J. Math. Kyoto Univ.
3-3 (1964) 347～361.

On the automorphisms of hypersurfaces*

By

Hideyuki MATSUMURA and Paul MONSKY

(Received June 23, 1964)

Introduction. When V is a projective variety, we denote by $\mathrm{Bir}(V)$ the group of birational transformations of V onto itself, by $\mathrm{Aut}(V)$ the group of automorphisms of V (i.e. the group of the biregular transformations of V onto itself), and by $\mathrm{Lin}(V)$ the subgroup of $\mathrm{Aut}(V)$ consisting of the elements induced by the projective transformations of the ambient space which leave V invariant. The last one is obviously an algebraic group, while $\mathrm{Aut}(V)$ has the structure of an "algebraic group with (eventually) countably-infinite number of components".

Let $H_{n,d}$ denote a hypersurface of degree d in the $(n+1)$-dimensional projective space $\boldsymbol{P}_{n+1}$, defined by an equation $f(X_0, X_1, \cdots, X_{n+1})=0$ of degree d. The main results of this memoir are:

(1) *If $H_{n,d}$ is non-singular and $n\geqslant 2$, $d\geqslant 3$, then $\mathrm{Aut}(H_{n,d})$ is finite except the case $n=2$, $d=4$.*

(2) *If $H_{n,d}$ is generic over the prime field and if $n\geqslant 2$, $d\geqslant 3$, then $\mathrm{Aut}(H_{n,d})$ is trivial except the following case: the ground field has characteristic $p>0$ and $n=2$, $d=4$.*

The exception in (1) is a real one, while in (2) it is likely that the theorem holds without exception, though we have to leave the question open. The main part of the proofs consists in showing that $\mathrm{Lin}(H_{n,d})$ is small. For the sake of completeness we have added a few known results.

---

* Major part of this work was done in 1962 at the University of Chicago when the firstnamed arthur was supported by the National Science Foundation, G-19801.

§1. Non-singular hypersurfaces

Let k be an algebraically closed ground field and let $k[X]=k[X_0, X_1, \cdots, X_{n+1}]$ be the homogeneous coordinate ring of $\boldsymbol{P}_{n+1}$. Assume that our hypersurface $H_{n,d}$: $f(X_0, X_1, \cdots, X_{n+1})=0$ is non-singular. This implies that the homogeneous ideal $(f, f_0, \cdots, f_{n+1})$ of $k[X]$ generated by $f(X)$ and by the partial derivatives $f_i(X)=\partial f(X)/\partial X_i$ is irrelevant (i.e. is a primary ideal beloging to the maximal ideal $(X_0, \cdots, X_{n+1})$).

Theorem 1. *If $H_{n,d}$ is non-singular and if $n\geqslant 2$, $d\geqslant 3$, then* $\mathrm{Lin}(H_{n,d})$ *is finite.*

Proof. Since $\mathrm{Lin}(H)$ is an algebraic group, it suffices to show that its connected component $\mathrm{Lin}(H)_0$ is trivial. So we consider a connected algebraic subgroup G of $\mathrm{GL}(n+2, k)$ which contains the scalar matrices $\{\alpha E \mid \alpha \in k^*\}$ and for which the form $f(X)$ is semi-invariant, and we wish to prove $G=\{\alpha E\}$. For this purpose we must distinguish two cases.

Case I. *The characteristic of k is either zero or a prime p not dividing the degree d.*

Let $\mathfrak{g}$ be the Lie algebra of G, identified with the tangent space of G at the origin E. Let $\xi\in\mathfrak{g}$, and let $g=(g_{ij})$ be a variable point of G. Then the g_{ij} are regular functions on G and we can identify ξ with the constant matrix (ξ_{ij}), where $\xi_{ij}=\langle \xi, g_{ij}\rangle$. Since $\dim G=\dim \mathfrak{g}$ and since $G\supset\{\alpha E\mid\alpha\in k^*\}$, G will coincide with $\{\alpha E\}$ if $\mathfrak{g}$ coincides with $\{\beta E\mid\beta\in k\}$.

Since $f(X)$ is semi-invariant under G we have a polynomial identity

$$(1)\qquad f\Big(\sum_{j=0}^{n+1} g_{0j}X_j, \cdots, \sum_{j=0}^{n+1} g_{n+1,j}X_j\Big)=\chi(g)f(X_0, \cdots, X_{n+1})$$

for $g=(g_{ij})\in G$, where χ is a rational character of G. Consider this equation (1) as a relation between the regular functions g_{ij} and χ on G, and apply an arbitrary tangent vector $\xi=(\xi_{ij})\in\mathfrak{g}$ to both sides. Then we obtain

$$(2)\qquad \sum_{i=0}^{n+1} f_i(X)\Big(\sum_{j=0}^{n+1}\xi_{ij}X_j\Big)=c'f(X),\quad c'=\langle\xi,\chi\rangle.$$

Using the Euler identity $f(X)=(1/d)\sum f_i(X)X_i$ and putting $c=c'/d$, we have

$$(3)\qquad \sum_{i=0}^{n+1} f_i(X)\Big(\sum_{j=0}^{n+1}\xi_{ij}X_j-cX_i\Big)=0.$$

