \pi_{21}(Sp(4)) \to \pi_{21}(S^{15}) \cong Z_2$ is an epimorphism. It follows from Lemma 3.4 that we have an exact sequence

$$0 \longrightarrow Z_4 \longrightarrow Z_{12} \oplus Z_2 \xrightarrow{i_*} \pi_{21}(Sp(4)) \longrightarrow Z_2 \longrightarrow 0.$$

Thus $\pi_{21}(Sp(4))$ is isomorphic to $Z_6 \oplus Z_2$ or Z_{12}. If, $\pi_{21}(Sp(4)) \cong Z_{12}$, then $i_*\pi_{21}(Sp(3)) = 2(\pi_{21}(Sp(4)))$. Then the injection homomorphism $i_*: \pi_{21}(Sp(3)) \to \pi_{21}(Sp(5))$ vanishes, since $\pi_{21}(Sp(5)) \cong Z_2$. The group $\pi_{21}(Sp(5))$ is generated by $\theta_{19}\circ\eta_{19}^2$ (Theorem 2.1). In §2, we have seen that $i_*: \pi_{20}(Sp(4)) \to \pi_{20}(Sp(5))$ is an epimorphism. We have seen also that $i_*: \pi_{20}(Sp(3)) \to \pi_{20}(Sp(4))$ is an isomorphism. Thus, there exists an element $\alpha \in \pi_{20}(Sp(3))$ such that $i_*(\alpha) = \theta_{19} \circ \eta_{19}$ for $i_*: \pi_{20}(Sp(3)) \to \pi_{20}(Sp(5))$. Consider the composition $\alpha \circ \eta_{20}$. Then $i_*(\alpha \circ \eta_{20}) = \theta_{19} \circ \eta_{19}^2 \neq 0$, on the other hand $i_*: \pi_{21}(Sp(3)) \to \pi_{21}(Sp(5))$ vanishes as stated above. Thus the assumption $\pi_{21}(Sp(4)) \approx Z_{12}$ leads us to the contradiction, and we have

$$\pi_{21}(Sp(4)) \approx Z_6 \oplus Z_2 .$$

Consider the exact sequence $(2.1)_4$:

$$0 = \pi_{23}(S^{19}) \longrightarrow \pi_{22}(Sp(4)) \longrightarrow \pi_{22}(Sp(5)) \xrightarrow{p_*} \pi_{22}(S^{19}) \longrightarrow \pi_{21}(Sp(4)) \xrightarrow{i^*} \pi_{21}(Sp(5)) .$$

The last homomorphism i_* is an epimorphism and its kernel is isomorphic to Z_6 by the above discussion. Then the image of p_* is isomorphic to $Z_4 \approx Z_{24}/Z_6$, since $\pi_{22}(S^{19}) \approx Z_{24}$. By Theorem 2.2, $\pi_{22}(Sp(5)) \approx Z_{2 \cdot 11!}$.

It follows that

$$\pi_{22}(Sp(4)) \approx Z_{11!/2} = Z_{19958400} .$$

Consider the exact sequence $(2.1)_3$:

$$\pi_{23}(S^{15}) \longrightarrow \pi_{22}(Sp(3)) \xrightarrow{i_*} \pi_{22}(Sp(4)) \xrightarrow{p_*} \pi_{22}(S^{15}) \xrightarrow{\Delta} \pi_{21}(Sp(3)) .$$

The group $\pi_{22}(S^{15})$ is isomorphic to Z_{240} [11] and the image of the last homomorphism Δ is Z_4 by Lemma 3.4. Thus the cokernel of i_* is isomorphic to $Z_{60} \approx Z_{240}/Z_4$. Since $\pi_{23}(S^{15}) \approx Z_2 \oplus Z_2$ [11], it follows that the sequence

$$(3.1) \qquad Z_2 \oplus Z_2 \longrightarrow \pi_{22}(Sp(3)) \longrightarrow Z_{332640} \longrightarrow 0$$

is exact.

Next consider the exact sequence $(2.1)_2$:

$$\pi_{23}(S^{11}) \longrightarrow \pi_{22}(Sp(2)) \xrightarrow{i_*} \pi_{22}(Sp(3)) \longrightarrow \pi_{22}(S^{11}) \longrightarrow \pi_{21}(Sp(2)) .$$

The group $\pi_{22}(S^{11})$ is isomorphic to Z_{504} [11]. It follows from

Lemma 3.3 that the cokernel of i_* is isomorphic to $Z_{63} \approx Z_{504}/Z_8$. We have $\pi_{22}(Sp(2)) \approx Z_{5280} \oplus Z_2 \oplus Z_2$ [7] and $\pi_{23}(S^{11}) \approx Z_2$ [11].

Now apply

Lemma 3.5. *The homomorphism* $\Delta: \pi_{23}(S^{11}) \to \pi_{22}(Sp(2))$ *is a monomorphism.*

Then we have an exact sequence

(3.2) $$0 \longrightarrow Z_2 \longrightarrow Z_{5280} \oplus Z_2 \oplus Z_2 \longrightarrow \pi_{22}(Sp(3)) \longrightarrow Z_{63} \longrightarrow 0.$$

There exists an element of order $332640 = 5280 \times 63$ in $\pi_{22}(Sp(3))$ by the exactness of (3.1), but the element cannot be divisible by 2 by the exactness of (3.2). Thus $\pi_{22}(Sp(3))$ has a direct factor isomorphic to Z_{332640}. It follows from the exactness of (3.2)

$$\pi_{22}(Sp(3)) \approx Z_{332640} \oplus Z_2 = Z_{11!/120} \oplus Z_2.$$

It follows immediately from the exact sequence $0 = \pi_{24}(S^{19}) \to \pi_{23}(Sp(4)) \to \pi_{23}(Sp(5)) \to \pi_{23}(S^{19}) = 0$ that $\pi_{23}(Sp(4)) \approx Z_2$, where $\pi_{23}(Sp(5)) \approx Z_2$ by Theorem 2.2.

Consider the exact sequence $(2.1)_3$:

$$\pi_{23}(Sp(3)) \xrightarrow{i_*} \pi_{23}(Sp(4)) \xrightarrow{p_*} \pi_{23}(S^{15}) \xrightarrow{\Delta} \pi_{22}(Sp(3)),$$

where $\pi_{23}(Sp(4)) \approx Z_2$ and $\pi_{23}(S^{15}) \approx Z_2 \oplus Z_2 = \{\bar{\nu}_{15}, \varepsilon_{15}\}$. By use of Lemma 5.3 in §5 we have $\Delta \bar{\nu}_{15} = 0$, hence we know that $\pi_{23}(Sp(4)) \approx Z_2$ is generated by $[\bar{\nu}_{15}]$, and that

(3.3) $i_*: \pi_{23}(Sp(3)) \longrightarrow \pi_{23}(Sp(4))$ *is trivial.*

By the isomorphism $i_*: \pi_{23}(Sp(4)) \approx \pi_{23}(Sp(5))$, we see that $\pi_{23}(Sp(5)) \approx Z_2$ is generated by $i_*[\bar{\nu}_{15}]$.

