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# Analytic manifolds admitting parallel fields of complex planes

By

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In this paper we discuss an $n$-dimensional analytic manifold[1) $M^n$ admitting a field of complex $r$-planes which is parallel with respect to a given affine connection and has only the zero vector in common with its complex conjugate plane field.

In the case where $n=2r$, we have the theorem due to Patterson [2], that, *if a Riemann manifold* $M^{2r}$ *admits a field of* $r$*-planes which is null and parallel with respect to a given positive definite metric* $g$*, the* $M^{2r}$ *admits a complex analytic structure in terms of which* $g$ *is a Kähler metric.* On the other hand, in the previous paper [1], we proved the theorem that, *if a Riemann manifold* $M^{2r+1}$ *admits a field of* $r$*-planes satisfying the similar conditions, the* $M^{2r+1}$ *admits an almost contact metric structure having the covariant constant* $\varphi$*-tensor.*

We will treat mainly the general case $r \leqq \left[\frac{n}{2}\right]$. Recently K. Yano [5] introduced the notion of an $f$-structure including an almost complex structure and an almost contact structure. Our main result is that there is a close relation between an $f$-structure and the existence of a field of complex $r$-planes satisfying the above conditions.

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1) Throughout the paper we assume the manifolds and tensors, including vectors, to be of class $C^\omega$.

## § 1. Plane fields in a manifold with an $f$-structure

Let us consider an $n$-dimensional analytic manifold $M^n$ with an $f$-structure of rank $2r$ or an $f_r$-structure, that is to say, a real non-zero tensor field $f$ of type (1,1) and of class $C^\omega$ such that

$$f^3+f=0, \quad rank\ of\ f=2r. \tag{1.1}$$

The condition (1.1) shows us that characteristic values of $f$ are $\sqrt{-1}, -\sqrt{-1}$ (each r-ple) and 0 (($n$-$2r$)-ple). In tangent space at each point of any coordinate neighbourhood $U$ in $M^n$ we can take three kinds of vector spaces $f^r$, $\bar{f}^r$ and $h^{n-2r}$ which are spanned by characteristic contravariant vectors corresponding to characteristic values $-\sqrt{-1}$, $\sqrt{-1}$ and 0 respectively, where each superscript means the dimension of vector spaces.

It is obvious that $\bar{f}^r$ is spanned by vectors whose components are complex conjugate components of vectors in $f^r$ , and that $f^r$ and $\bar{f}^r$ are both $r$-dimensional and satisfy the relation

$$f^r \cap \bar{f}^r = \{0\}. \tag{1.2}$$

Hence the direct sum $f^r+\bar{f}^r$ forms a field of complex $2r$-planes $\tilde{p}^{2r}$, and has a real basis, that is, $\tilde{p}^{2r}$ contains a field of real $2r$-planes $p^{2r}$. Let $\Lambda_{(\alpha)}$[2] be basic vectors of $f^r$, and write

$$\Lambda_{(\alpha)}=A_{(\alpha)}+\sqrt{-1}\ B_{(\alpha)}, \tag{1.3}$$

then $A_{(\alpha)}$ and $B_{(\alpha)}$ form a real basis of both $\tilde{p}^{2r}$ and $p^{2r}$.

**Definition.** *The complex $f$-plane-field is a field $f^r$ of characteristic contravariant compex vector spaces corresponding to characteristic value $-\sqrt{-1}$ of $f$, and real $f$-plane-field is a field $p^{2r}$ of real $2r$-planes spanned by $A_{(\alpha)}$ and $B_{(\alpha)}$.*

Let $C_{(A)}$ be basic vectors of $h^{n-2r}$, then $A_{(\alpha)}$, $B_{(\alpha)}$ and $C_{(A)}$ form a basis of real tangent space at each point of $M^n$. Now, the definition of $\Lambda_{(\alpha)}$ gives us

$$f\Lambda_{(\alpha)}=-\sqrt{-1}\ \Lambda_{(\alpha)}. \tag{1.4}$$

2) In this paper the indices $a, b, c, d, e$ run over the range $1,\cdots,2r$; $h, i, j,\cdots, r, s, t$ the range $1,\cdots, n$; $A, B, C, D, E, F$ the range $2r+1,\cdots, n$; $\alpha, \beta, \gamma, \delta$ the range $1,\cdots, r$; $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ the range $r+1,\cdots,2r$.

