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A remark on square integrable analytic semiexact differentials on open Riemann surfaces

Dedicated to Professor A. Kobori on his 60th birthday

By

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1. For canonical homology basis $\{A_n, B_n\}_{n=1,2,\cdots}$ on an open Riemann surface the necessary and sufficient conditions for the existence of a square integrable analytic semiexact differential with given A-periods were investigated by Virtanen [1], Kusunoki [2] and Sainouchi [3]. In this paper we shall give a condition for the uniqueness of the existence of such differentials, which contains my previous result in [3]. In part, we make use of the same method as that in the Ahlfors' proof (Ahlfors [4], Theorem 9) giving the condition which the surface should belong to the class O_{AD}.

2. Let $\bar{W}$ be a compact bordered Riemann surface of genus g and $\{A_i, B_i\}_{i=1,2,\cdots,g}$ be a cononical homology basis mod ∂W. We denote by $\Gamma_{ase}(\bar{W})$ the class of analytic semiexact differentials defined on W and also denote by $\Gamma_{ase}^A(\bar{W})$ the subclass of $\Gamma_{ase}(\bar{W})$ such that all A-periods of its element vanish. For the compact bordered surface $\Gamma_{ase}^A(\bar{W})\neq\{0\}$ and the period $\int_c \alpha$ $(\alpha\in\Gamma_{ase}^A(\bar{W}))$ to any chain c in W is the bounded linear functional on $\Gamma_{ase}^A(\bar{W})$, hence there exists a unique differential $\varphi_0(c)\in\Gamma_{ase}^A(\bar{W})$ such that

$$(\alpha, \varphi_0(c)) = 2\pi\int_c \alpha$$

for all differentials $\alpha\in\Gamma_{ase}^A(\bar{W})$.

By the Schwarz' inequality we have

$$|(\alpha, \varphi_0)|^2 \leqq \|\alpha\|^2 \|\varphi_0\|^2,$$

hence φ_0 has the following minimum property;

$$\min_\alpha \frac{\|\alpha\|^2}{\left|2\pi \int_c \alpha\right|^2} = \min_\alpha \frac{\|\alpha\|^2}{|(\alpha, \varphi_0)|^2} = \frac{1}{\|\varphi_0\|^2},$$

where α varies over the class $\Gamma^A_{ase}(\bar{W})$. We denote by $d_W(c)$ this minimum value. Now let R be an open Riemann surface of *infinite genus* and $\{R_n\}$ be a canonical exhaustion of R. For a chain c contained in R_n we have by the minimum property of $d_{R_n}(c)$

$$d_{R_n}(c) \leqq d_{R_{n+1}}(c).$$

Hence $\lim_{R_n \to R} d_{R_n}(c)$ is finite or infinite. We denote by Γ_{ase} the class of square integrable analytic differentials on R and by Γ^A_{ase} the subclass $\left\{\omega \in \Gamma_{ase} \,\middle|\, \int_{A_i} \omega = 0 \ (i=1, 2, \cdots)\right\}$.

Proposition. *If* $\lim_{R_n \to R} d_{R_n}(c) = \infty$ *for any finite chain* c, *then* $\Gamma^A_{ase} = \{0\}$, *that is,* $\omega \in \Gamma_{ase}$ *is determined uniquely by its A-periods. Conversely, if* $\Gamma^A_{ase} = \{0\}$, *then* $\lim_{R_n \to R} d_{R_n}(c) = \infty$ *for any finite chain.*

Proof. If $\alpha \in \Gamma^A_{ase}$ and $\alpha \not\equiv 0$, then for some chain c contained in R_{n_0}

$$(\alpha, \varphi_{n_0})_{R_{n_0}} = 2\pi \int_c \alpha \neq 0,$$

where $\varphi_{n_0} (\in \Gamma^A_{ase}(R_{n_0}))$ is the period reproducing differential to the chain c. By the definition of $d_{R_n}(c)$ we have

$$d_{R_n}(c) \leqq \frac{\|\alpha\|^2_{R_n}}{|(\alpha, \varphi_{n_0})_{R_{n_0}}|^2} \leqq \frac{\|\alpha\|^2}{|(\alpha, \varphi_{n_0})_{R_{n_0}}|^2} \qquad (n \geqq n_0).$$

Hence $\lim_{R_n \to R} d_{R_n}(c) < \infty$.

Conversely, if $\lim_{R_n \to R} d_{R_n}(c) < \infty$ for some $c(\subset R_{n_0})$, we put

$$\Phi_n = \frac{\varphi_n}{\|\varphi_n\|^2_{R_n}}.$$

Then

$$(\Phi_n, \varphi_n)_{R_n} = 1 \quad \text{and} \quad (\Phi_{n+p}, \varphi_n)_{R_n} = \frac{1}{\|\varphi_{n+p}\|^2_{R_{n+p}}}(\varphi_{n+p}, \varphi_n)_{R_n}$$

$$= \frac{2\pi\int_c \varphi_{n+p}}{2\pi\int_c \varphi_{n+p}} = 1,$$

and so

$$(\Phi_n, \Phi_n - \Phi_{n+p})_{R_n} = \frac{1}{\|\varphi_n\|^2_{R_n}}(\varphi_n, \Phi_n - \Phi_{n+p})_{R_n} = 0.$$