In the exact sequence $(2.1)_5$:

$$\pi_{25}(S^{23}) \longrightarrow \pi_{24}(Sp(5)) \longrightarrow \pi_{24}(Sp(6)) \longrightarrow \pi_{24}(S^{23}) \xrightarrow{\Delta} \pi_{23}(Sp(5)),$$

we have $\pi_{25}(S^{23}) \approx Z_2 = \{\eta_{23}^2\}$, $\pi_{24}(Sp(5)) \approx Z_2$ (Theorem 2.2) $\pi_{24}(Sp(6)) = 0$, $\pi_{24}(S^{23}) \approx Z_2 = \{\eta_{23}\}$, and $\pi_{23}(Sp(5)) \approx Z_2 = \{i_*[\bar{\nu}_{15}]\}$. Therefore, $\Delta \eta_{23} = i_*[\bar{\nu}_{15}]$, hence we have that $\Delta \eta_{23}^2 = i_*[\bar{\nu}_{15}] \circ \eta_{23}$ and $\pi_{24}(Sp(5)) \approx Z_2$ is generated by $i_*[\bar{\nu}_{15}] \circ \eta_{23}$.

Consider the homomorphism

$$p_* \circ \Delta : \pi_{25}(S^{19}) \longrightarrow \pi_{24}(Sp(4)) \longrightarrow \pi_{24}(S^{15}),$$

where $\pi_{25}(S^{19}) \approx Z_2 = \{\nu_{19}^2\}$ and $\pi_{24}(S^{15}) \approx Z_2 \oplus Z_2 \oplus Z_2 = \{\nu_{15}^3, \mu_{15}, \eta_{15}\varepsilon_{16}\}$. We have $\Delta\nu_{19}^2 = [\nu_{15}] \circ \nu_{18}^2$, since $\Delta : \pi_{19}(S^{19}) \to \pi_{18}(Sp(4))$ is an epimorphism and since $\pi_{18}(Sp(4):2) \approx Z_{128}$ is generated by $[\nu_{15}]$. Whence $p_*(\Delta\nu_{19}^2) = p_*([\nu_{15}] \circ \nu_{18}^2) = \nu_{15}^3$. Therefore $\pi_{24}(Sp(4))$ is isomorphic to $\pi_{25}(S^{19}) \oplus \pi_{24}(Sp(4))/\pi_{25}(S^{19})$, where $\pi_{25}(S^{19}) \approx Z_2$ and $\pi_{24}(Sp(4))/\pi_{25}(S^{19}) \approx \pi_{24}(Sp(5)) \approx Z_2$ by the exactness of the sequence:

$$\pi_{25}(S^{19}) \longrightarrow \pi_{24}(Sp(4)) \longrightarrow \pi_{24}(Sp(5)) \longrightarrow \pi_{24}(S^{19}) = 0.$$

Thus $\pi_{24}(Sp(4)) \approx Z_2 \oplus Z_2$. One of its generators is $[\nu_{15}] \circ \nu_{18}^2$. Let α be another one such that $p_*(\alpha) = 0$ $(: \pi_{24}(Sp(4)) \to \pi_{24}(S^{15}))$. Consider the element $[\bar{\nu}_{15}] \circ \eta_{23} = \alpha + x[\nu_{15}] \circ \nu_{18}^2$. Since $i_*[\bar{\nu}_{15}] \circ \eta_{23} \neq 0$ in $\pi_{24}(Sp(5))$, we have $[\bar{\nu}_{15}] \circ \eta_{23} \in \pi_{24}(Sp(4))$. Applying p_*, we have

$$\begin{aligned} p_*([\bar{\nu}_{15}] \circ \eta_{23}) &= \bar{\nu}_{15}\eta_{23} \\ &= \nu_{15}^3 \qquad \text{by Lemma 6.3 of [11]} \\ &= p_*(\alpha) + x\nu_{15}^3. \end{aligned}$$

Therefore $x = 1$, and

$$\alpha = [\bar{\nu}_{15}] \circ \eta_{23} + [\nu_{15}] \circ \nu_{18}^2.$$

The exact sequence $(2.1)_3$

$$\pi_{24}(Sp(4)) \xrightarrow{p_*} \pi_{24}(S^{15}) \longrightarrow \pi_{23}(Sp(3)) \longrightarrow \pi_{23}(Sp(4))$$

is reduced to the exact one:

$$0 \longrightarrow Z_2 \longrightarrow Z_2 \oplus Z_2 \oplus Z_2 \longrightarrow \pi_{23}(Sp(3)) \longrightarrow 0,$$

by the above discussion and by (3.3). Thus we obtain

$$\pi_{23}(Sp(3)) \approx Z_2 \oplus Z_2.$$

§ 4. Generators

In this section we shall study the generators of the 2-primary components of $\pi_i(Sp(n))$, $i \leq 23$, and prove the lemmas used in the previous section.

We omit those of $\pi_i(Sp(1))$ and $\pi_i(Sp(2))$, as they are already stated in [11], [6] and [7].

It follows easily from (2. 2) that

$$\begin{aligned}
\pi_3(Sp(n)) &= \{i_*\iota_3\} \cong Z & &\textit{for}\quad n\geq 1\,,\\
\pi_4(Sp(n)) &= \{i_*\eta_3\} \cong Z_2 & &\textit{for}\quad n\geq 1\,,\\
\pi_5(Sp(n)) &= \{i_*\eta_3^2\} \cong Z_2 & &\textit{for}\quad n\geq 1\,,\\
\pi_7(Sp(n)) &= \{i_*\theta_7 = i_*[12\iota_7]\} \cong Z & &\textit{for}\quad n\geq 2\,.
\end{aligned}$$

Consider the exact sequence $(2.1)_2$:

$$\pi_{11}(Sp(3)) \xrightarrow{p_*} \pi_{11}(S^{11}) \longrightarrow \pi_{10}(Sp(2)) \longrightarrow \pi_{10}(Sp(3)) = 0\,.$$

We have that $\pi_{11}(S^{11})=\{\iota_{11}\}\cong Z$ and $\pi_{10}(Sp(2))\cong Z_{5!}$ by (2. 2). Therefore $\pi_{11}(Sp(3))$ is generated by $[5!\,\iota_{11}]$, that is,

$$\pi_{11}(Sp(n)) = \{i_*\theta_{11} = i_*[5!\,\iota_{11}]\} \cong Z \qquad \text{for}\quad n\geq 3\,.$$

Next, $\pi_{10}(Sp(2):2)\cong Z_8$ is generated by $[\nu_7]$ by Theorem 5. 1 of [7]. Restricting our considerations to the 2-primary components, we may consider as (cf. Lemma 2 .3 of [7])

(4. 1) $\quad \Delta\iota_{11} = [\nu_7]\,,\quad$ *and* $\quad \Delta(E\alpha) = [\nu_7]\circ\alpha\quad$ *for* $\quad \alpha\in\pi_i(S^7:2)\,.$

