J. Math. Kyoto Univ.
4-2 (1965) 355-368

On R. Sulanke's method deriving H. Rund's connection in a Finsler space

By

Makoto MATSUMOTO

(Received Dec. 9, 1964)

In order to derive Rund's connection in a Finsler space, R. Sulanke [8] used the concept of direction-dependent affine connections, and determined the connection uniquely by the system of axioms, analogous to the case of a Riemann space.

The purpose of this paper is to give a global foundation of direction-dependent affine connections, and to discuss axioms introduced by R. Sulanke. Since there exists, in general, no direction field in the large on differentiable manifold, we shall first define the concept of f-relative linear connections, from which a Finsler connection is naturally constructed. R. Sulanke gave three axioms to determine Rund's connection uniquely, and remarked later on [9] that the second axiom was a result from the other two. It is well known that Berwald's connection [1] satisfies the first and second axioms, but not the third. In the final section of the present paper, we shall show that there are infinitely many connections of Berwald's type.

§ 1. A system of f-relative linear connections

Let $P(M, \pi, G)$ be the principal bundle of frames tangent to a differentiable n-manidold M, where π is the projection $P \rightarrow M$ and G the general linear group $GL(n, R)$. We consider a local coordinate (x^i), $i=1, 2, \cdots, n$, of M, and then a frame $p=(p_a)$, $a=1, 2, \cdots, n$, tangent to M at a point x is expressed as $p_a = p^i_a (\partial/\partial x^i)_x$. Thus, (x^i, p^i_a) is thought of as a local coordinate of $p \in P$, which is called a canonical coordinate. In terms of a canonical coordinate, the right translation R_g of P by an element $g=(g^a{}_b) \in G$ is expressed as $(x^i, p^i{}_a) \to (x^i, p^i{}_b g^b{}_a)$.

Let $B(M, \tau, F, G)$ be the bundle of tangent vectors to M, where τ is the projection $B \to M$ and F the real vector n-space. Analogous to the case of P, we have a canonical coordinate (x^i, b^i) of a point $b \in B$, where $b=b^i(\partial/\partial x^i)_x$. B is an associated bundle with P, and G acts on F by the rule $f=f^a e_a \in F \to g^a{}_b f^b e_a$, where (e_a), $a=1, 2, \cdots, n$, is a fixed base of F and $g=(g^a{}_b) \in G$. A point $p=(x^i, p^i{}_a) \in P$ is regarded as the admissible mapping $F \to B$, such that $f=f^a e_a \to (x^i, p^i{}_a f^a)$.

Definition. *A system of f-relative linear connections* $\{\Gamma(f)\}$ in $P(M, \pi, G)$ is a collection of distributions $\Gamma(f): p \in P \to \Gamma(f)_p$ on P corresponding to any non-zero element $f \in F$, such that the following two conditions are satisfied:

RC 1. $\quad P_p = P^v{}_p + \Gamma(f)_p$ (*direct sum*),

RC 2. $\quad R_g \Gamma(f)_p = \Gamma(g^{-1}f)_{pg}$,

where P_p is the tangent vector space to P at a point p and $P^v{}_p$ the vertical subspace of P_p.

It is to be remarked that a distribution $\Gamma(f)$ alone is not a linear connection in P, because *RC* 2 differs from the ordinary condition $R_g \Gamma_p = \Gamma_{pg}$ for a linear connection [6, (5. 2)]. It follows from *RC* 1 that any tangent vector $X \in P_p$ is written in the form $v(f)X + h(f)X$, where $v(f)X \in P^v{}_p$ and $h(f)X \in \Gamma(f)_p$ are called the f-relative vertical part and f-relative horizontal part of X respectively. If $v(f)X=0$, X is called f-relative horizontal, and if $h(f)X=0$, X is f-relative vertical. It is easy to show that those $v(f)$ and $h(f)$ satisfy the following equations respectively:

$$(1.\ 1) \qquad R_g v(f) = v(g^{-1}f) R_g, \quad R_g h(f) = h(g^{-1}f) R_g.$$

In similar manner with the case for a linear connection in P, we can define the $\hat{G}$(Lie algedra of G)-valued 1-form $\omega(f)$, which is given by