Now by the Euler identity we have $(f, f_0, \cdots, f_{n+1})=(f_0, \cdots, f_{n+1})$, and by the hypothesis of non-singularity the depth of this ideal is zero. (For the definition of depth, of. Nagata [8]. For a polynomial ideal it is equal to the affine dimension of the variety defined by the ideal.) Put $\mathfrak{a}_i=(f_0, \cdots, \widehat{f_i}, \cdots, f_{n+1})$, $0\leqslant i\leqslant n+1$. Then depth $\mathfrak{a}_i\geqslant 1$ and depth $(\mathfrak{a}_i, f_i)=0$, therefore depth $\mathfrak{a}_i=1$ because we are dealing with homogeneous ideals. It follows from this and from the unmixedness theorem of Macaulay that

$$\mathfrak{a}_i : f_i=\mathfrak{a}_i\,.$$

Hence we get

$$\sum_{j=0}^{n+1}\xi_{ij}X_j-cX_i\in\mathfrak{a}_i\,.$$

Since $\mathfrak{a}_i$ is generated by forms of degree $d-1>1$, the only linear forms in it is zero. Therefore we conclude $\xi_{ij}=c\delta_{ij}$, i.e. $\xi=cE$, *Q.E.D.*

Case II. *k is of characteristic $p>0$ and $d\equiv 0$ (mod p).*

Since G is generated by its Borel subgroups B, and since each B contains the normal subgroup $\{\alpha E\}$, we may assume that G is solvable. Then G is the semi-direct product of a torus T and a connected unipotent group U. We are going to prove $T=\{\alpha E\}$ and $U=\{E\}$.

1) *The case of a torus.* Assume that a torus T in $\mathrm{GL}(n+2)$ leaves the form $f(X)$ semi-invariant. After a suitable linear substitution of the variables $X_0, \cdots, X_{n+1}$ we may assume that

$$t=\begin{pmatrix}\chi_1(t) & & 0\\ & \chi_2(t) & \\ & & \ddots \\ 0 & & \chi_{n+1}(t)\end{pmatrix}\qquad (t\in T)$$

and $f(\chi_0(t)X_0, \cdots, \chi_{n+1}(t)X_{n+1})=\chi(t)f(X_0, \cdots, X_{n+1})$ where χ_i $(0\leqslant i\leqslant n+1)$ and χ are rational characters of T.

If $f(1, 0, \cdots, 0)\neq 0$ then f constains X_0^d and we have $d\chi_0=\chi$ (we write the product of characters additively). If $f(1, 0, \cdots, 0)=0$ then there exists at least one index i such that $f_i(1, 0, \cdots, 0)\neq 0$ since our $H_{n,d}$ is non-singular, and then f contains $X_0^{d-1}X_i$ and we have $\chi_i+(d-1)\chi_0=\chi$. In any case there exists an index i satisfying $\chi_i+(d-1)\chi_0=\chi$. Similarly, for any index $0\leqslant i\leqslant n+1$ there exists some index $j=j(i)$ with $\chi_j+(d-1)\chi_i=\chi$. Since the character group has no torsion we can easily see that there exists a sequence $i_0=0, i_1, i_2, \cdots, i_r$ such that

$$\begin{gathered} c\chi_0+\chi_{i_1}=\chi, \\ c\chi_{i_1}+\chi_{i_2}=\chi, \\ \cdots\cdots\cdots\cdots\cdots \\ c\chi_{i_r}+\chi_0=\chi, \end{gathered}$$

where $c=d-1$. Eliminating $\chi_{i_1}, \cdots, \chi_{i_r}$ we get $(1-(-c)^{r+1})\chi_0=(1-c+c^2-\cdots+(-c)^r)\chi$, hence $(1-(-c)^{r+1})(d\chi_0-\chi)=0$. Since $c=d-1>1$, we get $d\chi_0=\chi$. Similarly $\chi_0=\chi_1=\cdots=\chi_{n+1}$, and so $T=\{\alpha E\}$.

2) *The unipotents case.* Let U be a connected unipotent algebraic subgroup of $\mathrm{GL}(n+2)$ which leaves $f(X)$ semi-invariant. Since U has no non-trivial rational character, U actually leaves $f(X)$ invariant. By a suitable change of variables we may assume that U is in the upper triangular form

$$u=\begin{pmatrix} 1 & u_{0,1} & \cdots\cdots & u_{0,n+1} \\ & 1 & \cdots\cdots & u_{1,n+1} \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix} \qquad (u\in U).$$

Let $\eta=(\eta_{ij})$ be an arbitrary element of the Lie algebra of U. Then $\eta_{ij}=0$ for $i\geqslant j$. As in the Case I (but using the invariance of f), we obtain an identity

$$\sum_{i=0}^{n} f_i(X)\sum_{j=i+1}^{n+1}\eta_{ij}X_j=0. \tag{4}$$

On the other hand, in the present case the Euler identity shows that $\sum_{i=0}^{n+1} f_i(X)X_i=0$, and the two vectors (with linear forms as components) $(X_0, X_1, \cdots, X_{n+1})$ and $(\sum_{j=1}^{n+1}\eta_{0j}X_j, \sum_{j=2}^{n+1}\eta_{1j}X_j, \cdots, 0)$ are linearly independent over k if $\eta\neq 0$. Now we have the following