Consider the exact sequence $(2.1)_2$:

$$\pi_{13}(S^{11}) \xrightarrow{\Delta} \pi_{12}(Sp(2)) \longrightarrow \pi_{12}(Sp(3)) \longrightarrow \pi_{12}(S^{11}) \xrightarrow{\Delta} \pi_{11}(Sp(2)) \longrightarrow$$
$$\xrightarrow{i_*} \pi_{11}(Sp(3)) \cong Z\,,$$

where $\pi_{13}(S^{11})=\{\eta_{11}^2\}\cong Z_2$, $\pi_{12}(S^{11})=\{\eta_{11}\}\cong Z_2$ by [11], $\pi_{12}(Sp(2))=\{i_*\mu_3,\, i_*\eta_3\varepsilon_4\}\cong Z_2\oplus Z_2$, $\pi_{11}(Sp(2))=\{i_*\varepsilon_3\}=Z_2$ by Theorem 5. 1 of [7]. We have

(4. 2) $$\Delta\eta_{11} = i_*\varepsilon_3\,,$$

since $i_*\pi_{11}(Sp(2))=0$. By this relation, we have

$$\Delta\eta_{11}^2 = \Delta(\eta_{11})\circ\eta_{11} = i_*\varepsilon_3\eta_{11} = i_*\eta_3\varepsilon_4\,, \qquad \text{by (7. 5) of [11]}.$$

Hence $\qquad \pi_{12}(Sp(3)) = \{\theta_{11}\circ\eta_{11} = i_*\mu_3\} \cong Z_2\,,$

and $\qquad \pi_{12}(Sp(n)) = \{i_*\mu_3\} \cong Z_2 \qquad$ *for* $\quad n\geq 3\,.$

Consider the exact sequence:

$$0=\pi_{15}(S^{11})\longrightarrow\pi_{14}(Sp(2):2)\longrightarrow\pi_{14}(Sp(3):2)\longrightarrow\pi_{14}(S^{11}:2)$$
$$\xrightarrow{\Delta}\pi_{13}(Sp(2))\longrightarrow\pi_{13}(Sp(3))\xrightarrow{p_*}\pi_{13}(S^{11})\xrightarrow{\Delta}\cdots.$$

It follows from the above discussion that the last homomorphism p_* is trivial. We have that $\pi_{14}(Sp(2):2)=\{[2\sigma']\}\cong Z_{16}$, $\pi_{13}(Sp(2))=\{[\nu_7]\circ\nu_{10},\ i_*\eta_3\mu_4\}\cong Z_4\oplus Z_2$ by Theorem 5.1 of [7] and $\pi_{14}(S^{11}:2)=\{\nu_{11}\}\cong Z_8$. So we have

$$\pi_{13}(Sp(3))=\{i_*\eta_3\mu_4\}\cong Z_2,$$

since $\Delta\nu_{11}=[\nu_7]\circ\nu_{10}$ by (4.1).

As the order of $\Delta\nu_{11}$ is 4, we have, by Theorem 2.2,

$$\pi_{14}(Sp(3):2)=\{[4\nu_{11}]\}\cong Z_{32}.$$

Here note that, for suitable choice of $[4\nu_{11}]$,

$$(4.3)\qquad \begin{aligned} &i_*[2\sigma']=2[4\nu_{11}],\\ &315\Delta\iota_{15}\equiv[4\nu_{11}]. \qquad \bmod\ 2[4\nu_{11}].\end{aligned}$$

The exactness of the sequence

$$0=\pi_{16}(S^{11})\longrightarrow\pi_{15}(Sp(2))\longrightarrow\pi_{15}(Sp(3))\longrightarrow\pi_{15}(S^{11})=0,$$

where $\pi_{15}(Sp(2))=\{[\sigma'\eta_{14}]\}\cong Z_2$, implies that

$$\pi_{15}(Sp(3))=\{i_*[\sigma'\eta_{14}]\}\cong Z_2.$$

Consider the exact sequence $(2.1)_3$:

$$\pi_{15}(Sp(4))\longrightarrow\pi_{15}(S^{15})\longrightarrow\pi_{14}(Sp(3))\longrightarrow\pi_{14}(Sp(4))=0,$$

where $\pi_{14}(Sp(3))=Z_{2\cdot7!}$ and $\pi_{15}(S^{15})=\{\iota_{15}\}\cong Z$. Whence we have

$$\pi_{15}(Sp(4))=\{\theta_{15}=[2\cdot7!\,\iota_{15}]\}\cong Z.$$

So $\quad\pi_{15}(Sp(n))=\{i_*\theta_{15}=i_*[2\cdot7!\,\iota_{15}]\}\cong Z\qquad for\quad n\geq4.$

In the exact sequence $(2.1)_3$:

$$\pi_{17}(S^{11})\xrightarrow{\Delta}\pi_{16}(Sp(2))\longrightarrow\pi_{16}(Sp(3))\longrightarrow\pi_{16}(S^{11})=0,$$

we have that $\pi_{16}(Sp(2))=\{[\nu_7]\circ\nu_{10}^2,\ [\sigma'\eta_{14}]\circ\eta_{15}\}\cong Z_2\oplus Z_2$ and $\Delta(\nu_7^2)=[\nu_7]\circ\nu_{10}^2$ for the generator ν_7^2 of $\pi_{17}(S^{11})$. It follows that

$$\pi_{16}(Sp(3)) = \{i_*[\sigma'\eta_{14}]\circ\eta_{15}\} \cong Z_2.$$

Consider the exact sequence $(2.1)_2$:

$$\pi_{18}(Sp(2)) \longrightarrow \pi_{18}(Sp(3)) \longrightarrow \pi_{18}(S^{11}) \xrightarrow{\Delta} \pi_{17}(Sp(2)) \longrightarrow \pi_{17}(Sp(3)) \longrightarrow \cdots,$$

where $\pi_{17}(Sp(2)) \cong Z_{40}$, $\pi_{18}(S^{11}) \cong Z_{240}$, $\pi_{17}(Sp(2):2) = \{[\nu_7]\circ\sigma_{10}\}$ and $\pi_{18}(S^{11}:2) = \{\sigma_{11}\}$ by [7] and [11].

We have $$\Delta\sigma_{11} = [\nu_7]\circ\sigma_{10} \qquad \text{by (4.1).}$$

The exactness of $(2.1)_3$:

$$\pi_{18}(S^{15}) \cong Z_{24} \longrightarrow \pi_{17}(Sp(3)) \longrightarrow \pi_{17}(Sp(4)) = 0$$

implies that $\pi_{17}(Sp(3))$ has no 5-components. Thus the above homomorphism Δ is an epimorphism. This proves Lemma 3.1.

Since $\Delta\sigma_{11}$ is of order 8, we have that $\pi_{18}(Sp(3):2) \cong Z_{16}$ is generated by $[8\sigma_{11}]$.