Hence we have $f^2\Lambda_{(\alpha)}=-\Lambda_{(\alpha)}$, that is, $f^2A_{(\alpha)}=-A_{(\alpha)}$ and $f^2 B=-B_{(\alpha)}$.

Thus real plane field $p^{2r}$ becomes a distribution $L$ corresponding to the projection operator $l=-f^2$ defined by K. Yano. Thus we have

**Proposition 1.** *In a manifold $M^n$ with an f-structure of rank 2r, the complex f-plane-field $f^r$ satisfies the condition $f^r\cap\bar{f}^r=\{0\}$ at each point of $M^n$, and the distribution L corresponding to the projection operator $l=-f^2$ is a real f-plane-field.*

In adding to the $L$-distribution, Yano defined a complementary distribution $M$ corresponding to projection operator $m=f^2+1$ and found a positive definite Riemann metric $g$ such that

$$g=gm+{}^tfgf \tag{1.5}$$

with respect to which the distributions $L$ and $M$ are mutually orthogonal.

**Definition.** *A manifold with an $(f_r\text{-}g)$-structure or an $(f\text{-}g)$-structure of rank 2r is a Riemann manifold with a positive definite metric g and an $f_r$-structure satisfying the relation* (1.5).

In a manifold with an $(f_r\text{-}g)$-structure we see

$$l\Lambda_{(\alpha)}=\Lambda_{(\alpha)},\quad m\Lambda_{(\alpha)}=0, \tag{1.6}$$

that is to say, $\Lambda_{(\alpha)}$ are orthogonal to the distribution $M$. Moreover the relations (1.5), (1.6) and (1.4) lead us to

$${}^t\Lambda_{(\alpha)}\, g\Lambda_{(\alpha)}={}^t\Lambda_{(\alpha)}gm\Lambda_{(\beta)}+{}^t(f\Lambda_{(\alpha)})\ g\ f\Lambda_{(\beta)}=-{}^t\Lambda_{(\alpha)}g\Lambda_{(\beta)}.$$

Then the vectors $\Lambda_{(\alpha)}$ are null and $f^r$ is a field of complex null planes. Thus we obtain

**Proposition 2.** *In a manifold with an $(f_r\text{-}g)$-structure, vectors in complex f-plane-field are complex null vectors and orthogonal to the distribution M.*

## § 2. Manifolds with *f*-structure admitting a parallel plane field

**Definition.** *A manifold with an $(f_r\text{-}\Gamma)$-structure is a mani-*

*fold in which an $f_r$-structure and a symmetric affine connection $\Gamma$ are given globally.*

In the first place we prove

**Theorem 1.** *In a manifold with an $(f_r\text{-}\Gamma)$-structure, a necessary and sufficient condition for complex $f$ -plane-field $f^r$ to be parallel with respect to a given symmetric affine connection $\Gamma$ is that the tensor field $f$ satisfies the condition $\nabla f\cdot f=0$, where, expressed in terms of local coordinate $(x^i, U)$, $\nabla f\cdot f$ means that $\nabla_k f^i{}_l\, f^l{}_j$.*

*Proof.* $A_{(\alpha)}$ being basic vectors of $f^r$, the condition for $f^r$ to be parallel can be written, by *A. G.* Walker's result [4], as

$$\nabla A_{(\alpha)}=\xi_{(\alpha)}^{(\beta)}\cdot A_{(\beta)} \tag{2.1}$$

for $r^2$ local complex covariant vector fields $\xi_{(\alpha)}^{(\beta)}$. Differentiating (1.4) covariantly and using (2.1), we get

$$\nabla f\cdot A_{(\alpha)}=0,\ \text{i.e.}\ \nabla f\cdot A_{(\alpha)}=0\ \text{ and }\ \nabla f\cdot B_{(\alpha)}=0. \tag{2.2}$$