$$(1.\ 2) \qquad \omega(f)\, F(A) = A, \quad A \in \hat{G},$$

$$(1.\ 3) \qquad \omega(f)\, \Gamma(f) = 0,$$

where $F(A)$ is the fundamental vector field on P corresponding to an element $A \in \hat{G}$. The 1-form $\omega(f)$ is called the f-relative connection form [4, (2. 5)]. As is easily verified from (1. 1), $\omega(f)$ satisfies the equation [4, (2. 6)]

$$\omega(f)_{pg} R_g = ad(g^{-1})\,\omega(gf)_p, \qquad g \in G. \tag{1. 4}$$

The lift $l(f)_p X$ of a tangent vector $X \in M_x$ to a point $p \in P$ is by definition the tangent vector to P at p, such that $l(f)_p X$ is f-relative horizontal and $\pi l(f)_p X = X$. Making use of this $l(f)$, we obtain the f-relative basic vector field $B_f(f_1)$ corresponding to $f_1 \in F$, which is defined by

$$B_f(f_1)_p = l(f)_p\, p f_1, \tag{1. 5}$$

where $p \in P$ is regarded as an admissible mapping $F \to B$. Let $\theta = p^{-1}\,\pi$ be the basic form on P, then we have

$$\theta\, B_f(f_1) = f_1. \tag{1. 6}$$

It is obvious that the following equation holds from *RC* 2:

$$R_g B_f(f_1)_p = B_{g^{-1}f}(g^{-1}f_1)_{pg}. \tag{1. 7}$$

On the other hand, we can define the absolute basic vector field $B(f)$ corresponding to $f \in F$ which is given by the special f-relative basic vector field $B_f(f)$ [6, § 5].

Now, if we put $\hat{g}^b{}_a = (\partial/\partial g^a{}_b)_e$, the set $(\hat{g}^b{}_a)$ is considered as a base of the Lie algebra $\hat{G}$. Hence the f-relative connetion form $\omega(f)$ is expressed by $\omega(f)^a{}_b\, \hat{g}^b{}_a$. From (1. 2) and (1. 4) it follows that those componets $\omega(f)^a{}_b$ are written in the form [4, (2. 10)]

$$\omega(f)^a{}_b = p^{-1a}{}_i (dp^i{}_b + p^j{}_b F_j{}^i{}_k(pf) dx^k), \tag{1. 8}$$

in terms of a canonical coordinate $(x^i, p^i{}_a)$, where $(p^{-1a}{}_i)$ is the inverse matrix of $(p^i{}_a)$ and $F_j{}^i{}_k$ are functions of (x^i, b^i), $b^i = p^i{}_a f^a$, $f = f^a e_a$. Those $F_j{}^i{}_k$ are called coefficients of the f-relative linear connection $\Gamma(f)$.

Next, by making use of (1. 3), we obtain a local expression of the f-relative basic vector field $B_f(f_1)$ as follows:

$$B_f(f_1)_p = p^i{}_a f^a{}_1 \left(\frac{\partial}{\partial x^i} - p^j{}_b F_j{}^k{}_i(pf) \frac{\partial}{\partial p^k_b} \right) \tag{1. 9}$$

at p. Especially the absolute basic vector field $B(f)$ is written in the form

$$(1.\ 10)\qquad B(f)_p=p^i{}_a f^a\left(\frac{\partial}{\partial x^i}-p^j{}_b F_j{}^k{}_i(pf)\frac{\partial}{\partial p^k_b}\right).$$