Lemma 1. *Let k be a field and let $f_0(X), \cdots, f_{n+1}(X)$ be forms of the same degree d' in $k[X_0, \cdots, X_{n+1}]$. Put $\mathfrak{a}=\sum_{i=0}^{n+1} f_i(X)k[X]$ and assume*

$$\text{i) depth } \mathfrak{a}\leqslant 1, \qquad \text{ii) } \sum_{i=0}^{n+1} f_i(X)X_i=0, \qquad \text{iii) } n\geqslant 2,\ d'\geqslant 2.$$

Then ii) *is the only linear relation between $\{f_i(X)\}$ with linear forms as coefficient. Namely, if $l_0(X), \cdots, l_{n+1}(X)$ are linear forms satisfying $\sum f_i(X)l_i(X)=0$, then there exists a constant $c\in k$ such that $l_0(X)=cX_0, \cdots, l_{n+1}(X)=cX_{n+1}$.*

From this lemma and from (4) we get $\eta=0$, hence $U=\{E\}$, which was to be proved. For the proof of the lemma, we first note that depth $\mathfrak{a}=1$, because if depth $\mathfrak{a}=0$ then as in the proof of Case I we can conclude from $\sum f_i(X)X_i=0$ that $X_0\in(f_1, \cdots, f_{n+1})$, which is absurd. Thus depth $\mathfrak{a}=1$. Without loss of generality we may assume that k is an infinite field. Then there exists a matrix $(s_{ij})\in \mathrm{GL}(n+2, k)$ such that, putting $f_i'=\sum_{j=0}^{n+1} s_{ij}f_j$ $(0\leqslant i\leqslant n+1)$, we have depth $(f_1', \cdots, f_{n+1}')=1$. (Obviously, any "sufficiently general" (s_{ij}) has this property.) Let $(s_{ij})^{-1}=(a_{ij})$. Then $f_i=\sum a_{ij}f_j'$. Put $\sum_i a_{ij}X_i=Y_j$, $\sum_i a_{ij}l_i(X)=h_j(Y)$, $f_j'(X)=G_j(Y)$. We have

$$(*)\quad \sum G_j(Y)Y_j=0, \qquad (**)\quad \sum G_j(Y)h_j(Y)=0.$$

Suppose the vector $(l_0(X), \cdots, l_{n+1}(X))$ is not proportional to $(X_0, \cdots, X_{n+1})$. Then $(h_0(Y), \cdots, h_{n+1}(Y))$ is not proportional to $(Y_0, \cdots, Y_{n+1})$. By renumbering $Y_1, \cdots, Y_{n+1}$ we may assume that $h_0(Y)$ contains Y_1. Then

$$(5)\qquad \sum_{j=1}^{n+1} G_j(Y)(Y_jh_0-Y_0h_j)=0,\ \ Y_1h_0-Y_0h_1\neq 0.$$

Since depth $(G_1, \cdots, G_{n+1})=1$ we see as in Case I that the quadratic form $Y_1h_0-Y_0h_1$ lies in the ideal generated by $G_2, \cdots, G_{n+1}$. This is a contradiction if $d'>2$.

Now we assume $d'=2$. (This case was proved by Prof. M. Nagata for the first time.) Put $\varphi_j=Y_jh_0-Y_0h_j$ $(1\leqslant j\leqslant n+1)$. Then the φ_j are linear combinations of $G_1, \cdots, G_{n+1}$ with coefficients in k. On the other hand φ_j contains Y_jY_1 but does not contain Y_iY_1 $(i\neq j, 0)$, hence $\varphi_1, \cdots, \varphi_{n+1}$ are linearly independent over k. Therefore $G_1, \cdots, G_{n+1}$ are linear combinations of $\varphi_1, \cdots, \varphi_{n+1}$. Then we have

$$\begin{aligned}(f_1'(X), \cdots, f_{n+1}'(X)) &= (G_1(Y), \cdots, G_{n+1}(Y))\\ &= (\varphi_1(Y), \cdots, \varphi_{n+1}(Y))\subset(Y_0, h_0(Y)).\end{aligned}$$

It follows that

$$1=\operatorname{depth}(f_1', \cdots, f_{n+1}')\geqslant\operatorname{depth}(Y_0, h_0(Y))\geqslant n+2-2=n,$$

this contradicts the assumption $n\geqslant 2$. Thus the Lemma and the Theorem 1 are completely proved.

Theorem 2. *Let $H_{n,d}$ $(n\geqslant 2, d\geqslant 3)$ be non-singular. Then* $\operatorname{Aut}(H_{n,d})=\operatorname{Lin}(H_{n,d})$ *except the case $n=2$, $d=4$.*

Proof. If $n\geqslant 3$, any positive divisor on the non-sigular $H_{n,d}$ is cut out by a hypersurface of $\boldsymbol{P}_{n+1}$ according to a theorem of Severi-Lefschetz-Andreotti ([1], [5]). Therefore the linear system L_1 of hyperplane sections on $H_{n,d}$ is complete and is a unique base of the additive semi-group of the linear equivalence classes of positive divisors. Hence L_1 is invariant under $\operatorname{Aut}(H_{n,d})$. From this it follows easily that $\operatorname{Aut}(H_{n,d})=\operatorname{Lin}(H_{n,d})$.