It follows easily from the exact sequence $(2.1)_3$:

$$\pi_{18}(Sp(4):2) \longrightarrow \pi_{18}(S^{15}:2) = \{\nu_{15}\} \longrightarrow \pi_{17}(Sp(3)) = 0$$

that $\pi_{18}(Sp(4):2) \cong Z_{128}$ is generated by $[\nu_{15}]$.

In the exact sequence $(2.1)_2$:

$$\pi_{19}(Sp(3)) \longrightarrow \pi_{19}(S^{11}) \longrightarrow \pi_{18}(Sp(2):2),$$

where $\pi_{19}(S^{11}) = \{\bar{\nu}_{11}, \varepsilon_{11}\} \cong Z_2 \oplus Z_2$, we have

$$\begin{aligned} (4.4) \qquad \Delta(\bar{\nu}_{11}+\varepsilon_{11}) &= \Delta(\eta_{11}\sigma_{11}) && \text{by Lemma 6.4 of [12]} \\ &= \varepsilon_3\sigma_{11} && \text{by (4.2)} \\ &= 0 \end{aligned}$$

Hence $\pi_{19}(Sp(3)) \cong \pi_{19}(Sp(4)) \cong Z_2$ are genarated by $[\eta_{11}\sigma_{12}]$.

Consider the exact sequence $(2.1)_4$:

$$\pi_{20}(S^{19}) \xrightarrow{\Delta} \pi_{19}(Sp(4)) \xrightarrow{i_*} \pi_{19}(Sp(5)) \longrightarrow \pi_{19}(S^{19}) \xrightarrow{\Delta} \pi_{18}(Sp(4)) \longrightarrow \pi_{18}(Sp(5)) = 0.$$

It follows that $\pi_{19}(Sp(5))$ is generated by $[9!\,\iota_{19}]$, since $\pi_{18}(Sp(4)) \approx Z_{9!}$ by Theorem 2.2. And we have

$$\Delta\eta_{19} = i_*[\eta_{11}\sigma_{12}] \tag{4.5}$$

for the generator η_{19} of $\pi_{20}(S^{19})$, since $\pi_{19}(Sp(5)) \approx Z$ implies the triviality of $i_*\pi_{19}(Sp(4))$. Thus

$$\pi_{19}(Sp(n)) = \{i_*\theta_{19} = i_*[9!\,\iota_{19}]\} \approx Z, \qquad \textit{for} \quad n \geq 5.$$

Next consider the homomorphism $\Delta: \pi_{21}(S^{11}) \to \pi_{20}(Sp(2))$, where $\pi_{21}(S^{11}:2) = \{\sigma_{11}\nu_{18}, \eta_{11}\mu_{12}\}$ and $\pi_{20}(Sp(2)) = \{[\nu_7]\circ\sigma_{10}\nu_{17}, i_*\bar{\mu}_3, i_*\eta_3\mu_4\sigma_{13}\}$. To prove Lemma 3.2 it is sufficient to show the following two relations:

$$\Delta(\sigma_{11}\nu_{17}) = [\nu_7]\circ\sigma_{10}\nu_{17},$$
$$\Delta(\eta_{11}\mu_{12}) = i_*\eta_3\mu_4\sigma_{13}.$$

The first relation is easily obtained by (4.1). Since we have $\Delta(\eta_{11}\mu_{12}) = i_*\varepsilon_3\mu_{11}$ by (4.2), we shall show

$$i_*\varepsilon_3\mu_{11} = i_*\eta_3\mu_4\sigma_{13} \qquad \text{in} \quad \pi_{20}(Sp(2))$$

in order to prove the second relation.

By Theorem 14.1 of [11] we have $\varepsilon\mu = \eta\mu\sigma$. As the kernel of $E^\infty: \pi_{20}(S^3:2) \to (G^{17}:2)$ is $\bar{\varepsilon}'$, we obtain

$$i_*\varepsilon_3\mu_{11} \equiv i_*\eta_3\mu_4\sigma_{13} \qquad \text{mod } i_*\bar{\varepsilon}' \qquad \text{in} \quad \pi_{20}(Sp(2)).$$

Here $i_*\bar{\varepsilon}' = 0$ in $\pi_{20}(Sp(2))$. This shows the above relation.

In the exact sequence $(2.1)_2$:

$$\pi_{12}(S^{11}) \longrightarrow \pi_{20}(Sp(2)) \xrightarrow{i_*} \pi_{20}(Sp(3)) \xrightarrow{p_*} \pi_{20}(S^{11}) \xrightarrow{\Delta} \pi_{19}(Sp(2)),$$

we have that $p_*([\eta_{11}\sigma_{12}]\circ\eta_{19}) = \eta_{11}\sigma_{12}\eta_{19} = \nu_{11}^3 + \eta_{11}\varepsilon_{12} \neq 0$. So, considering $i_*\pi_{20}(Sp(2)) = \{i_*\bar{\mu}_3\}$, we see that

$$\pi_{20}(Sp(3)) = \{i_*\bar{\mu}_3, [\eta_{11}\sigma_{12}]\circ\eta_{19}\} \approx Z_2 \oplus Z_2.$$

We have also

$$\pi_{20}(Sp(4)) = \{i_*\bar{\mu}_3, i_*[\eta_{11}\sigma_{12}]\circ\eta_{19}\} \approx Z_2 \oplus Z_2,$$

which follows from the exactness of the sequence $(2.1)_3$:

$$\pi_{21}(S^{15}) \xrightarrow{\Delta} \pi_{20}(Sp(3)) \longrightarrow \pi_{20}(Sp(4)) \longrightarrow \pi_{20}(S^{15}) = 0\,,$$

since $\Delta(\nu_{15}^2) = \Delta(\nu_{15}) \circ \nu_{17} \subset \pi_{17}(Sp(3)) \circ \nu_{17} = 0$ for a generator ν_{15}^2 of $\pi_{21}(S^{15})$.

Consider the exact sequence $(2.1)_4$:

$$\pi_{21}(S^{19}) \xrightarrow{\Delta} \pi_{20}(Sp(4)) \longrightarrow \pi_{20}(Sp(5)) \longrightarrow \cdots\,,$$

where $\pi_{21}(S^{19}) = \{\eta_{19}^2\} \cong Z_2$ and $\pi_{20}(Sp(4)) = \{i_*[\eta_{11}\sigma_{12}] \circ \eta_{19},\ i_*\bar{\mu}_3\}$.

