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On orbit spaces by torus groups

By

Kayo OTSUKA

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Let V be an affine variety with universal domain K and let T be a torus acting on V in the usual sense.

Consider the set U of points of V whose orbits are of maximal dimension. Then we can think of orbit space U/T, which may not be a variety in general but is a prescheme. For simplicity, we denote by V/T the orbit space U/T. Let R be the coordinate ring of V over K. Then T acts also on R. The set $I_T(R)$ of T-invariants in R is finitely generated over K, hence defines an affine variety W.

The main result of our present article is that V/T is covered by a finite number of projective varieties over W.[1)2)]

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1. Formulation of the result

Let V be an affine variety with coordinate ring $R=K[x_1, \cdots, x_n]$. A variety X is called a *projective variety over* V if there is a set of elements $u_0, \cdots, u_m$ of a field containing R such that X is covered by affine varieties X_i defined by $R[u_0/u_i, \cdots, u_m/u_i]$ $(i=0, 1, \cdots, m)$. If one u_i (hence every u_j which is not zero) is transcendental over the function field of X then $R[u_0, \cdots, u_m]$ is called a *homogeneous coordinate ring* of X. $R[u_0, \cdots, u_m]$ is a graded ring in which (1)

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1) The definition will be recalled in §1 below.

2) Though we treat the case of usual varieties for the simplicity of formulation, this can be adapted easily to the case of affine schemes whose rings are finitely generated over K. The reason is that Theorem 2. 1 in [2] can be adapted to the case.

elements of R are of degree 0 and (2) the u_i are of degree 1. For the necessity in order to apply induction argument on the dimension of a torus to treat with, and for a generality in appearance, we consider the case where a torus T acts on the projective variety X over the affine variety V.

The action of T is assumed to induce an automorphism group of a homogeneous coordinate ring $R[u_0, \cdots, u_m] = K[x_1, \cdots, x_n, u_0, \cdots, u_m]$ of X. Under the circumstance, we may assume that x_i and u_j are all T-semi-invariants, because every rational T-module is generated by semi-invariants. Then, in particular, T acts on each affine variety X_i, where we can think of X_i/T in the sense we stated in the introduction. Therefore we can consider orbit space X/T,[3] as a prescheme which is covered by X_i/T.

On the other hand, we consider the set $I_T(R)$ of T-invariants in R. This is an affine ring over K, hence defines an affine variety W. Now our main theorem is formulated as follows:

Main Theorem. *X/T is covered by a finite number of projective varieties over W.*

Excat meaning of this theorem is that: if $P \in X$ is such that its orbit has a maximal dimension, then there is a T-stable open subset U of X, such that U/T exists in the usual sense and such that U/T is a projective variety over W.

2. Preliminary lemmas.

Before proving the theorem, we explain some lemmas. One basic result we use often in this article is the following well known fact:[4]

Lemma 1. *Let a torus T acts on an affine variety V with coordinate ring R. If every T-orbit on V is closed, then V/T is the affine variety defined by $I_T(R)$. In particular, if dim $T=1$, and if there is no T-invariant point on V, then V/T is the affine variety defined by $I_T(R)$. In the general case, the set of closed*

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3) X/T is not the set of all orbits but is the set of orbits of maximal dimension.

4) The first and the last assertion can be generalized to the case where T is a semi-reductive algebraic group, see [3].

orbits on V is naturally identified with the affine variety defined by $I_T(R)$.

Lemma 2. *Let V be an affine variety and let T be a torus acting on V. Then for $P\in V$ the orbit PT has a maximal dimension if and only if the dimension of PT is equal to the dimension of T/H, where $H=\{\sigma\mid\sigma\in T, P^{\sigma}=P \text{ for } {}^{\forall}P\in V\}$.*

Proof. We may assume that $H=\{1\}$ and that every element t of the torus T is a diagonal matrix. Let $H_P=\{t\mid P^t=P, P\in V\}$, then $\dim PT=\dim T-\dim H_P$. If $t=\begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n\end{pmatrix}\in H_P$ and $P=(p_1,\cdots,p_n)$, then $P^t=(p_1t_1, p_2t_2, \cdots, p_nt_n)=P$. So if $p_i\neq 0$ then $t_i=1$. Namely, if $\Pi p_i\neq 0$, then $H_P=\{1\}$ hence $\dim H_P=0$. It is clear that the orbit PT has a maximal dimension.