If $n=2$, anyway L_1 is complete because the non-singular $H_{n,d}$ is projectively normal. If $n=2$ and $d=3$ then L_1 is the anti-canonical system $-K$. Hence L_1 is again invariant under $\operatorname{Aut}(H_{n,d})$. If $n=2$ and $d>4$, then the canonical system $K=L_{d-4}$ is invariant under $\operatorname{Aut}(H_{n,d})$. Let $\sigma\in\operatorname{Aut}(H_{n,d})$. If $\sigma L_1\neq L_1$, take a divisor $D\in L_1$. Then $\sigma D-D$ is not ~ 0, while $(d-4)(\sigma D-D)\sim 0$. Put $m=d-4$, $m(\sigma D-D)=(\psi)$. Then the algebraic function $\psi^{1/m}$ defines an unramified covering of $H_{n,d}$, which is a contradiction because the fundamental group $\pi_1(H_{n,d})$ is trivial (Cf. [3], [5]). Therefre L_1 is invariant under $\operatorname{Aut}(H_{n,d})$, and the proof is completed.

Theorem 3. *Non-singular surfaces $H_{2,d}$ in $\boldsymbol{P}_3$ are minimal models for $d \geqslant 4$. Hence we have* $\mathrm{Bir}(H_{2,d}) = \mathrm{Aut}(H_{2,d})$ $(d \geqslant 4)$.

Proof. Assume $d \geqslant 4$. The canonical class K of $H_{2,d}$ is $(d-4)$ times hyperplane section. Hence $l(K) = p_g > 0$. Therefore $H_{2,d}$ is neither rational nor birationally equivalent to a ruled surface. By a fundamental theorem of Castelnuovo-Enriques-Zariski ([12]) $H_{2,d}$ has a minimal model. If $H_{2,d}$ is not minimal then it must contain an exceptional curve C of the first kind. But then $p_a(C) = 0$, $(C^2) = -1$, $2p_a(C) - 2 = (C^2) + (CK)$, hence $-1 = (CK)$. This is impossible because $C > 0$ and $(CK) = (d-4)\deg(C)$. Therefore $H_{2,d}$ is minimal.

Theorem 4. *The group* $\mathrm{Bir}(H_{2,4}) = \mathrm{Aut}(H_{2,4})$ *of a non-singular quartic surface in $\boldsymbol{P}_3$ is discrete (i.e.* $\mathrm{Aut}(H_{2,4})_0 = \{e\}$*), but there exist examples of non-singular $H_{2,4}$ with infinite number of automorphisms.*

Proof. The first assertion follows from [6] (because $p_g > 0$ and $h^{01} = 0$) or simply from the elementary fact that, if V is a normal projective variety on which the linear system of hyperplane sections is complete, then the linear part of $\mathrm{Aut}(V)_0$ (and if V is regular, $\mathrm{Aut}(V)_0$ itself) coincides with $\mathrm{Lin}(V)_0$. The second assertion is classical ([4], [9], [10]). The example of Fano-Severi is as follows (for other examples, see [10] p. 279); We assume that the characteristic is zero. Let F be a non-singular quartic surface in $\boldsymbol{P}_3$ containing a non-sigular curve C of genus 2 and of degree 6. Let C' be a hyperplane section of F. Then $(C^2) = 2$, $(CC') = 6$, $(C'^2) = 4$. Let (t, u), $t > 0$, be a solution of the Pell equation $t^2 - 7u^2 = 1$ and put $|D| = |(t-3u)C + uC'|$. Then we have $(D^2) = 2$, $\deg(D) > 0$, $p_a(D) = 2$, $l(D) \geqslant 3$. One can prove, by the theory of moduli of $K3$-surfaces, that C and C' form a base of the divisor class group $\mathrm{Pic}(F)$ provided that F is sufficiently general. (Cf. [10] p. 275.) Assuming this, it follows that there is no positive cycle X with $(X^2) = -2$. Hence there is no positive cycle X with $(X^2) < 0$. By Riemann-Roch on F we have $l(X) \geqslant 2$ for $X > 0$. In particular, $|D|$ has no fixed component. $|D|$ is irreducible, for otherwise it would be composite with a pencil $\{D_1\}$ and D would be algebaically equivalent to sD_1, $s>1$, hence $2=(D^2)=s^2(D_1^2)$, which is absurd. A generic member of $|D|$ is non-singular, because if it has a singular point Q then Q is a base point of $|D|$ by Bertini (characteristic zero), hence $(D^2)\geqslant 4$, contradiction. By Riemann-Roch we have $l(D\cdot D)=2$, therefore $l(D)=3$. Thus $|D|$ determines a rational surjective mapping $\varphi: F\to \boldsymbol{P}_2$. Since F is not rational φ is not birational, and since $(D^2)=2$ we have $[k(F):k(\varphi(F))]=2$. Therefore there exists an automorphism of $k(F)$ which induces (since F is a minimal model) an automorphism σ of F satisfying $\sigma^2=e$, $D^\sigma=D$. There are infinitely many solutions (t, u) of $t^2-7u^2=1$. If two different solutions (t, u) and (t', u') define one and the same automorphism σ, then we have $|C'^\sigma|=|C'|$, hence $\sigma\in \mathrm{Lin}(F)$. But F depends on 33 parameters. (Proof: Non-singular curves of genus 2 depend on 3 parametes of moduli. On each such curve C there are ∞^2 complete linear systems of degree 6, and each of which defines embeddings of C in $\boldsymbol{P}_4$. By means of generic projection from $\boldsymbol{P}_4$ into $\boldsymbol{P}_3$ we get embeddings of C in $\boldsymbol{P}_3$, which depend on 19 parameters. Therefore non-singular sextic curves of genus 2 in $\boldsymbol{P}_3$ depend on $3+2+19=24$ parameters. Given such a curve C, the linear system L (resp. M) of cubic (resp. quartic) surfaces passing through C has dimension 2 (resp. 11) at least. One can show that a generic member of L is non-singular. Then it is easy to see that a generic member F of M is non-singular. Now on F we have $\dim|C|=2$. Hence F depends on $24+11-2=33$ parameters.) On the other hand it is easy to see that a quartic F' in $\boldsymbol{P}_3$ of which the group $\mathrm{Lin}(F')$ contains an element of order 2 depends on 27 parameters at most (Cf. §2). Thus, if we take a sufficiently general F, then $\mathrm{Aut}(F)$ contains infinitely many elements of order 2.