In the case of an ordinary linear connection in P, the associated connection is defined in the associated bundle B. In the similar way, we obtain the associated connection in B with the system of f-relaive linear connections $\{\Gamma(f)\}$. In fact, if we take the mapping $K_f: P\to B,\ p\in P\to pf\in B$, the associated connection $H: b\in B\to H_b$ is defined by

$$(1.\ 11)\qquad H_b=K_f\Gamma(f)_p,\qquad pf=b.$$

It is to be noted that H is uniquely determined by $\{\Gamma(f)\}$, independent of the choice of $f\in F$, which is easily seen from the equation $K_{g^{-1}p}=K_pR_{g^{-1}}$ As thus defined H is called the *associated non-linear connection* in B. It is easy to verify that the tangent vector space B_b to B at b is decomposed in the form $B^v{}_b+H_b$ (direct sum), where $B^v{}_b$ is the vertical subspace of B_b. Hence $X\in B_b$ is written in the form $v'X+h'X$ where $v'X\in B^v{}_b$ and $h'X\in H_b$. From (1. 9) it follows that a local expression of $X\in H_b$ is given by

$$(1.\ 12)\qquad X=X^i\left(\frac{\partial}{\partial x^i}-b^jF_j{}^k{}_i(x^h,\ b^h)\frac{\partial}{\partial b^k}\right),$$

where $b=(x^h,\ b^h)$.

Similar to the definition of a path in the case of an ordinary linear connection, we have an *absolute path* on M, which is a projection of an integral curve of the associated non-linear connection H. It is seen easily that an absolute path is a projection of an integral curve of an absolute basic vector field $B(f)$. According to (1. 10), the system of differential equations representing an absolute path is

$$(1.\ 13)\qquad \frac{d^2x^i}{dt^2}+F_j{}^i{}_k\left(x^h,\ \frac{dx^h}{dt}\right)\frac{dx^j}{dt}\frac{dx^k}{dt}=0.$$

On the other hand, an integral curve of the f-relative basic vector field $B_f(f_1)$ may be considered, but its projection on M is meaningless, because of (1. 7). From this point of view, relative path due to R. Sulanke [8] seems to be defined locally alone.

§ 2. Vertical connections

Let $Q(B, \bar{\pi}, G)$ be the induced bundle from $P(M, \pi, G)$ by the projection τ of $B(M, \tau, F, G)$ [4]. The bundle space Q is defined by $Q=\{(b, p)\in B\times P, \tau(b)=\pi(p)\}$. We consider the closed submanifold $Q(x)=\{(b, p)\in Q, \tau(b)=\pi(p)=x\}$ of Q for a fixed point $x\in M$. Then $Q(x)$ has the structure of a principal bundle over $B(x)=\{b\in B, \tau(b)=x\}$ (the fibre over $x\in M$ of the tangent vector bundle B). The group of structure of $Q(x)$ is G clearly.

Definition. *A vertical connection* in Q is a distribution $C: q\in Q\to C_q$, such that every restriction $C|Q(x)$ of C to $Q(x)$ for $x\in M$ are a connection $C(x)$ in $Q(x)$.

With respect to the connection $C(x)$, we obtain a operation of lift, which is denoted by l^c, and further the basic vector field $B^c(f)$ corresponding to $f\in F$ is defined as follows:

$$B^c(f)_q=l^c_q dp j_{f_1} f,\quad q=(b, p),\quad f_1=p^{-1}b, \tag{2. 1}$$

where dp is the differential of an admissible mapping p and j_{f_1} is the identification of F with the tangent vector space F_{f_1} to F at f_1 [4, *p.* 3].

In terms of a canonical coordinate (b^i, p^i_a) of $Q(x)$, components $(\omega^c(x))^a_b$ of the connection form $\omega^c(x)$ of the connection $C(x)$ are expressed in the form

$$(\omega^c(x)_q)^a_b=p^{-1a}_i(dp^i_b+p^j_b C_j{}^i{}_k(b)db^k),\qquad q=(b,p), \tag{2. 2}$$

where $C_j{}^i{}_k$ are functions of (x^h, b^h), canonical coordinate of b. Consequently a tangent vector $X\in C(x)_q$ is expressible by

$$X=X^i\left(\frac{\partial}{\partial b^i}-p^j_a C_j{}^k{}_i(b)\frac{\partial}{\partial p^k_a}\right). \tag{2. 3}$$