3. Reduction to one dimensional case

Now we shall go back to our main theorem. We may assume that some orbits have dimension equal to $\dim T$. We use the induction argument on the dimension of T. Let T_1 and T_2 be tori such that $T=T_1\times T_2$ with $\dim T_1=1$, $\dim T_2=\dim T-1$. Let $I_{T_2}(R)=T_2$-invariants in R. Then by the induction assumption we can assume that the orbit space X/T_2 is covered by a finite number of projective varieties X_i over W', where W' is the affine variety defined by $I_{T_2}(R)$. On the other hand we can see that $X/T=(X/T_2)/T_1=UX_i/T_1$ and $I_T(R)=I_{T_1}(I_{T_2}(R))$. Therefore if we can prove that each X_i/T_1 is covered by projective varieties over W, then our proof come to an end. Namely it is sufficient to prove the assertion in the case where the dimension of T is one.

4. One dimensional case

From now on, we shall assume that $\dim T=1$. Let $P\in V$ be such that $\dim PT=1$. When f is a T-semi-invariant, f defines a character χ so that $f^{\sigma}=\chi(\sigma)f$. Since T is a torus of dimension 1, there is an isomorphism t from T onto multiplicative group of K and

$\chi=t^{\alpha}$ with a natural number α. α is called the exponent of χ. Now we take one of u_i $(i=0, 1, \cdots, m)$ whose character has minimal exponent, say u_0. Then we may assume that u_0 is a T-invariant and then the character defined by each u_i has non-negative exponent.

When M is a homogeneous element of positive degree, say d, of $R[u_0, \cdots, u_m]$, we denote by R_M the affine ring of the affine variety $X-$(closed set defined by $M=0$), which is denoted by X_M. Namely, R_M is the ring generated by all elements of the form (homogeneous form of degree d)$/M$.

We call a monomial $M=u_{i_1}^{\beta_1}\cdots u_{i_r}^{\beta_r}x_{j_1}^{\gamma_1}\cdots x_{j_s}^{\gamma_s}$ is of *type* (1) if x_{j*} $(*=1, \cdots, s)$ are invariants and $\alpha_{i_1}=\cdots=\alpha_{i_r}$ where α_{i_k} are exponents of character defined by u_{i_k} $(k=1, \cdots, r)$. We call M is of *type* (2) when M is not of type (1).

Lemma 3. *If M is of type* (2), *then X_M has no fixed point.*

Proof. Assume that $M=u_{i_1}^{\beta_1}\cdots u_{i_r}^{\beta_r}x_{j_1}^{\gamma_1}\cdots x_{j_s}^{\gamma_s}$ and assume that x_{j_1} is not invariant. If there is a fixed point in X_M, then the proper semi-invariants in R_M can be specialized to zero simultaneously on X_M. But x_{j_1} can not be zero on X_M if $s\geqslant 1$. Otherwise, there is a pair (k, l) such that $\beta_k\neq\beta_l$ $(k, l\leqslant r)$ and the proper semi-invariant u_{i_k}/u_{i_l} can not be zero on X_M. Therefore X_M has no fixed point.

Let X be a projective variety over V and let $K[x_1, \cdots, x_n, u_0, \cdots, u_m]$ be a homogeneous coordinate ring of X, where the degree of each x_i is zero and the degree of each u_j is 1.

Let $\boldsymbol{P}$ be a projective variety defined by $I_T(R)[M_0, \cdots, M_1]$ where M_k are the monomials on x and u which have same character and of the same degree (in u).