Remarks. 1. The equality $\mathrm{Bir}(H_{n,d})=\mathrm{Aut}(H_{n,d})$ and the finiteness of this group is obvious if $d>n+2$, because in that case the canonical system on $H_{n,d}$ is ample (cf. [6]).

2. Let Θ be the sheaf of germes of regular sections of the tangent bundle of a vaiety V. Then there is a canonical monomorphism of

the Lie algebra of $\mathrm{Aut}(V)_0$ to $\mathrm{H}^0(V, \Theta)$ ([7]), which is an isomorphism in characteristic zero as is well known. Kodaira-Spencer (Lemma 14.2 of [13]) showed $\mathrm{H}^0(H_{n,d}, \Theta)=0$ ($n\geqslant 2$, $d\geqslant 3$) in the classical case by an analytic method. Our proof of Th. 1 in the classical case is more algebraic. We do not know whether $\mathrm{H}^0(H_{n,d}, \Theta)=0$ in the abstract case.

§2. Generic hypersurfaces

Lek k be the universal domain of characteristic $p\geqslant 0$, and let k_0 be the prime field in k. A hypersuface $H_{n,d}$ is called generic if it is generic over k_0, i.e. if it is defined by a homogeneous equation $f(X_0, \cdots, X_{n+1})=0$ of which the $\binom{n+d+1}{d}$ coefficients are algebraically independent over k_0. A generic $H_{n,d}$ is non-singular.

Theorem 5. *If $H_{n,d}$ is generic and if $n\geqslant 2$, $d\geqslant 3$, then* $\mathrm{Lin}(H_{n,d}) = \{e\}$.

Proof. Putting $m=n+2$, we consider a generic form $f(X)$ of degree $d\geqslant 3$ in $k[X_1, \cdots, X_m]$, where $m\geqslant 4$. We wish to prove that if $A=(a_{ij})\in\mathrm{GL}(m, k)$ leaves $f(X)$ semi-invariant:

$$f(A(X))=\alpha f(X), \quad \alpha\in k^*,$$

then $A=cE_m$ for some $c\in k^*$. Write $A=A_sA_u$, where A_s and A_u commute and are respectively semi-simple and unipotent. Then A_s and A_u also leave $f(X)$ semi-invariant. This is a standard fact from the theory of algebraic groups. So it suffices to consider semi-simple matrices and unipotent matrices.

(I) *Semi-simple case.* Let $A\in\mathrm{GL}(m, k)$ be semi-simiple and assume $f(A(X))=cf(X)$. By a suitable matrix T we bring A into the diagonal form;

$$TAT^{-1}=B=\begin{pmatrix} \alpha_1 E_{r_1} & & 0 \\ & \alpha_2 E_{r_2} & \\ & & \ddots \\ 0 & & & \alpha_s E_{r_s} \end{pmatrix}$$

where E_{r_i} is the unit matrix of size r_i, $\sum_{i=1}^{s} r_i = m$, and $\alpha_i\neq\alpha_j$ $(i\neq j)$.