These results and (1.4) give us $\nabla f\cdot fA_{(\alpha)}=0$. As the tensors $f$ and $\nabla f$ are real, we have the relations $\nabla f\cdot fA_{(\alpha)}=0$ and $\nabla f\cdot f\,B_{(\alpha)}=0$. On the other hand, as vectors $C_{(A)}$ of basis of $h^{n-2r}$ are characteristic vectors corresponding to characteristic value 0 of $f$, the relation $f\,C_{(A)}=0$ holds good. Then we have $\nabla f\cdot f\,C_{(A)}=0$. These results give us

$$\nabla f\cdot f=0.$$

Conversely, if the last relation holds good, then by differentiating (1.4) covariantly and using (1.4) we have $-\sqrt{-1}\,\nabla A_{(\alpha)}=f\nabla A_{(\alpha)}$. This equation gives us that $\nabla A_{(\alpha)}$ are contained in complex $f$-plane field. Consequently it follows that $\nabla A_{(\alpha)}$ have, for r$^2$ covariant vectors $\xi_{(\alpha)}^{(\beta)}$, the form $\nabla A_{(\alpha)}=\xi_{(\alpha)}^{(\beta)}A_{(\beta)}$. Thus the $f$-plane-field is a parallel plane field with respect to a given symmetric affine connection $\Gamma$, and the theorem is completely proved.

In a manifold with an $f_r$-structure there exist affine connections $\overset{*}{\Gamma}$ which are not always symmetric but leave $f$ covariant constant. In fact, denoting by $\delta$ covariant differentiation with respect to an arbitrary affine connection $\overset{\circ}{\Gamma}$, we can define $\overset{*}{\Gamma}$, for example, in terms of $f$ and $\overset{\circ}{\Gamma}$, as follows

$$(2.3)\qquad \overset{*}{\Gamma}=\overset{\circ}{\Gamma}+\delta f\cdot f-f\cdot\delta f+\tfrac{3}{2}f^2\cdot\delta f\cdot f.$$

For, (1.1) gives us

$$(2.4)\qquad f^2\cdot\delta f\cdot f^2=f\cdot\delta f\cdot f.$$

From (2.3), (2.4) and (1.1), denoting by $\partial$ and $\overset{*}{\nabla}$ partial differentiation and covariant differentiation with respect to $\overset{*}{\Gamma}$ respectively, we get

$$\overset{*}{\nabla}f=\partial f+\overset{*}{\Gamma}\cdot f-f\cdot\overset{*}{\Gamma}=\delta(f+f^3)=0.$$

However, the affine connection preserving $f$ covariant constant is not unique. For example, if $v$ is a covariant vector field, then $\overset{*}{\Gamma}+f\cdot v$ also preserves $f$ covariant constant.

Summarizing the above remarks, we get

**Proposition 3.** *In a manifold with an $f_r$-strcture there exist (not necessarily symmetric) affine connections with respect to which the structure tensor $f$ is covariant constant.*

Next, we consider an $(f_r-g)$-structure *in* $M^n$ and get the following result:

**Theorem 2.** *In a manifold with an $(f_r$-$g)$-structure, a necessary and sufficient condition for the complex $f$-plane-field to be parallel in Levi-Civita's sense is that the tensor field $f$ is covariant constant over $M^n$ with respect to $g$.*

*Proof.* If the complex $f$-plane-field $f^r$ is parallel, so is its conjugate $\overline{f^r}$ .Hence real $f$-plane-field $L$ is also parallel. As the distribution $M$ is orthogonal to $L$, $M$ is parallel, too. Then, for the basic vectors $N_{(A)}$ of $M$, there exist $(n-2r)^2$ covariant real vectors $C_{(A)}^{(B)}$ satisfying

$$(2.5)\qquad \nabla N_{(A)}=C_{(A)}^{(B)}\cdot N_{(B)}.$$

It follows, from the definition of $M$, that

(2.6) $$lN_{(A)}=0, \quad mN_{(A)}=N_{(A)}.$$

The last relation and the relation $fm=0$ give us $fN_{(A)}=0$.

Moreover, by differentiating this result covariantly and using (2.5) we find

(2.7) $$\nabla f\cdot N_{(A)}=0.$$

Hence the relations (2.2) and (2.7) give us $\nabla f=0$.