§ 3. Construction of a Finsler connection

The definition of a Finsler connection in Q has been given in the previous paper [4]. The purpose of the present section is to construct a Finsler connection from a system of f-relative connection $\{\Gamma(f)\}$, together with a vertical connection C. To do this, we first prove

Proposition 1. *The operations*

$$(h^h)_q=\overline{K}_f h(f)\eta, \quad (h^v)_q=l^c{}_q v'\bar{\pi},$$

applied to the tangent vector space to Q are projection operators, where $q=(b, p)$, $f=p^{-1}b$, η is the induced mapping $Q\to P$, $(b, p)\to p$, and $\overline{K}_f$ is the mapping $P\to Q$, $p\to(pf, p)$.

Proof. We have

$$(h^h)^2{}_q=\overline{K}_f h(f)\eta\,\overline{K}_f h(f)\eta=\overline{K}_f h(f)h(f)\eta=\overline{K}_f h(f)\eta=(h^h)_q,$$

and

$$(h^v)^2{}_q=l^c{}_q v'\bar{\pi}l^c{}_q v'\bar{\pi}=l^c{}_q v'v'\bar{\pi}=l^c{}_q v'\bar{\pi}=(h^v)_q.$$

Therefore the proof is complete.

Thus, let us consider distributions Γ^h and Γ^v corresponding to projection operators h^h and h^v respectively:

(3. 1) $$\Gamma^h{}_q=\{X\in Q_q,\ h^h X=X\},$$

(3. 2) $$\Gamma^v{}_q=\{X\in Q_q,\ h^v X=X\}.$$

In the first place, we shall show that

(3. 3) $$Q_q=Q^v{}_q+\Gamma^h{}_q+\Gamma^v{}_q \qquad \text{(direct sum)},$$

where $Q^v{}_q$ is the vertical subspace of the tangent vector space Q_q.

Given $X\in Q_q$, we put

$$X^h=h^hX, \quad X^v=h^vX,$$
$$Y=X-X^h-X^v.$$

It follows from Proposition 1 that $X^h\in\Gamma^h{}_q$ and $X^v\in\Gamma^v{}_p$. Then we obtain

$$\begin{aligned}\bar{\pi}Y&=\bar{\pi}X-\bar{\pi}\overline{K}_f h(f)\eta X-\bar{\pi}l_q{}^c v'\bar{\pi}X\\&=\bar{\pi}X-K_f h(f)\eta X-v'\bar{\pi}X=\bar{\pi}X-h'K_f\eta X-v'\bar{\pi}X,\end{aligned}$$

where K_f is the mapping $P\to B$ used in (1. 11). Putting $Z=\bar{\pi}X$

$-K_f\eta X \in B_b$, it is easy to see that Z is vertical. Hence we have

$$\bar{\pi} Y=\bar{\pi}X-h'(\bar{\pi}X-Z)-v'\bar{\pi}X=\bar{\pi}X-(h'+v')\bar{\pi}X=0.$$

Consequently, we have the decomposition of X as (3. 3):

$$X=Y+X^h+X^v, \quad Y\in Q^v{}_q, \quad X^h\in \Gamma^h{}_q, \quad X^v\in \Gamma^v{}_q.$$

It is easy to see that (3. 3) is direct sum. Hence the proof of (3. 3) is complete.

Next, we shall show that

(3. 4) $$R_g\Gamma^h{}_q=\Gamma^h{}_{qg}, \quad R_g\Gamma^v{}_q=\Gamma^v{}_{qg}.$$

Let us consider $X\in \Gamma^h{}_q$, $q=(b, p)$. By means of the definition of Γ^h, $X=\overline{K}_f h(f)\eta X$, and, making use of the relation $\overline{K}_{g^{-1}f}R_g=R_g\overline{K}_f$ and (1. 1), we have

$$\overline{K}_{g^{-1}f}h(g^{-1}f)\eta R_gX=\overline{K}_{g^{-1}f}h(g^{-1}f)R_g\eta X=\overline{K}_{g^{-1}f}R_gh(f)\eta X$$
$$=R_g\overline{K}_fh(f)\eta X=R_gX.$$

Since $(pg)^{-1}=g^{-1}f$, the above equation shows that $R_gX\in \Gamma^h{}_{qg}$, and hence the first of (3. 4) is true. Next, given $X\in \Gamma^v{}_q$, we have $X=l^c{}_qv'X$, and it follows from the property of the l^c that

$$l^c{}_{qg}v'\bar{\pi}R_gX=l^c{}_{qg}v'\bar{\pi}X=R_gl^c{}_qv'\bar{\pi}X=R_gX,$$

which shows that the second of (3. 4) is true.