The centralizer H of B in $\mathrm{GL}(m, k)$ is $\mathrm{GL}(r_1, k)) \times \cdots \times \mathrm{GL}(r_s, k)$. The homogeneous space $\mathrm{GL}(m)/H$ is a variety defined over k_0 and its dimension is $m^2 - \sum r_i^2 = 2\sum_{i<j} r_i r_j$. Consider the natural map $\varphi : \mathrm{GL}(m) \to \mathrm{GL}(m)/H$ and put $\varphi(T) = t$. $TH = \varphi^{-1}(t)$ is a variety defined over $k_0(t)$. Take a point $S \in TH$ which is algebraic over $k_0(t)$. Then $SAS^{-1} = B$ and

$$\mathrm{t.d}(S/k_0) \leqslant \mathrm{t.d.}(t/k_0) \leqslant 2\sum_{i<j} r_i r_j$$

where t.d. means transcendence degree. Put $f(S^{-1}(X)) = g(X)$. Then $g(B(X)) = cg(X)$. We are going to prove that, if $s > 1$ (i.e. if A is not a scalar matrix), then more than $2\sum r_i r_j$ monomials of degree d are missing in $g(X)$. Then, since $f(X) = g(S(X))$, the original form $f(X)$ is not generic, contrary to our assumption. Therefore A must be scalar.

In order to prove the assertion, we change the notation and denote the variables by

$$X_{1,1}, \cdots, X_{1,r_1};\ X_{2,1}, \cdots, X_{2,r_2};\ \cdots;\ X_{s,1}, \cdots, X_{s,r_s}.$$

(X_i) will denote $(X_{i,1}, \cdots, X_{i,r_i})$. Then we can express the equation $g(B(X)) = cg(X)$ as follows: $g(\alpha_1(X_1), \cdots, (\alpha_s(X_s)) = cg((X_1), \cdots, (X_s))$.

Among the monomials of degree d, we consider only those which are divisible by $X_{1,1}^{d-3}$, and compare their coefficients in the equation, trying to find that more than $2\Sigma r_i r_j$ of such monomials are absent. Therefore we may assume $d = 3$. We classify the cubic monomials into four classes as follows.

$C_i = \{\text{cubic in } (X_i)\}$. $\quad \sharp C_i = r_i(r_i+1)(r_i+2)/6$

$C_{ij} = \{\text{quadratic in } (X_i) \text{ and linear in } (X_j)\}$. $i<j$.
$\quad \sharp C_{ij} = r_i r_j (r_i+1)/2$

$D_{ij} = \{\text{linear in } (X_i) \text{ and quadratic in } (X_j)\}$. $i<j$.
$\quad \sharp D_{ij} = r_i r_j (r_j+1)/2.$

$C_{ijl} = \{\text{linear in } (X_i), (X_j) \text{ and } (X_l)\}$. $i<j<l$.
$\quad \sharp C_{ijl} = r_i r_j r_l.$

Here $\sharp C_i$ implies the number of elements in C_i, etc.

C_{ij} and D_{ij} cannot co-exist in $g(X)$ since $\alpha_i^2\alpha_j \neq \alpha_i\alpha_j^2$. Similarly, for each $1 \leqslant i \leqslant s$, at most one out of the classes

$$D_{1i}, \cdots, D_{i-1,i}, C_i, C_{i,i+1}, \cdots, C_{i,s}$$

can appear in $g(X)$. Now, for any pair (i, j) with $1 \leqslant i < j \leqslant s$, we define E_{ij} as follows:

a) If both C_{ij} and D_{ij} are absent in $g(X)$, then

$$E_{ij} = C_{ij} \cup D_{ij}\,.$$

b) If C_{ij} is absent but D_{ij} is present in $g(X)$, then

$$E_{ij} = C_{ij} \cup C_j\,.$$

c) If C_{ij} is present in $g(X)$, then

$$E_{ij} = D_{ij} \cup C_i\,.$$

If $C_i \subset E_{ij} \cap E_{il}$ $(i<j<l)$, then C_{ij} and C_{il} co-exist in $g(X)$, which is impossible. Similarly $C_i \subset E_{ji} \cap E_{li}$ $(j<l<i)$ is impossible. If $C_i \subset E_{ij} \cap E_{li}$ $(l<i<j)$, then both C_{ij} and D_{li} appear in $g(X)$, which is again impossible. Therefore the sets E_{ij} $(i<j)$ are mutually disjoint. On the other hand, one can easily verify

$$\sharp E_{ij} \geqslant 2r_ir_j$$

where the equality can hold only when $r_i = r_j = 1$. Since all E_{ij} are absent in $g(X)$, at least $\sum \sharp E_{ij}$ monomials are missing and we have $\sum \sharp E_{ij} \geqslant 2\sum r_ir_j$, the equality holds only when $s=m$, $r_1=r_2=\cdots=r_m=1$. In this last case, since $m \geqslant 4$, at least one of $X_1X_2X_3$ and $X_1X_2X_4$ is absent in $g(X)$ also. Thus our assertion is proved.