On the other hand the converse follows at once from Theorem 1.

## § 3. Manifolds admitting a parallel plane field

**Definition.** *A complex $\pi^r$ -plane-field in a manifold $M^n$ is a field of complex r-dimensional planes $\pi^r$ satisfying the relation $\pi^r\cap\bar{\pi}^r=\{0\}$ at each point of $M^n$. A manifold with a $(\pi^r\text{-}\Gamma)$-structure is a manifold admitting a complex $\pi^r$-pane-field and a symmetric affine connection $\Gamma$ with respect to which $\pi^r$ is parallel.*

It follows apparently from Theorem 1 that a manifold with an $(f_r\text{-}\Gamma)$-structure satisfying the condition $\nabla f\cdot f=0$ has a $(\pi^r\text{-}\Gamma)$ structure. In this section we shall consider the convrese of this fact.

Assume $M^n$ to admit a $(\pi^r\text{-}\Gamma)$-structure. Since the relation $\pi^r\cap\bar{\pi}^r=\{0\}$ holds good in each point of $M^n$, the direct sum $\pi^r+\bar{\pi}^r=\tilde{\varphi}^{2r}$ forms a field of complex $2r$-dimensional planes and contains a real base, that is to say, at each point of $M^n$, for $r$ basic contravariant vectors $\lambda_{(\alpha)}=a^i{}_{(\alpha)}+\sqrt{-1}\ b^i{}_{(\alpha)}$ of $\pi^r$, $2r$ real contravariant vectors $c_{(\alpha)}=(a_{(\alpha)}, b_{(\alpha)})$ are linearly independent over the real number field. Then we take a field $\varphi^{2r}$ of real $2r$-dimen sional planes spanned by $c_{(\alpha)}$, and call it $\pi^r$-plane-field hereafter. It is clear that the real $\pi^r$-plane field is contained in $\tilde{\varphi}^{2r}$ and is independent to the choice of basis $\lambda_{(\alpha)}$ of $\pi^r$. By the assumption, complex $\pi^r$-plane-field is parallel. Then there exist $r^2$ complex covariant vector fields $\eta^{(\beta)}_{(\alpha)}$ for which the relation

$$\nabla \lambda_{(\alpha)} = \eta^{(\beta)}_{(\alpha)} \cdot \lambda_{(\beta)} \tag{3.1}$$

holds good. Considering the real part and imaginary part of (3.1), we have also

$$\nabla c_{(a)} = B^{(b)}_{(a)} \cdot c_{(b)}, \tag{3.2}$$

where $B^{(b)}_{(a)}$ are $(2r)^2$ real covariant vector fields determined by $\eta^{(\beta)}_{(\alpha)}$. Consequently $\psi^{2r}$ is also a parallel field.

Now, in each local coordinate neighbourhood, let us consider the system of real partial differential equations

$$X_a f \equiv c^i{}_{(a)} \frac{\partial f}{\partial x^i} = 0, \quad (a = 1,2,\cdots,2r), \tag{3.3}$$

and the system of complex partial differential equations

$$X_\alpha f \equiv \lambda^i{}_{(\alpha)} \frac{\partial f}{\partial x^i} = 0, \quad (\alpha = 1,2,\cdots,r). \tag{3.4}$$

These are both completely integrable from the conditions (3.1) and (3.2). Denoting $(n-2r)$ independent solutions of (3.3) by $w^A$, the $w^A$ are solutions of (3.4), too. And actually, for $n-r$ independent solutions of (3.4), real solutions of (3.4) are $w^A$ only, because of the definition of $c_{(a)}$.