Finally we shall show that

(3. 5) $$\bar{\pi}\Gamma_q^v=B_b^v, \quad q=(b, p).$$

It is clear from (3. 2) that $\bar{\pi}\Gamma^v{}_b\subset B^v{}_q$. Conversely, if we take any $X\in B^v{}_b$, it is obvious that $Y=l^c{}_qX\in \Gamma^v{}_q$ and $\bar{\pi}Y=X$. Thus we obtain (3. 5).

As a consequence of equations (3. 3), (3. 4) and (3. 5), it is concluded that the pair of distributions (Γ^v, Γ^h) as defined by (3.1) and (3. 2) is a Finsler connection in Q certainly.

In the general theory of our Finsler connection in Q, we obtain the induced non-linear connection in B and further the quasi-connections in P [4]. In the case of the Finsler connection as above constructed, it is easy to show that the induced non-linear connection coincides with H as defined by (1. 11), that is,

(3. 6) $$\tilde{\pi}\Gamma^{h}{}_{q}=H_{b}, \qquad q=(b, p).$$

and that the quasi-connectons coincide with the original f-relative linear connections $\Gamma(f)$, that is

(3. 7) $$\eta\Gamma^{h}{}_{q}=\Gamma(f)_{p}, \qquad q=(b, p), \quad f=p^{-1}b.$$

It is to be noticed that the Finsler connection as above constructed satisfies the condition F [5], because this condition F is expressed by $K_{f}\Gamma(f)_{p}=H_{b}$, $pf=b$, which is (1. 11) itself.

Gathering the foregoing results together we obtain

Proposition 2. *When a system of f-relative linear connections* $\{\Gamma(f)\}$ *in P and a vertical connection C in Q are given, the pair of distributions* (Γ^{v}, Γ^{h}) *on Q as defined by* (3. 1) *and* (3. 2) *is a Finsler connection satisfying the condition F. The induced non-linear connection coincides with the one associated with* $\{\Gamma(f)\}$ *and the quasi-f-connections coincide with the original* $\{\Gamma(f)\}$.

In terms of a canonical coordinate, it is easy to show that

(3. 8) $$\Gamma^{h}{}_{q}\ni X=X^{i}\left(\frac{\partial}{\partial x^{i}}-b^{j}F_{j}{}^{k}{}_{i}(b)\frac{\partial}{\partial b^{k}}-p^{j}{}_{a}F_{j}{}^{k}{}_{i}(b)\frac{\partial}{\partial p^{k}_{a}}\right),$$

(3. 9) $$\Gamma^{v}{}_{q}\ni X=X^{i}\left(\frac{\partial}{\partial b^{i}}-p^{i}{}_{a}C_{j}{}^{k}{}_{i}(b)\frac{\partial}{\partial p^{k}{}_{a}}\right),$$

where $(x^{i}, b^{i}, p_{a}{}^{i})$ is the canonical coordinate of $q=(b, p)$. Comparing the general expressions of $X\in\Gamma^{h}{}_{q}$ and $\in\Gamma^{v}{}_{q}$, we see that, in place of coefficients $F^{i}{}_{k}$, coefficients $b^{j}F_{j}{}^{i}{}_{k}$ appear in (3. 8) and (3.9), which shows that the condition F holds for the Finsler connection as above.

On the other hand, the condition of homogeneity (*Cond. H*) is essential for a Finsler connection [5], and, however, the condition does not always hold good for the Finsler connection as above. This condition is expressed by the fact that any v-basic vector field $B^{v}(f)$ is positively homogeneous of degree 1, and any h-basic vector field $B^{h}(f)$ is of degree 0.