Hence

$$\|\Phi_n - \Phi_{n+p}\|^2_{R_n} = \|\Phi_{n+p}\|^2_{R_n} - \|\Phi_n\|^2_{R_n} \leqq \|\Phi_{n+p}\|^2_{R_{n+p}} - \|\Phi_n\|^2_{R_n}$$

$$= d_{R_{n+p}}(c) - d_{R_n}(c).$$

Therefore

$$\|\Phi_n - \Phi_{n+p}\|^2_{R_n} \to 0 \qquad (R_n \to R).$$

Thus we may conclude in usual way that Φ_n tend to an analytic semiexact differential Φ. Since $\Phi_n \in \Gamma^A_{ase}(\bar{R}_n)$, Φ belongs to Γ^A_{ase} and $2\pi\int_c \Phi = 1$. q.e.d.

Remark. (1) If $\lim_{R_n \to R} d_{R_n}(c) = d(c) < \infty$, then for any $\alpha \in \Gamma^A_{ase}$

$$(\alpha, \Phi_n - \Phi)_{R_n} = (\alpha, \Phi_n)_{R_n} - (\alpha, \Phi)_{R_n}$$

$$= \frac{2\pi\int_c \alpha}{\|\varphi_n\|^2_{R_n}} - (\alpha, \Phi)_{R_n}.$$

On the other hand, since

$$|(\alpha, \Phi_n - \Phi)_{R_n}| \leqq \|\alpha\| \, \|\Phi_n - \Phi\|_{R_n} \to 0 \qquad (R_n \to R),$$

we have

$$(\alpha, \Phi) = \lim_{R_n \to R} \frac{2\pi\int_c \alpha}{\|\varphi_n\|^2_{R_n}} = d(c)\cdot 2\pi\int_c \alpha.$$

Hence $\Phi/d(c)$ is the period reproducing differential in Γ^A_{ase} to the chain c.

(2) Let $d'_{R_n}(c)$ and $d''_{R_n}(c)$ be the extremal values corresponding to $\Gamma_{ase}(\bar{R}_n)$ and $\Gamma_{ae}(\bar{R}_n)$, respectively, then

$$d'_{R_n}(c) \leqq d_{R_n}(c) \leqq d''_{R_n}(c)\,.$$

We can show easily that $d'_{R_n}(c)$ is always convergent. On the other hand $d''_{R_n}(c)$ is not always convergent (cf. Ahlfors [4], Weill [5]).

3. When we make use of B-cycle in canonical homology basis $\{A_i, B_i\}_{n=1,2,\dots}$ of R, we obtain

Proposition. *Let R belongs to the class O_{AD}. A necessary and sufficient condition in order that $\omega \in \Gamma_{ase}$ is determined by its A-periods is $\lim_{R_n \to R} d_{R_n}(B_i) = \infty$ for every B-cycles.*

Proof. If $\alpha \in \Gamma^A_{ase}$ and $\alpha \not\equiv 0$, since R belongs to O_{AD}, there exists a B-cycle B_i such that $\int_{B_i} \alpha \neq 0$. Hence we have $\lim_{R_n \to R} d_{R_n}(B_i) < \infty$ as before.

4. The generalized analytic modulus $K(\bar{R}_n - R_1)$ associated with $\bar{R}_n - R_1$ is defined as follows (cf. [3]):

$$K(\bar{R}_n - R_1) = \inf_{\omega} \frac{\int_{\partial R_n} u\bar{\omega}}{\int_{\partial R_1} u\bar{\omega}},$$

where ω varies over $\Gamma^A_{ase}(\bar{R}_n - R_1)$ such that $i\int_{\partial R_1} u\bar{\omega} > 0$ and $u(p) = \int_{p_i}^{p} \omega$ $(p, p_i \in \alpha_n^{(i)})$ is the function defined separately on each contour $\alpha_n^{(i)}$ of $\partial(\bar{R}_n - R_1)$. If ω belongs to $\Gamma^A_{ase}(\bar{R}_n)$, then $\|\omega\|^2_{R_n} = i\int_{\partial R_n} u\bar{\omega}$ and so we have for a chain $c (\subset \bar{R}_n)$

$$K(\bar{R}_n - R_1) \leqq \frac{\|\varphi_n(c)\|^2_{R_n}}{\|\varphi_n(c)\|^2_{R_1}} = \frac{d_{R_n}(c)}{\|\Phi_n\|^2_{R_1}},$$

where $\Phi_n = \dfrac{\varphi_n(c)}{\|\varphi_n(c)\|^2_{R_n}}$.

Now let $\lim_{R_n \to R} d_{R_n}(c)$ be finite, then $\Phi_n \to \Phi$ $(\not\equiv 0)$ and so

$$\lim_{R_n \to R} \|\Phi_n\|^2_{R_1} = \|\Phi\|^2_{R_1} > 0\,.$$

Hence $\lim_{R_n \to R} K(\bar{R}_n - R_1)$ is finite. Thus we have

Proposition ([3]). *If* $\lim_{R_n \to R} K(\bar{R}_n - R_1) = \infty$, *then* $\omega \in \Gamma_{ase}$ *is uniquely determined by its A-periods.*

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