Now let $z^\alpha = u^\alpha + \sqrt{-1}\ v^{\bar{\alpha}} (\bar{\alpha} = \alpha + \mathrm{r})$ be the other $r$ complex solutions of (3.4). Then it is easy to show that there is no functional relationship of the form $F(u^1, \cdots u^r, v^{r+1}, \cdots v^{2r}, w^{2r+1}, \cdots w^n) = 0$. Therefore we take $(u^\alpha, v^{\bar{\alpha}}, w^A)$ as a new coordinate system in each local coordinate neighbourhood $U$, and call it *a canonical coordinate system* hereafter. From the equations (3.4) and (3.3), in the canonical coordinate system, $a_{(\alpha)}$ and $b_{(\alpha)}$ must have the following components

$$^t a_{(\alpha)} = (\cdots, a^\beta_{(\alpha)}, \cdots, a^{\bar{\beta}}_{(\alpha)}, \cdots, 0, \cdots) \text{ and } {}^t b_{(\alpha)} = (\cdots, -a^{\bar{\beta}}_{(\alpha)} \cdots, a^\beta_{(\alpha)}, \cdots, 0, \cdots). \tag{3.5}$$

It is clear that an integral manifold of (3.3) is a maximal integral manifold of real $\pi^r$-plane-field $\psi^{2r}$, and is expressed locally by $w^A = \text{const.}$, and $(u^\alpha, v^{\bar{\alpha}})$ can be regarded as a local coordinate system of this submanifold.

Furthermore let us consider transformations of canoical coordinate systems. For every pair $U$, $U'$ of intersecting neighbourhoods admitting canonical coordinate systems $(x^i)=(u^\alpha, v^{\bar\alpha}, w^A)$ and $(x^{i'})=(u^{\alpha'}, v^{\bar\alpha'} w^{A'})$ respectively, in $U\cap U'$, $w^A$ and $w^{A'}$ are solutions of the system (3.3). Then we can represent $w^{A'}$ as $w^{A'}=F^{A'}(w^{2r+1},\cdots,w^n)$ where $F^{A'}$ are real analytic functions. Similarly, $u^{\bar\alpha}$ and $v^{\bar\alpha'}$ are represented as real analytic functions of $u^1,\cdots,u^r$, $v^{r+1},\cdots,v^{2r}$.

From now on, we shall show that $z^{\alpha'}=u^{\alpha'}+\sqrt{-1}\ v^{\bar\alpha'}$ are complex analytic functions of $z^\alpha=u^\alpha+\sqrt{-1}\ v^{\bar\alpha}$. For this purpose we consider the components of the vectors $\lambda_{(\alpha)}$. From relation (3.5), in a canonical coordinate system, $\lambda_{(\alpha)}$ have the form

$$(3.6)\qquad {}^t\lambda_{(\alpha)}=(\cdots,\ a^\beta_{(\alpha)}-\sqrt{-1}\ a^{\bar\beta}_{(\alpha)},\cdots,\ a^{\bar\beta}_{(\alpha)}+\sqrt{-1}\ a^\beta_{(\alpha)},\cdots,0,\cdots).$$

Then they satisfy the relation

$$(3.7)\qquad \lambda^\beta_{(\alpha)}+\sqrt{-1}\ \lambda^{\bar\beta}_{(\alpha)}=0.$$

Since relation (3.7) also holds good in $U'$, we get

$$\left(\lambda^\gamma_{(\alpha)}\frac{\partial u^{\beta'}}{\partial u^\gamma}+\lambda^{\bar\gamma}_{(\alpha)}\frac{\partial u^{\beta'}}{\partial v^{\bar\gamma}}\right)+\sqrt{-1}\left(\lambda^\gamma_{(\alpha)}\frac{\partial v^{\bar\beta'}}{\partial u^\gamma}+\lambda^{\bar\gamma}_{(\alpha)}\frac{\partial v^{\bar\beta'}}{\partial v^{\bar\gamma}}\right)=0.$$

Substituting (3.6) and (3.7) in the last equations, we get

$$\left[a^\gamma_{(\alpha)}\left(\frac{\partial u^{\beta'}}{\partial u^\gamma}-\frac{\partial v^{\bar\beta'}}{\partial v^{\bar\gamma}}\right)+a^{\bar\gamma}_{(\alpha)}\left(\frac{\partial u^{\beta'}}{\partial v^{\bar\gamma}}+\frac{\partial v^{\bar\beta'}}{\partial u^\gamma}\right)\right]$$
$$+\sqrt{-1}\left[b^\gamma_{(\alpha)}\left(\frac{\partial u^{\beta'}}{\partial u^\gamma}-\frac{\partial v^{\bar\beta'}}{\partial v^{\bar\gamma}}\right)+b^{\bar\gamma}_{(\alpha)}\left(\frac{\partial u^{\beta'}}{\partial v^{\bar\gamma}}+\frac{\partial v^{\bar\beta'}}{\partial u^\gamma}\right)\right]=0.$$