Proposition 3. *The necessary and sufficient condition for the*

Finsler connection as given in Proposition 2 to satisfy the condition of homogeneity is that any basic vector field $B^c(f)$ *of the vertical connection C is positively homogeneous of degree* 1 *and any f-basic vector field* $B_f(f_1)$ *of the system of f-relative linear connections* $\{\Gamma(f)\}$ *is* R^+*-invariant, that is,*

$$B_{zf}(f_1)=B_f(f_1), \qquad z\in R^+. \tag{3. 10}$$

Proof. It is clear that l^c (operation of lift with respeet to C) and l (the one with respect to (Γ^v, Γ^h)) satisfy the equation $l^c v'=lv'$. It follws from (2. 1) and the definition of $B^v(f)$ that $B^v(f)$ is equal to $B^c(f)$. Therefore the first condition is necessary. Next, we shall first show that

$$B^h(f_1)_q=\overline{K}_f B_f(f_1)_p, \qquad q=(b, p), \quad f=p^{-1}b. \tag{3. 11}$$

In fact, it follows from (3. 1) that $\overline{K}_f B_f(f_1)_p \in \Gamma^h{}_q$. Let θ^h be the h-basic form of the Finsler connection [4], and we know from (1. 6) that

$$\begin{aligned}\theta^h(\overline{K}_f B_f(f_1)_p)&=p^{-1}\tau\bar{\pi}(\overline{K}_f B_f(f_1)_p)=p^{-1}\tau K_f B_f(f_1)_p\\ &=p^{-1}\pi B_f(f_1)_p=\theta B_f(f_1)_p=f_1.\end{aligned}$$

Therefore (3. 11) is proved. By means of the relation $\overline{h}_z\overline{K}_f=\overline{K}_{zf}$, it follows from (3. 11) that

$$\overline{K}_{zf}B_f(f_1)_p=\overline{h}_z B^h(f_1)_q=B^h(f_1)_{zq}=\overline{K}_{zf}B_{zf}(f_1)_p,$$

and, acting the induced mapping η on the above equation, we have (3. 10) directly. The sufficiency of the conditions will be shown without difficulty.

§ 4. On axioms given by R. Sulanke

To illustrate the notion of relative covariant derivative, we take a tensor T of (1, 1)-type, which is a mapping $q\in Q\rightarrow T(q)\in F\otimes F^*$. The f_1-relative covariant differential of T with respect to the system of f-relative linear connections $\{\Gamma(f)\}$ is by definition $(\overline{K}_f B_{f_1}(f))_q(T)$, where $q=(b,p)$, $f=p^{-1}b$, and $B_{f_1}(f)$ is the f_1-relative basic vector field corresponding to $f\in F$ with respect to the f_1-relative linear connection $\Gamma(f_1)$. It follows from (1. 9) that the covariant differential is expressed by

(4. 1) $$(\overline{K}_f B_{f_1}(f))_q(T) = T^i_{j|k}(f_1) p_i^{-1a} p^j_b b^k e_a \otimes e^b,$$

where $q=(x^i, b^i, p^i_a)$, and $T=T^a_b e_a \otimes e^b$, $T^i_j = T^i_j p^{-1a}_i p^j_b$. Coefficients $T^i_{j|k}(f_1)$ of the right hand of (4. 1) are called components of the *f_1-relative covariant derivative* of T, which are respresented by

(4. 2) $$T^i_{j|k}(f_1) = \frac{\partial T^i_j}{\partial x^k} - \frac{\partial T^i_j}{\partial b^l} F_h{}^l{}_k(pf_1) b^h + T^l_j F_l{}^i{}_k(pf_1) - T^i_l F_j{}^l{}_k(pf_1).$$

On the other hand, the absolute covariant differential of T is by definition $(\overline{K}_f B(f))_q(T)$, where $q=(b, p)$, $f=p^{-1}$b, and $B(f)$ is the absolute basic vector field on P. Coefficients $T^i_{j|k}(p^{-1}b)$ are components of the *absolute covariant derivative* of T.