We consider the set $\mathfrak{M}_{\boldsymbol{P}}$ of monomials M_i such that $\boldsymbol{P}_{M_i}=\boldsymbol{P}-$(the closed set in $\boldsymbol{P}$ defined by $M_i=0$) is defined by $I_T(R_{M_i})$. Now we consider the union of such affine open set $X_{M_i}(M_i\in\mathfrak{M}_{\boldsymbol{P}})$, and denote it by $U_{\boldsymbol{P}}$.

Lemma 4. *When $P\in V$ is given so that $\dim PT=1$, then there is a $\boldsymbol{P}$ such that $P\in U_{\boldsymbol{P}}$.*

Proof. Since P is not a fixed point, there is a monomial M of positive degree and of type (2) such that $M(P)\neq 0$. Then, we consider $I_T(R_M)$. This is generated by a finite number of elements of the form M_i/M^γ (M_i being monomials, $i=1,\cdots,t$). Then the projective variety $\boldsymbol{P}$ with homogeneous coordinate ring $I_T(R)[M^\gamma, M_1,\cdots,M_t]$ contains a point which corresponds to the orbit of P.

We consider the set of $\boldsymbol{P}$ such that $P\in U_{\boldsymbol{P}}$ and chose a member in the set which has a maximal $U_{\boldsymbol{P}}$. We denote it again by the same $\boldsymbol{P}$. Then we wish to prove that $\boldsymbol{P}\subseteq U_{\boldsymbol{P}}/T$.

Assume that $\boldsymbol{P}\ni Q\notin U_{\boldsymbol{P}}/T$ and assume that $M_j(Q)\neq 0$. First we consider the case where M_j is of type (2). Then by Lemma 3, X_{M_j} has no fixed point. We consider $I_T(R_{M_j})$. This is generated by elements of the form M'/M_j^γ where M' is of same degree and defines the same character as M_j^γ. Let a set of generators be $M_0'/M_j^\gamma,\cdots, M_r'/M_j^\gamma$. Next we consider the projective variety $\boldsymbol{P}'$ which defined by $M_0',\cdots,M_r'$, and all monomials of degree γ in $M_0,\cdots,M_t$, say $M_{r+1}',\cdots,M_s'$. Now we can see that $M_0'/M_j^\gamma,\cdots,M_s'/M_j^\gamma$ are all T-invariants, hence $I_T(R_{M_j})=K[M_0'/M_j^\gamma,\cdots,M_s'/M_j^\gamma]$. Therefore $M_j^\gamma\in\mathfrak{M}_{\boldsymbol{P}'}$. On the other hand, when $M_i\in\mathfrak{M}_{\boldsymbol{P}}$, then $M_i^\gamma\in\mathfrak{M}_{\boldsymbol{P}'}$ and $\boldsymbol{P}_{M_j}=\boldsymbol{P}'_{M_j^\gamma}$ as is easily seen. Therefore $U_{\boldsymbol{P}'}\subsetneqq U_{\boldsymbol{P}}$, and then this fact contradicts to the maximality of $\boldsymbol{P}$. Next consider the case where M_j is of type (1). We consider the set $A=\{M_iM_k\,|\,M_i$ or M_k is of type (2)$\}$, and let $\boldsymbol{P}'$ be the projective variety over W with homogeneous coordinates $\{M_iM_k\,|\,M_iM_k\in A\}$. Then all members of A are of type (2). If $M_j\in\mathfrak{M}_{\boldsymbol{P}}$, then $M_j^2\in A$, and $\boldsymbol{P}'_{M_j^2}=\boldsymbol{P}_{M_j}$ as is easily seen. Thus we can reduce to the first case, and we complete the proof.

5. Remarks

Orginal motivation of the present study was to observe the following question:

Let G be a connected linear algebraic group and let H be an algebraic subgroup of G. Is it true that G/H has no everywhere regular non-constant rational function if and only if G/H is a projective variety (i.e., if and only if H contains a Borel subgroup of G)?

As will be shown later, the answer of this question is not affirmative. But, because of the following lemma, we see a rather close connection between the above question and our main result, as will be shown below.