We have $\Delta(\eta_{19}^2) = i_*[\eta_{11}\sigma_{12}] \circ \eta_{19}$ by (4.5). Thus

$$\pi_{20}(Sp(5)) = \{i_*\bar{\mu}_3\} \cong Z_2$$

and $$\pi_{20}(Sp(n)) = \{i_*\theta_{19} \circ \eta_{19} = i_*\bar{\mu}_3\} \cong Z_2\,, \qquad \textit{for} \quad n \geq 5\,.$$

By Theorem 2.1 we have

$$\pi_{21}(Sp(5)) = \{i_*\eta_3\bar{\mu}_4\} \cong Z_2\,,$$

since $\eta_3\bar{\mu}_4 = \bar{\mu}_3\eta_{20}$. Hence we have

$$\pi_{21}(Sp(n)) = \{i_*\theta_{19} \circ \eta_{19}^2 = i_*\eta_3\bar{\mu}_4\} \cong Z_2\,, \qquad \textit{for} \quad n \geq 5\,.$$

For $\Delta : \pi_{22}(S^{11}) \rightarrow \pi_{21}(Sp(2))$ and $p_* : \pi_{21}(Sp(2)) \rightarrow \pi_{21}(S^7)$ we have

$$\begin{aligned} p_*\Delta(\zeta_{11}) &= \nu_7 \circ \zeta_{10} && \text{by (4.1)} \\ &= E^2\sigma''' \circ \sigma_{14} && \text{by Lemma 9.2 of [11]} \\ &= 4(\sigma'\sigma_{14}) && \text{by Lemma 5.14 of [11]} \\ &\in \pi_{21}(S^7 : 2) \end{aligned}$$

So we obtain, by the exactness of $(2.1)_1$,

$$\Delta\zeta_{11} = 4[\sigma'\sigma_{14}] \quad \textit{or} \quad \Delta\zeta_{11} = 4[\sigma'\sigma_{14}] + i_*\eta_3\bar{\mu}_4\,.$$

In either case, $\Delta\zeta_{11}$ is of order 8 in $\pi_{21}(Sp(2))$.

Assume that $\Delta\zeta_{11} = 4[\sigma'\sigma_{14}] + i_*\eta_3\bar{\mu}_4$, then we have

$$\pi_{21}(Sp(3):2) = \{i_*[\sigma' \circ \sigma_{14}]\} \cong Z_8\,.$$

But this contradicts that $\pi_{21}(Sp(5))$ is generated by $\theta_{19} \circ \eta_{19}^2 = i_*\eta_3\bar{\mu}_4$. Whence we have $\Delta\zeta_{11} = 4[\sigma'\sigma_{14}]$ and have proved Lemma 3.3.

Easily we have

$$\pi_{21}(Sp(3):2) = \{[\sigma'\sigma_{14}],\ i_*\eta_3\bar{\mu}_4\} \cong Z_4 \oplus Z_2\,.$$

We shall prove Lemma 3.4. We obtain by use of (4.3)

$$\Delta(2\sigma_{15}) = 2[4\nu_{11}]\circ\sigma_{14} = [2\sigma']\circ\sigma_{14}.$$

We have that

$$p_*([2\sigma']\circ\sigma_{14}) = p_*(2[\sigma'\sigma_{14}]) = 2\sigma'\sigma_{14}.$$

Hence we get, in $\pi_{21}(Sp(2))$,

$$[2\sigma']\circ\sigma_{14} \equiv 2[\sigma'\sigma_{14}] \qquad \bmod \ \{i_*\eta_3\bar{\mu}_4,\ 8[\sigma'\sigma_{14}]\},$$

since the kernel of $p_*: \pi_{21}(Sp(2)) \to \pi_{21}(S^7)$ is generated by $i_*\eta_3\bar{\mu}_4$ and $8[\sigma'\sigma_{14}]$. So in $\pi_{21}(Sp(3))$

$$[2\sigma']\circ\sigma_{14} \equiv 2[\sigma'\sigma_{14}] \qquad \bmod \ i_*\eta_3\bar{\mu}_4.$$

Thus the order of $\Delta\sigma_{15}$ is 4.

Consider the homomorphism:

$$\pi_{22}(S^{15}:3) \xrightarrow{\Delta} \pi_{21}(Sp(3):3) \xrightarrow{p_*} \pi_{21}(S^{11}:3).$$

The last homomorphism p_* is already known to be isomorphic. By Proposition 13.6 and Theorem 13.9 of [11] we have that

$$\pi_{22}(S^{15}:3) = \{\alpha_2(15)\} \cong Z_3 \quad \text{and} \quad \pi_{21}(S^{11}:3) = \{\beta_1(11)\} \cong Z_3.$$

We have

$$\begin{aligned} p_*(\Delta(\alpha_2(15))) &= (p_*(\Delta\iota_{15}))\circ\alpha_2(14), && \text{since } \alpha_2(15) \text{ is a suspension element} \\ &\subset \pi_{14}(S^{11}:3)\circ\alpha_2(14) && \\ &= \{\alpha_1(11)\circ\alpha_2(14)\} && \text{by Proposition 13.6 of [11]} \\ &= \{3\beta_1(11)\} && \text{by Lemma 13.8 of [11]} \\ &= 0. && \end{aligned}$$

Thus $\Delta(\alpha_2(15))=0$. As $\pi_{21}(Sp(3))$ has no 5-components, we have proved Lemma 3.4.

Consider the exact sequence $(2.1)_3$:

$$\pi_{22}(S^{15}:2) \xrightarrow{\Delta} \pi_{21}(Sp(3):2) \longrightarrow \pi_{21}(Sp(4):2) \xrightarrow{p_*} \pi_{21}(S^{15}:2),$$

where $\pi_{21}(Sp(3):2) \cong Z_4 \oplus Z_2 = \{[\sigma'\sigma_{14}],\ i_*\eta_3\bar{\mu}_4\}$. $\pi_{21}(S^{15}) \cong Z_2 = \{\nu_{15}^2\}$.

We have known in §3 that p_* is an epimorphism and the order of the image of Δ is 4. It follows that $\pi_{21}(Sp(4):2)\cong Z_2\oplus Z_2$ is generated by $[\nu_{15}]\circ\nu_{18}$ and $i_*\eta_3\bar{\mu}_4$.

Next we shall prove Lemma 3.5. We have, for $p_*:\pi_{22}(Sp(2))\to\pi_{22}(S^7)$,

$$\begin{aligned}
p_*\Delta\theta' &= p_*\Delta\{\sigma_{11}, 2\nu_{18}, \eta_{21}\}_1 && \text{(see page 141 of [11])}\\
&\subset p_*\{\Delta\sigma_{11}, 2\nu_{17}, \eta_{20}\} && \text{by Theorem 5.2 in 5}\\
&\subset \{p_*\Delta\sigma_{11}, 2\nu_{17}, \eta_{20}\} && \text{by Proposition 1.2 of [11]}\\
&= \{\nu_7\sigma_{10}, 2\nu_{17}, \eta_{20}\} && \text{by (4.1)}\\
&\supset \{\nu_7\sigma_{10}\nu_{17}, 2\iota_{20}, \eta_{20}\} && \text{by Proposition 1.2 of [11]}\\
&= \{\sigma'\nu_{14}^2, 2\iota_{20}, \eta_{20}\} && \text{by (7.19) of [11]}\\
&\supset \sigma'\{\nu_{14}^2, 2\iota_{20}, \eta_{20}\} && \text{by Proposition 1.2 of [11]}\\
&\ni \sigma'\varepsilon_{14} && \text{by (6.1) of [11].}
\end{aligned}$$