With respect to the vertical connection C, we also have the covariant differential of T, which is defined by $B^v(f)(T)=B^c(f)(T)$, and further, we have the covariant derivative $T^i_j|_k$ of T.

Now, let a Finsler fundamental metric function L be given. Then, the vertical connection C is uniquely determined by the hypothesis that the connection be symmetric and the covariant derivative of the metric tensor G induced from L as usual vanish, that is,

$$\frac{\partial g_{ij}}{\partial b^k} - g_{lj} C_i{}^l{}_k - g_{il} C_j{}^l{}_k = 0,$$

where g_{ij} are components of the metric tensor G. Then $C_j{}^l{}_k$ are given by

(4. 3) $$C_{ijk}(x^h, b^h) = g_{jl} C_i{}^l{}_k = \frac{1}{2} \frac{\partial g_{ij}}{\partial b^k}.$$

In order to determine the system of f-relative linear connections $\{\Gamma(f)\}$, R. Sulanke requires the following three axioms:

I. *The f-relative linear connections are symmetric,* that is,

$$F_j{}^i{}_k = F_k{}^i{}_j.$$

II. *The extremal of the variation problem of the length integral coincides with the absolute path,* namely

$$F_{j\,k}^{\,i}(x^h,\ b^h)b^jb^k=\gamma_{j\,k}^{\,i}(x^h,\ b^h)b^jb^k,$$

where $\gamma_{j\,k}^{\,i}$ are Christoffel's symbols constructed from g_{ij}.

III. *The absolute covariant derivative of the metric tensor G vanishes,* namely

$$\frac{\partial g_{ij}}{\partial x^k}-\frac{\partial g_{ij}}{\partial b^l}F^l{}_k-g_{lj}F_{i\,k}^{\,l}-g_{il}F_{j\,k}^{\,l}=0,$$

where we put $F^l{}_k(x^h,\ b^h)=b^iF_{i\,k}^{\,l}(x^h,\ b^h)$.

Later on, R. Sulanke show [9] that the axiom II is a result from the other two.

As for the axiom III, it is well known that both of Cartan's and Rund's connections satisfy the axiom, while Berwald's connection does not so [1]. We will be concerned, in the sequel, with deriving connetions, which do not satisfy the axiom III. Putting

$$P_{ijk}=g_{ij|k}=\frac{\partial g_{ij}}{\partial x^k}-\frac{\partial g_{ij}}{\partial b^l}F^l{}_k-g_{lj}F_{i\,k}^{\,l}-g_{il}F_{j\,k}^{\,l},\tag{4. 4}$$

we say that a connection is *of Berwald' type,* if P_{ijk} does not vanish. It must be supposed, of course, that P_{ijk} be positively homogeneous of degree 0 and symmetric with respect to the first two indices.

Suppose that P_{ijk} be the tensor determined by the given Finsler metric L, and we shall show that a connection of Berwald's type is determined uniquely by L, such that (4. 4) and the axiom I hold. To do this, we put

$$Q_{ijh}=\frac{1}{2}(P_{ijh}+P_{ikj}-P_{jki}).\tag{4. 5}$$

It follows from (4. 4) and the axiom I that

$$\gamma_{jik}+C_{jkl}F^l{}_i-C_{ijl}F^l{}_k-C_{ikl}F^l{}_j-F_{jik}=Q_{ijk}.\tag{4. 6}$$

Contracting (4. 6) by the element of support b^k, we have

$$\gamma_{jio}-C_{ijl}F^l{}_o-F_{jio}=Q_{ijo},\tag{4. 7}$$

where we denote by the index o the contraction by b^k. Moreover, contraction of (4. 7) by b^j gives

$$(4.8)\qquad \gamma_{olo}-F_{olo}=Q_{loo}.$$

Thus we have $F_{olo}=\gamma_{olo}-Q_{loo}$, and, inserting these in (4. 7), we obtain

$$F_{jlo}=\gamma_{jlo}-Q_{ljo}-C_{ljl}(\gamma_{o}{}^{l}{}_{o}-Q^{l}{}_{\cdot oo}).$$