Lemma. 5. *When G acts rationally on a module M, then an element a of M is G-invariant if and only if a is B-invariant with a suitable Borel subgroup B of G.*

Proof. We may assume that M is a finite module over the universal domain K of G, and we regard M as an affine space on which G acts. Then G-orbit of a is quasi-affine. On the other hand, since a is B-invariant, G-orbit of a is projective. Hence a is G-invarisnt.

Now, in the above question, we can replace H with its connected component of the identity, and we assume that H is connected. Then, applying Lemma 5 (for H acting on G), we see that G/H has a non-constant regular function if and only if G/B_H does with a Borel subgroup B_H of H. Let U be the unipotent part of B_H. Then G/U is a quasi-affine varity (see [1]) on which the torus B_H/U acts. Thus G/U is an open subset of an affine variety V on which B_H/U acts. Therefore we see that, under the assumption that G/H is not a projective variety,

Proposition 1. *If V can be chosen so that every point P of V which is not in G/U has B_H/U-orbit whose dimension is less than* $\dim B_H/U$, *then G/H has a non-constant regular function.*

Now, we shall give a counter-example to the question stated above. Set $G=SL(3,K)$ and let H be the subgroup of G consisting of all matrices of the form

$$\begin{pmatrix} t & a & b \\ 0 & t & c \\ 0 & 0 & t^{-2} \end{pmatrix}.$$

H is properly contained in a Borel bubgroup of G. With this pair of G and H, we have:

Proposition 2. *The facter space $G/H=\{gH \mid g\in G\}$ has no* *non-constant everywhere regular rational function.*

Proof. Let

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

be a generic point of G. Then the affine ring R of G is $K[x_{11}, \cdots, x_{33}]$ with the unique relation det $|x_{ij}|=1$. Let H_u be the unipotent part of H. We first consider H_u-invariants in R (under the right multiplication by elements of H_u). Obviously, x_{11}, x_{21}, x_{31},

$$y_1=\begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix}, \quad y_2=\begin{vmatrix} x_{31} & x_{32} \\ x_{11} & x_{12} \end{vmatrix}, \quad y_3=\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$$

are H_u-invariants. We want to show that

Lemma 6. $K[x_{11}, x_{21}, x_{31}, y_1, y_2, y_3]$ *is the set of H_u-invariants in R.*

Proof of the Lemma 6. One sees easily that if, for two $P, Q\in G$, $x_{i1}(P)=x_{i1}(Q)$ and $y_i(P)=y_i(Q)$ for $i=1, 2, 3$, then $PH_u=QH_u$. Therefore these x_{11}, x_{21}, x_{31}, y_1, y_2, y_3 separates all cosets from each other. Since $\dim G/H_u=8-5$, the obvious relation $\sum x_{i1}y_i=0$ is a unique relation for these elements, and we see that $K[x_{11}, x_{21}, x_{31}, y_1, y_2, y_3]$ is a normal ring. Furthermore, $K(x_{11}, x_{21}, x_{31}, y_1, y_2, y_3, x_{12}, x_{13}, x_{23})$ is the function field $K(G)$ of G, as is easily seen. Thus $K(G)$ is purely transcendental over $K(x_{11}, x_{21}, x_{31}, y_1, y_2, y_3)$ and therefore the normality of $K[x_{11}, x_{21}, x_{31}, y_1, y_2, y_3]$ implies that this affine ring is integrally closed in $K(G)$. Thus we prove the lemma.

Now we go back to the proof of Proposition 2. The action of H on R induces and action of the torus H/H_u. We denote by (t) the class of

$$\begin{pmatrix} t & a & b \\ 0 & t & c \\ 0 & 0 & t^{-2} \end{pmatrix}$$

in H/H_u. Then $x_{i1}(t)=tx_{i1}$ and $y_i(t)=ty_i$. Therefore there is no non-constant H-invariant in $K[x_{11}, x_{21}, x_{31}, y_1, y_2, y_3]$. Since H-

invariants are H_u-invariants, we complete the proof of the proposition.

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Kyoto Prefectural University