(II) *Unipotent case.* Let $A \in \mathrm{GL}(m, k)$ be a unipotent matrix, $A \neq E$, and let $f(X)$ be a form of degree $d \geqslant 3$ which is semi-invariant under A. Then actually f is invariant under A. We wish to show that $f(X)$ is not generic. Let J be the Jordan normal form of A. We assume that the blocks in J are arranged in the order of increasing size. For $1 \leqslant i \leqslant m-1$ we have $J(X_i) = X_i + \varepsilon_i X_{i+1}$, and $J(X_m) = X_m$, where $\varepsilon_i = 1$ or 0. We say J is of type $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{m-1})$. We shall say that an index i is *regular* if $\varepsilon_i = 1$. We also define a number $\alpha(J)$ by

$$\alpha(J) = \sum \left\{ \binom{i+1}{2} + 1 \right\},$$

where the sum runs over the regular indices of J. The proof of our theorem depends upon the follwing estimates.

Lemma 2. *Let $g(X)$ be a form of degree $d \geqslant 3$ which is transformed into itself by J. Then the coefficients of g satisfy at least $\alpha(J)$ linearly independent linear relations with coefficients in k_0.*

Lemma 3. *Let A be a unipotent matrix with Jordan form J. Then $\alpha(J) >$* t.d.(A/k_0).

Assuming these lemmas, we choose a non-singular matrix T with algebraic coefficients over $k_0(A_{ij})$ such that $TA = JT$. Let $f(X) = g(T(X))$. Then g is transformed into itself by J. Since the coefficients of f depend rationally on the coefficients of g and the A_{ij}, lemmas 1 and 2 tell us that f is not a generic form, proving the theorem.

To prove Lemma 1, order the monomials of degree d lexicographically; $\Pi X_i^{a_i} < \Pi X_i^{b_i}$ if $a_i = b_i$ $(i < s)$, $a_s < b_s$. Now observe that if $\mu = \Pi X_i^{a_i}$, then:

$$\mu(J(X)) = \Pi (X_i + \varepsilon_i X_{i+1})^{a_i} = \mu(X) + \sum_{\nu < \mu} c_{\mu\nu} \cdot \nu \tag{6}$$

In particular, regarded as a transformation on the space spanned by the monomials, J has the form $E + \Delta$, where $\Delta = (c_{\mu\nu})$ is strictly triangnlar. Now suppose that $g = \sum_{\mu} a_\mu \cdot \mu$. Then:

$$\sum_{\mu} a_\mu \cdot \mu + \sum_{\nu} \left(\sum_{\mu} c_{\mu\nu} \cdot a_\mu \right) \cdot \nu = \sum_{\mu} a_\mu \cdot \mu \tag{7}$$

Comparing coefficients we find that $\sum_{\nu < \mu} c_{\mu\nu} \cdot a_\mu = 0$ for every monomial ν of degree d. Thus the coefficients of g satisfy rank $(c_{\mu\nu})$ linearly independent linear equations with coefficients in k_0. If μ is any monomial, let μ' be its predecesar in the lexicographic order. Say that μ is *regular* for J if $c_{\mu\mu'} \neq 0$. Since $(c_{\mu\nu})$ has strict triangular form, rank $(c_{\mu\nu})$ is at least equal to the number of regular μ. Thus we must show:

Lemma 4. *There are at least $\alpha(J)$ regular monomials.*

Proof. Suppose s is a regular index for J, and let $\mu=\left(\prod_{i=1}^{s} X_i^{a_i}\right)\cdot X_m^{a_m}$. Then $\mu'=(X_{s+1}/X_s)\cdot\mu$ and we see easily that $c_{\mu\mu'}=a_s$. In particular, if the characteristic p does not divide a_s then μ is regular. Now fix a regular index s. If $a_s=1$, the number of monomials of the form $\prod_{i=1}^{s} X_i^{a_i}\cdot X_m^{a_m}$ is just the number of monomials of degree $d-1$ in $X_1,\cdots,X_{s-1}$ and X_m, i.e. $\binom{s+d-2}{d-1}$. Furthermore, since $d\geqslant 3$, there is a regular monomial of the form $X_s^2X_m^{d-2}$, or $X_s^3X_m^{d-3}$, depending on the characteristic. Thus there are at least $\sum_s\left(\binom{s+d-2}{d-1}+1\right)$ regular monomials in all where s runs over the regular indices for J. Since this function is monotonic increasing in d, and $d\geqslant 3$, the lemma is proved.

The idea behind the proof of Lemma 3 is the following. Each regular index in J gives a contribution of about $m^2/2$ to $\alpha(J)$, and t.d.$(k_0(A_{ij})/k_0)\leqslant m^2$. Thus if there are three regular indices, things are easy. If there are fewer than three regular indices, one gets finer estimates on $k_0(A_{ij})/k_0$ which again establish the lemma. The actual proof involves consideration of four separate cases.

We begin with a lemma giving an upper bound for t.d.$(k_0(N_{ij})/k_0)$ where N is a nilpotent matrix.

Suppose that N is an m by m nilpotent matrix. Let $V_i=$ image of N^i, and $\beta_i=\dim V_i$. Then $\beta_0>\beta_1>\beta_2>\cdots$. We say that N is of type $(\beta_0,\beta_1,\cdots)$. Let $\beta(N)=2\sum_0^\infty(\beta_i-\beta_{i+1})\beta_{i+1}$. The following lemma is basic:

Lemma 5. *Let the notation be as above. Then*

$$\text{t.d.}(k_0(N_{ij})/k_0)\leqslant\beta(N).$$

Proof. N is determined by the subspace V_1 of V_0, by its restriction to V_1 and by the images in V_1 of a set of generators of V_0/V_1 under N. V_1 depends on at most $(\beta_0-\beta_1)\cdot\beta_1$ parameters, the same holds true for the images of the $\beta_0-\beta_1$ generators of V_0/V_1. An induction argument now gives the desired result, since N restricted to

V_1 is of type $(\beta_1, \beta_2, \cdots)$.