As $2r$ vectors $a_{(\alpha)}$ and $b_{(\alpha)}$ are linearly independent, these equations can be reduced to the well known Cauchy-Riemann's differental equations

$$(3.8)\qquad \frac{\partial u^{\beta'}}{\partial u^\gamma}=\frac{\partial v^{\bar\beta'}}{\partial v^{\bar\gamma}},\quad \frac{\partial u^{\beta'}}{\partial v^{\bar\gamma}}=-\frac{\partial v^{\bar\beta'}}{\partial u^\gamma}.$$

Consequently, by Hartogs's theorem, $z^{\alpha'}$ have the form $z^{\alpha'}=\Phi^{\alpha'}(z^1,\cdots z^r)$ and $\Phi^{\alpha'}$ are complex analytic. Thus we have

**Theorem 3.** *In a manifold with a $(\pi^r$-$\Gamma)$-structure, real $\pi^r$-plane-field is integrable and its integral submanifolds have* *complex analytic structures.*

In the canonical coordinate system of any coordinate neighbourhood $U$, consider a non-zero real tensor $f$ of type (1,1) whose components are given by the following form;

$$(f^i{}_j)=\begin{pmatrix} 0, & E_r, & 0 \\ -E_r, & 0, & 0 \\ 0, & 0, & 0 \end{pmatrix}, \tag{3.9}$$

where $E_r$ is an $r$-dimensional unit matrix. Take another coordinate neighbourhood $U'$ which intersects with $U$ and consider, in the canonical coordinate system on $U'$, a tensor $f$ of type (1,1) given by, as well as (3.9),

$$(f^{i'}_{j'})=\begin{pmatrix} 0, & E_r, & 0 \\ -E_r, & 0, & 0 \\ 0, & 0, & 0 \end{pmatrix}. \tag{3.9'}$$

Then in $U\cap U'$ we can easily verify the usual transformation law of the tensor of type (1,1) i. e. $f^{i'}_{j'}=f^p_q\dfrac{\partial x^{i'}}{\partial x^p}\dfrac{\partial x^q}{\partial x^{j'}}$, by making use of (3.8). Hence we can find a non-zero real tensor $f$ of type (1,1) over $M^n$ whose components in the canonical coordinate system on $U$ are given by (3.9). Of course it can be easily verified that

$$f^3+f=0, \quad \text{rank of } f=2r.$$

Thus the $M^n$ admits an $f_r$-structure.

Finally direct calculation leads us to

$$f\lambda_{(\alpha)}=-\sqrt{-1}\,\lambda_{(\alpha)} \tag{3.10}$$

by virtue of equations (3.6) and (3.9). From equation (3.10), it follows that the $\pi^r$ is nothing but the complex $f$-plane-field $f^r$ corresponding to the $f$-structure given by (3.9). Since the $\pi^r$ is parallel with respect to $\Gamma$, we obtain, from these results and by Theorem 1, the following

**Theorem 4.** *In order that a manifold $M^n$ admits a $(\pi^r$-$\Gamma)$-*

*structure, it is necessary and sufficient that the $M^n$ has an $(f_r\text{-}\Gamma)$-structure satisfying the condition $\nabla f\cdot f=0$. In this case the field $\pi^r$ coincides with the complex f-plane-field.*

**Remark:** The condition $\pi^r\cap\bar{\pi}^r=\{0\}$ implies $2r\leqq n$. When $n=2r$, the rank of $f$ becomes $n$, from which the tensor $f$ is almost, complex one and the condition $\nabla f\cdot f=0$ is reducible to $\nabla f=0$. Thus we obtain the famous theorem (Patterson [3]) i. e. *"If a manifold $M^{2r}$ admits a field $\pi^r$ of complex r-planes such that $\pi^r$ and $\bar{\pi}^r$ at each point have only the zero vector in common, and a symmetric affine connection $\Gamma$ with respect to which $\pi^r$ is parallel, then the $M^{2r}$ is a complex manifold.".*