We have $p_*\Delta\theta'\equiv\sigma'\varepsilon_{14}$ mod $\pi_{21}(S^7)\circ\eta_{21}=\{\sigma'\bar{\nu}_{14}+\sigma'\varepsilon_{14}, \kappa_7\eta_{21}\}$. By (10.23) of [11] we have $\kappa_9\eta_{23}=\bar{\varepsilon}_9$. The kernel of $E^2: \pi_{22}(S^7:2)\to\pi_{24}(S^9:2)$ is generated by $\sigma'\bar{\nu}_{14}$ and $\sigma'\varepsilon_{14}$. So we have $\kappa_7\eta_{21}=\bar{\varepsilon}_7+a\sigma'\bar{\nu}_{14}+b\sigma'\varepsilon_{14}$, where $a, b=0, 1$. Thus we obtain

$$p_*\Delta\theta' = \sigma'\varepsilon_{14}+x(\sigma'\bar{\nu}_{14}+\sigma'\varepsilon_{14})+y(\bar{\varepsilon}_7+a\sigma'\bar{\nu}_{14}+b\sigma'\varepsilon_{14}),$$

where $x, y=0, 1$. Apply the boundary homomorphism $\Delta:\pi_{22}(S^7)\to\pi_{21}(S^3)$ to the above equality, where $\Delta(\bar{\varepsilon}_7)=\nu'\bar{\varepsilon}_6\neq 0$ and $\Delta(\sigma'\varepsilon_{14})=\Delta(\sigma'\bar{\nu}_{14})=0$ by Proposition 3.2 of [7], and $\Delta p_*\Delta\theta'=0$. It follows that $\Delta\theta'=[\sigma'\varepsilon_{14}]+x([\sigma'\bar{\nu}_{14}]+[\sigma'\varepsilon_{14}])=[\sigma'\varepsilon_{14}]$ or $[\sigma'\bar{\nu}_{14}]$.

In either case we have proved Lemma 3.5. Assume that, $x=0$, then, $\Delta\theta'=[\sigma'\varepsilon_{14}]$. Consider the exact sequence $(2.1)_2$:

$$(4.6)\qquad \pi_{24}(S^{11})\longrightarrow\pi_{23}(Sp(2))\longrightarrow\pi_{23}(Sp(3))\xrightarrow{p_*}\pi_{23}(S^{11}),$$

where $\pi_{24}(S^{11})\cong Z_6\oplus Z_2$, $\pi_{24}(S^{11}:2)=\{\theta'\eta_{23}, \sigma_{11}\nu_{18}^2\}$, $\pi_{23}(Sp(2))\cong Z_2\oplus Z_2\oplus Z_2=\{[\sigma'\mu_{14}], [\sigma'\eta_{14}]\circ\varepsilon_{15}, [\nu_7]\circ\sigma_{10}\nu_{17}^2\}$ and p_* is trivial by Lemma 3.5.

We have
$$\begin{aligned}
\Delta(\theta'\eta_{23}) &= [\sigma'\varepsilon_{14}]\circ\eta_{22} && \text{by the assumption}\\
&= [\sigma'\eta_{14}]\circ\varepsilon_{15} &&\\
\Delta(\sigma_{11}\nu_{18}^2) &= [\nu_7]\circ\sigma_{10}\nu_{17}^2 && \text{by (4.1).}
\end{aligned}$$

Therefore $\pi_{23}(Sp(2))/\Delta\pi_{24}(S^{11})\approx Z_2$ and $\pi_{23}(Sp(3))\approx Z_2$. But this contradicts to the result obtained in §3. Therefore $x=1$, i.e.,

$$\Delta\theta' = [\sigma'\bar{\nu}_{14}].$$

Now it is obvious that $\pi_{22}(Sp(3):2)\approx Z_{32}\oplus Z_2$ is generated by $i_*[\rho'']$ and $i_*[\sigma'\varepsilon_{14}]$.

It follows from Lemma 3.4 that $\pi_{22}(Sp(4):2)\approx Z_{128}$ is generated by $[4\sigma_{15}]$.

Since $\Delta\nu_{19}=[\nu_{15}]\circ\nu_{18}$ is of order 2 in the exact sequence

$$\pi_{22}(Sp(5)) \xrightarrow{p_*} \pi_{22}(S^{19}) \longrightarrow \pi_{21}(Sp(4)),$$

we have that $\pi_{22}(Sp(5):2)\approx Z_{512}$ is generated by $[2\nu_{19}]$.

In the exact sequence (4.6) we have that $\Delta(\theta'\eta_{23})=[\sigma'\bar{\nu}_{14}]\circ\eta_{22}$ and $\Delta(\sigma_{11}\nu_{18}^2)=[\nu_7]\circ\sigma_{10}\nu_{17}^2$. It follows immediately that $\pi_{23}(Sp(3))\approx Z_2\oplus Z_2$ is generated by $i_*[\sigma'\mu_{14}]$ and $i_*[\sigma'\eta_{14}]\circ\varepsilon_{15}$.

The generators of $\pi_{23}(Sp(4))$ and $\pi_{23}(Sp(5))$ are already stated in §3.

§5. Boundary homomorphism and secondary composition

Let Y be a CW-complex with a base point y_0. Let $S^nY=Y\mathbin{\times\!\!\!\!\times} S^n$ the reduced join of Y and the unit n-spheres S^n and let $E^nY=Y\mathbin{\times\!\!\!\!\times} E^n$, where E^n is the unit n-cube bounding S^{n-1}.

For topological pairs (A, B, a_0) and (C, D, c_0), we denote by $\pi(A, B; C, D)$ the set of the homotopy classes of maps $f:(A, B, a_0)\to(C, D, c_0)$.

We have the following exact sequence for an arbitrary topological space X and its subspace A with a base point a_0, as usual:

$$\cdots\longrightarrow\pi(S^{n+1}Y,\ X)\xrightarrow{j_*}\pi(E^{n+1}Y, S^nY; X, A)\xrightarrow{\partial}\pi(S^nY,\ X)\longrightarrow\cdots.$$

Let $p:X\to B$ be a fibre map with a fibre $A=p^{-1}(b_0)$, $b_0\in B$. Then p induces isomorphims $p_*:\pi(E^{n+1}Y, S^nY; X, A)\approx\pi(S^{n+1}Y, B)$ for all $n\geq 0$.