We are now ready to prove Lemma 3. Let $A=E+N'$, $J=E+N$. Then $\mathrm{t.d.}(k_0(A_{ij})/k_0) \leqslant \beta(N') = \beta(N)$. There are four cases to consider, according to the maximum of the sizes of the blocks of J. (Remember that the blocks are arranged in the increasing order of size.)

(1) If J is of type $(\cdots 1, 1, 1)$ then $\mathrm{t.d.}(k_0(A_{ij})/k_0) \leqslant m^2 - m$. For $\beta(N) \leqslant m^2 - m$ always.

(2) If J is of type $(\cdots 0, 1, 1)$ then $\mathrm{t.d.}(k_0(A_{ij})/k_0) \leqslant \frac{2}{3} m^2$. For $N^3=0$ in this case, and N is of type $(m, \gamma, \delta, 0)$.

(3) If J is of type $(\cdots 1, 0, 1)$ then $\mathrm{t.d.}(k_0(A_{ij})/k_0) \leqslant \frac{1}{2} m^2$. For $N^2=0$ in this case, and N is of type $(m, \gamma, 0)$.

(4) If J is of type $(\cdots 0, 0, 1)$ then $\mathrm{t.d.}(k_0(A_{ij})/k_0) \leqslant 2m-2$. In this case, N is of type $(m, 1, 0)$.

Now let us estimate $\alpha(J)$ in the 4 cases. In cases (1), $m-1$, $m-2$ and $m-3$ are all regular for J. Thus $\alpha(J) \geqslant \binom{m}{2} + \binom{m-1}{2} + \binom{m-2}{2} + 3$. The other cases are similar. Thus we are reduced to proving the following four inequalities. If $m \geqslant 4$, then:

(1) $\binom{m}{2} + \binom{m-1}{2} + \binom{m-2}{2} + 3 > m^2 - m$,

(2) $\binom{m}{2} + \binom{m-1}{2} + 2 > \frac{2}{3} m^2$,

(3) $\binom{m}{2} + \binom{m-2}{2} + 2 > \frac{1}{2} m^2$,

(4) $\binom{m}{2} + 1 > 2m-2$.

These just squeak through. This completes the proof of Lemma 3.

Combining Theorems 2 and 5, we see that $\mathrm{Aut}(H_{n,d}) = \{e\}$ for a generic $H_{n,d}$ except the case $n=2$, $d=4$. On the other hand, in characteristic zero there is a theorem of M. Noether-Andreotti-Salmon ([1]) according to which the Picard group of a generic surface of $\deg \geqslant 4$ in $\boldsymbol{P}_3$ is generated by the hyperplane section. This proves $\mathrm{Aut}(H_{2,4}) = \mathrm{Lin}(H_{2,4}) = \{e\}$ for generic quartics in characteristic zero.

It seems that the theorem of Noether-Andreotti-Salmon can be extended to the positive characteristic case by modifying the existing proofs. We wish to come back to this problem in future.

BIBLIOGRAPHY

[1] A. Andreotti and P. Salmon: *Anelli con unica decomponibilità in fattori primi ed un problema di intersezioni complete*. Monatshefte f. Math. 61 (1957).

[2] A. Borel: *Groupes linéaires algébriques*. Ann. of Math. 64 (1956).

[3] W. L. Chow: *On the fundamental group of an algebraic variety*. Amer. J. Math. 74 (1952).

[4] G. Fano: *Sopra alcune superficie del 4° ordine rappresentabili sul piano doppio*. Rendiconti del Reale Istituto Lombardo di Scienze e Lettere, 39 (1906).

[5] A. Grothendieck: *Séminaire de Géométrie Algébrique*. IHES 1962. Exposé 12.

[6] H. Matsumura: *On algebraic groups of birational transformations*. Rendiconti Accad. Naz. Lincei 34 (1963).

[7] H. Matsumura and F. Oort: *On the automorphism group schemes of algebraic schemes*. To appear.

[8] M. Nagata: *Local Rings*. Interscience Publishers. (1962) New York.

[9] F. Severi: *Complementi alla teoria della base per la totalità delle curve di una superficie algebrica*. Rendiconti Cir. Mat. Palermo 30 (1910).

[10] ——: *Geometria dei sistemi algebrici* ... Vol. II, Cap. V. (1959) Roma.

[11] B. L. v. d. Waerden: *Moderne Algebra*. Vol. II.

[12] O. Zariski: *The problem of minimal models in the theory of algebraic surfaces*. Amer. J. Math. 80 (1958).

[13] K. Kodaira and D. C. Spencer: *On deformations of complex analytic structures*, II. Ann. of Math. 67 (1958).