Recently, Ishihara and Yano [6] proved that a manifold with an $f_r$-structure admits a coordinate system with respect to which the tensor $f$ has components of the form (3.9), if and only if the Nijenhuis tensor *of* $f$ vanishes, i. e. $f$ is integrable. Then, in our case, the proof of the Theorem 4 shows us directly

**Corollary.** *If a manifold $M^n$ admits a $(\pi^r\text{-}\Gamma)$-structure, then the $M^n$ admits an integrable $f_r$-structure.*

**Remark:** Even though a manifold $M^n$ admits an $(f_r\text{-}\Gamma)$-structure satisfying the condition $\nabla f\cdot f=0$, the $f_r$-structure is not necessarily integrable. For, in this case, the $M^n$ admits an integrable $f_r^*$-structure defined by (3.9), *but*, the new tensor $f^*$ does not necessarily coincide with the original structure tensor $f$, though both $f$ and $f^*$ admit in common only the distribution $L$.

## § 4. Riemann manifolds admitting a field of parallel null planes

**Definition.** *A manifold with a $(\pi^r\text{-}g)$-structure is a Riemann manifold admitting a field $\pi^r$ of r-planes and a positive definite Riemann metric $g$ with respect to which the $\pi^r$ is null and parallel.*

It is clear, as a consequence of Theorem 2 and Proposition 2, that a manifold with an $(f_r\text{-}g)$-structure satisfying $\nabla f=0$ admits a $(\pi^r\text{-}g)$-structure.

Conversely we assume, in the following, a manifold $M^n$ to admit a $(\pi^r\text{-}g)$-structure.

Since $g$ is positive definite and the $\pi^r$ is null, the relation $\pi^r \cap \bar{\pi}^r = \{0\}$ holds good in each point of $M^n$. Then the $M^n$ admits a $(\pi^r\text{-}\Gamma)$-structure, and from Theorem 4, the $M^n$ admits an $(f_r\text{-}\Gamma)$-structure satisfying $\nabla f \cdot f = 0$, and the $\pi^r$ coincides with the complex $f$-plane-field. Moreover, in canonical coordinate system, the relations (3.5) and (3.9) hold good.

For the basic vectors $\lambda_{(\alpha)}$ of $\pi^r$, since $\pi^r$ is a field of null planes, the relation ${}^t\lambda_{(\alpha)}\ g\lambda_{(\beta)} = 0$ holds good, from which it follows that

$$ {}^t a_{(\alpha)} g a_{(\beta)} - {}^t b_{(\alpha)} g b_{(\beta)} = 0, \quad {}^t a_{(\alpha)} g b_{(\beta)} + {}^t b_{(\alpha)} g a_{(\beta)} = 0. $$

By means of relation (3.5), these results are reduced to

$$ (4.1) \qquad g_{\gamma\delta} = g_{\bar{\gamma}\bar{\delta}}, \quad g_{\gamma\bar{\delta}} = -g_{\bar{\gamma}\delta}. $$

From the equations (3.9) and (4.1), it is easy to verify that relation (1.5) holds good. Consequently the given metric $g$ and the tensor $f$ whose components have the form (3.9) in canonical coordinate system constitute an $(f_r\text{-}g)$-structure. Moreover, in our case, the complex $f$-plane-field is $\pi^r$ and is parallel with respect to the given metric $g$. Thus we obtain, from Theorem 2,

**Theorem 5.** *In order that a manifold $M^n$ admits a $(\pi^r\text{-}g)$-structure, it is necessary and sufficient that the $M^n$ admits an $(f_r\text{-}g)$-structure satisfying the condition $\nabla f = 0$. In this case the field $\pi^r$ coincides with the complex $f$-plane-field.*

**Remark:** When we confine ourselves to consider cases where $n = 2r$ and $n = 2r+1$ in this theorem, we obtain Patterson's theorem and Ichijyô's described in the introduction.

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