Define a boundary homomorphism $\Delta:\pi(S^{n+1}Y, B)\to\pi(S^nY, A)$ by the commutativity of the following diagram:

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \pi(S^{n+1}Y, X) \arrow[r, "j_*"] \arrow[dr, "p_*"'] & \pi(E^{n+1}Y, S^nY; X, A) \arrow[r, "\partial"] \arrow[d, "p_*", "\cong"'] & \pi(S^nY, A) \arrow[r, "i_*"] & \pi(S^nY, X) \arrow[r] & \cdots \\
& & \pi(S^{n+1}Y : B) \arrow[ur, "\Delta"'] & & &
\end{tikzcd}$$

For this Δ, we have the following

Theorem 5.1. *Let Z be a CW-complex with a base point z_0. Then $\Delta(\alpha\circ E\beta)=\Delta\alpha\circ\beta$ for $\alpha\in\pi(S^{n+1}Y, B)$ and $\beta\in\pi(S^nZ, S^nY)$.*

This theorem is, as it were, a generalization of (2.2) in [7], but the proof is easy and omitted.

We shall prove the following theorem, which is the purpose of this section.

Theorem 5.2. *Assume that $\alpha\circ E\beta=\beta\circ\gamma=0$ for $\alpha\in\pi(S^{n+1}Y, B)$, $\beta\in\pi(S^nZ, S^nY)$ and $\gamma\in\pi(S^nW, S^nZ)$, where Y, Z, W are CW-complexes with base points. Then we have*

$$\Delta\{\alpha, E\beta, E\gamma\}_1\subset\{\Delta\alpha, \beta, \gamma\}.$$

Proof. We denote by Ext (α) an extension of $\alpha: S^{n+1}Y\bigcup_{E\beta}CS^{n+1}Z\to B$ and denote by Coext(γ) a coextension of $\gamma: S^{n+1}W\to S^{n+1}Y\bigcup_{E\beta}CS^{n+1}Z$.

By Proposition 1.7 of [11], any element of $\{\alpha, E\beta, E\gamma\}_1$ can be represented as Ext $(\alpha)\circ E$(Coext (γ)). By Theorem 5.1 we obtain

$$\Delta\,(\mathrm{Ext}\,(\alpha)\circ E\,(\mathrm{Coext}\,(\gamma)))=\Delta\,(\mathrm{Ext}\,(\alpha))\circ\mathrm{Coext}\,(\gamma).$$

We have a commutative diagram, by naturality,

$$\begin{tikzcd}
\pi(S^{n+1}Y\cup CS^{n+1}Z, B) \arrow[r, "\Delta"] \arrow[d, "i_1^*"'] & \pi(S^nY\cup CS^nZ, A) \arrow[d, "i_2^*"'] \\
\pi(S^{n+1}Y, B) \arrow[r, "\Delta"] & \pi(S^nT, A),
\end{tikzcd}$$

where i_1 and i_2 are inclusions: $S^{n+1}Y\to S^{n+1}Y\bigcup_{E\beta}CS^{n+1}Z$ and $S^nY\to S^nY\bigcup_{\beta}CS^nZ$ respectively. Therefore

$$\begin{aligned}
i_2^*\Delta\,(\mathrm{Ext}\,(\alpha)) &= \Delta\,(i_1^*\,(\mathrm{Ext}\,(\alpha))) \\
&= \Delta\,(\mathrm{Ext}\,(\alpha)\circ i_1) \\
&= \Delta\,(\alpha)
\end{aligned}$$

by the definition of the extension. This shows that $\Delta(\mathrm{Ext}(\alpha))$ is an extension of $\Delta\alpha$, that is,

$$\Delta(\mathrm{Ext}(\alpha)\circ E(\mathrm{Coext}(\gamma))) = \mathrm{Ext}(\Delta\alpha)\circ\mathrm{Coext}(\gamma).$$

Therefore we have

$$\Delta\{\alpha, E\beta, E\gamma\}_1 \subset \{\Delta\alpha, \beta, \gamma\},$$

since the indeterminacy subgroups of the right hand side include those of the left hand side. q.e.d.

We shall prove the following special lemma which has been used in the previous sections.

Lemma 5.3. *For the homomorphisms* $\Delta: \pi_{24}(S^{23}) \to \pi_{23}(Sp(5))$, $i_*: \pi_{23}(Sp(4)) \xrightarrow{\cong} \pi_{23}(Sp(5))$ *and* $p_*: \pi_{23}(Sp(4)) \to \pi_{23}(S^{15})$, *we have*

$$p_* i_*^{-1}\Delta(\eta_{23}) = \bar{\nu}_{15}.$$

Proof. The following diagram is commutative:

$$\begin{array}{ccccccc}
\pi_{24}(S^{23}) & \xrightarrow{\Delta} & \pi_{23}(Sp(5)) & \underset{\cong}{\xrightarrow{i_*}} & \pi_{23}(Sp(4)) & \xrightarrow{p_*} & \pi_{23}(S^{15}) \\
\| & & \downarrow p_* & & \downarrow p_* & & \| \\
\pi_{24}(S^{23}) & \xrightarrow{\Delta} & \pi_{23}(Sp(5)/Sp(3)) & \underset{\cong}{\xrightarrow{i_*}} & \pi_{23}(Sp(4)/Sp(3)) & = & \pi_{23}(S^{15}),
\end{array}$$

where $Sp(4)/Sp(3) = S^{15}$ and Δ in the lower sequence is the boundary homomorphism for the bundle $(Sp(6)/Sp(3), p, S^{23} = Sp(6)/Sp(5))$. we remark that the above two injection homomorphisms are isomorphisms since $\pi_{24}(S^{19}) = \pi_{23}(S^{19}) = 0$. $Sp(5)/Sp(3)$ is a bundle over S^{19} with a fibre S^{15}. Then there is a cellular decomposition $S^{15}\cup e^{19}\cup e^{34}$ of $Sp(5)/Sp(3)$ such that the class of the attaching map of e^{19} is $\alpha = \Delta(\iota_{19})$ $(\Delta: \pi_{19}(S^{19}) \to \pi_{18}(S^{15}))$. Furthermore $Sp(6)/Sp(3)$ has a cellular decomposition $Sp(5)/Sp(3)\cup e^{24}\cup e^{38}\cup\cdots$ such that the class of the attaching map of e^{23} is $\Delta(\iota_{23}) \in \pi_{22}(Sp(5)/Sp(3))$.

Here we consider the homotopy groups of dimension up to 24. Then we may consider that

$$Sp(5)/Sp(3) = S^{15} \bigcup_{\alpha} e^{19} \quad \text{and} \quad Sp(6)/Sp(3) = S^{15} \bigcup_{\alpha} e^{19} \bigcup_{\Delta\iota_{23}} e^{23}.$$

We see in §2 and §3 that $\Delta: \pi_{19}(S^{19}) \to \pi_{18}(Sp(4))$ and

$p_*: \pi_{18}(Sp(4)) \to \pi_{18}(S^{15})$ are epimorphisms. It follows that $\Delta: \pi_{19}(S^{19}) \to \pi_{18}(S^{15})$ is an epimorphism and $\alpha = \Delta(\iota_{19})$ is a generator of $\pi_{18}(S^{15}) \cong Z_{24}$.

We see also in §2 and §3 that $\Delta: \pi_{23}(S^{23}) \to \pi_{22}(Sp(5))$ is an epimorphism and $p_*\pi_{22}(Sp(5)) = 6\pi_{22}(S^{19}) = \{2\nu_{19}\} \approx Z_4$. It follows $p_*\Delta(\iota_{23}) = \pm 2\nu_{19}$ for the above $\Delta\iota_{23}$. Then we have that $\Delta\iota_{23}$ is a coextension

$$\Delta\iota_{23} = \text{Coext}\,(\pm 2\nu_{18}) \in \pi_{22}(S^{15} \bigcup_{\alpha} e^{19})$$

of $\pm 2\nu_{18}$. Since $E^{10}: \pi_8(S^5) \to \pi_{18}(S^{15})$ is an isomorphism, there exists an element (generator) α' of $\pi_8(S^5)$ such that $E^{10}\alpha' = \alpha$. $S^{15} \bigcup_{\alpha} e^{19}$ is homotopy equivalent to 10-fold suspension $S^{10}(S^5 \bigcup_{\alpha'} e^9)$ of $S^5 \bigcup_{\alpha'} e^9$. Thus we may consider that $S^{15} \bigcup_{\alpha} e^{19} = S^{10}(S^5 \bigcup_{\alpha'} e^9)$.

Since $\alpha' \circ (\pm 2\nu_8) \in 2\pi_{11}(S^5) = 0$, there exists a coextension

$$\text{Coext}\,(\pm 2\nu_8) \in \pi_{12}(S^5 \cup e^9)$$

of $\pm 2\nu_8$. Then we have

$$\Delta\iota_{23} = \text{Coext}\,(\pm 2\nu_{18}) = E^{10}\,\text{Coext}\,(\pm 2\nu_8) + i_*\beta$$

for some element $\beta \in \pi_{22}(S^{15})$ and the injection $i: S^{15} \subset S^{15} \bigcup_{\alpha} e^{19}$, since $\text{Coext}\,(\pm 2\nu_{18})$ and $E^{10}(\text{Coext}\,(\pm 2\nu_8))$ are both coextensions of the same element $\pm 2\nu_{18}$.

Now consicer $\Delta\eta_{23}$. Then

$$\begin{aligned}
\Delta\eta_{23} &= (\Delta\iota_{23}) \circ \eta_{22} \\
&= (E^{10}\,\text{Coext}\,(\pm 2\nu_8) + i_*\beta) \circ \eta_{22} \\
&= E^{10}(\text{Coext}\,(\pm 2\nu_8) \circ \eta_{12}) + i_*(\beta \circ \eta_{22}) \\
&\in E^{10} i_* \{\alpha, \pm 2\nu_8, \eta_{11}\}_1 + i_*(\beta \circ \eta_{22}) \qquad \text{by Proposition 1.8 of [11].}
\end{aligned}$$

α and ν_5 generates $\pi_8(S^5) \cong Z_{24}$ and $\pi_8(S^5:2) \cong Z_8$. Thus there is an odd integer t such that $t\alpha = \pm\nu_5$. Then, by (6.1) of [11],

$$\begin{aligned}
\{\alpha, \pm 2\nu_8, \eta_{11}\}_1 &= \{\alpha, \pm 2t\nu_8, \eta_{11}\}_1 \\
&\equiv \{\pm t\alpha, 2\nu_8, \eta_{11}\}_1 \qquad \text{mod } G \\
&= \{\nu_5, 2\nu_8, \eta_{11}\}_1 \\
&\equiv \varepsilon_5 \qquad \text{mod } G,
\end{aligned}$$

where $G = \alpha \circ E\pi_{12}(S^8:2) + \pi_{12}(S^5:2) \circ \eta_{12} = \{\sigma''' \circ \eta_{12}\} = 0$ (cf. [11]). We have obtained

$$\Delta\iota_{23} = i_*(\mathcal{E}_{15}+\beta\circ\eta_{22}).$$

Next consider about β. Assume that $\beta\notin E^8\pi_{14}(S^7)=2\pi_{22}(S^{15})$. Then $\beta\in\sigma_{15}+2\pi_{22}(S^{15})$ and we have by Proposition 8.1 of [11]

$$Sq^8 \neq 0 \quad \text{in} \quad S^{15}\bigcup_{\beta} e^{23}.$$

Note that in $Sq^8=0$ in $S^{15}\bigcup_{\alpha} e^{19}\bigcup_{\gamma} e^{23}$, $\gamma=E^{10}\mathrm{Coext}(\pm 2\nu_8)$, since it is a 10-fold suspension of $S^5\bigcup_{\alpha'} e^9\bigcup_{\gamma'} e^{13}$, $\gamma'=\mathrm{Coext}(\pm 2\nu_8)$. Then it is verified without difficulty that

$$Sq^8 \neq 0 \quad \text{in} \quad S^{15}\bigcup_{\alpha} e^{19}\bigcup_{\Delta\iota_{23}} e^{23}$$

and

$$Sq^8H^{15}(Sp(6)/Sp(3)\,;Z_2)\neq 0.$$

Similarly we have $Sq^8H^{15}(Sp(6)/Sp(3)\,;Z_2)=0$ if $\beta\in E^8\pi_{14}(S^7)=2\pi_{22}(S^{15})$. The projection homomorphisms

$$p_*:\ H^i(Sp(6)/Sp(3)\,;Z_2)\longrightarrow H^i(Sp(6)\,;Z_2)$$

are isomorphisms for $i=15$ and 23 since $H^*(Sp(6);Z_2)\cong H^*(Sp(3);Z_2)\otimes H^*(Sp(6)/Sp(3)\,;Z_2)$ and $H^8(Sp(3)\,;Z_2)=0$, By Corollary 13.5 of [1],

$$Sq^8(v_4)=b_2^{4,12}v_6,$$

where v_4 and v_6 are generators of $H^{15}(Sp(6)\,;Z_2)\cong Z_2$ and $H^{23}(Sp(6)\,;Z_2)\cong Z_2$, and $b_2^{4,12}$ is the coefficient of σ_{12} in the expression

$$\sum x_1^2\cdots x_4^2x_5\cdots x_8 = B_p^{4,12}(\sigma_1,\ \cdots,\ \sigma_{12})\equiv\sigma_4\cdot\sigma_8+\sigma_{12} \qquad \mathrm{mod}\ 2.$$

Thus $Sq^8(v_4)=v_6$ and $Sq^8H^{15}(Sp(6)/Sp(3)\,;Z_2)\neq 0$. By the above discussion, we have obtained

$$\beta\equiv\sigma_{15} \quad \mathrm{mod} \quad 2\pi_{22}(S^{15}).$$

Since $2\pi_{22}(S^{15})\circ\eta_{22}=\pi_{22}(S^{15})\circ 2\eta_{22}=0$,

$$\begin{aligned}\Delta\eta_{23} &= i_*(\mathcal{E}_{15}+\beta\circ\eta_{22})\\ &= i_*(\mathcal{E}_{15}+\sigma_{15}\eta_{22})\\ &= i_*\bar{\nu}_{15}.\end{aligned}$$

q.e.d.

Kyoto University

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