Consequently we have from (4. 6) that

$$(4.9)\qquad F_{j}{}^{i}{}_{k}=\underset{\circ}{F}_{j}{}^{i}{}_{k}-Q^{i}{}_{\cdot jk}-C_{j}{}^{l}{}_{k}Q_{l\cdot}{}^{i}{}_{o}+C_{j}{}^{i}{}_{l}Q^{l}{}_{\cdot ko}+C_{k}{}^{i}{}_{l}Q^{l}{}_{\cdot jo}$$
$$+Q^{h}{}_{\cdot oo}(C_{j}{}^{l}{}_{k}C_{l}{}^{i}{}_{h}-C_{j}{}^{i}{}_{l}C_{k}{}^{l}{}_{h}-C_{k}{}^{i}{}_{l}C_{j}{}^{l}{}_{h}),$$

where we put

$$(4.10)\qquad \underset{\circ}{F}_{j}{}^{i}{}_{k}=\gamma_{j}{}^{i}{}_{k}+C_{j}{}^{l}{}_{k}\gamma^{i}{}_{\cdot lo}-C_{j}{}^{i}{}_{l}\gamma_{o}{}^{l}{}_{k}-C_{k}{}^{i}{}_{l}\gamma_{o}{}^{l}{}_{j}$$
$$-\gamma_{o}{}^{h}{}_{o}(C_{j}{}^{l}{}_{k}C_{l}{}^{i}{}_{h}-C_{j}{}^{i}{}_{l}C_{k}{}^{l}{}_{h}-C_{k}{}^{i}{}_{l}C_{j}{}^{l}{}_{h}).$$

Consequently we have

Proposition 4. *Suppose that a tensor* P_{ijk} *be given by the Finsler metric L, such that* P_{ijk} *be symmetric with respect to the first two indices and positively homogeneous of degree* 0. *Then a connection of Berwald's type satisfying* (4. 4) *and the axiom I is uniquely determined by L, coefficients of the connection* $F_{j}{}^{i}{}_{k}$ *being given by* (4. 9).

In particular, if we put $P_{ijk}=0$, coefficients of the connection $\underset{\circ}{F}_{j}{}^{i}{}_{k}$ as defined by (4. 10) are obtained, which coincide with Cartan's $\Gamma_{j}{}^{*i}{}_{k}$ [3] or Rund's $P_{j}{}^{*i}{}_{k}$ [7] or Sulanke's $P_{j}{}^{i}{}_{k}$ [8].

If the desirable axiom II is further imposed, it follows from (4. 8) that Q_{ioo} must vanish, and hence we have from (4. 5) that P_{ijk} must satisfy the condition

$$(4.11)\qquad 2P_{ioo}-P_{ooi}=0.$$

Thus we have from Proposition 4

Corollary. *Assuming that* P_{ijk} *be the tensor such as mentioned in Proposition* 4 *and satisfy* (4. 11), *a connection of Berwald's type satisfying* (4. 4) *and axioms I, II is uniquely determined by the Finsler metric. Coefficients of the connection* $F_{j}{}^{i}{}_{k}$ *are given by*

$$(4.12)\qquad F_{j}{}^{i}{}_{k}=\underset{\circ}{F}_{j}{}^{i}{}_{k}-Q^{i}{}_{\cdot jk}-C_{j}{}^{l}{}_{k}Q_{l}{}^{i}{}_{\cdot o}+C_{j}{}^{i}{}_{l}Q^{l}{}_{\cdot ko}+C_{k}{}^{i}{}_{l}Q^{l}{}_{\cdot jo}.$$

In the case of the connection given by L. Berwald, the tensor P_{ijk} is equal to $-2\ A_{ijk|0}$ (symbols used by E. Cartan) [2], [3], [7], which satisfies (4. 11) clearly, but it seems somewhat complicated. In the following we shall give two examples of a connection of Berwald's type.

A. $$P_{ijk}=g_{ij}l_k.$$

Following E. Cartan, we denote by l_k covariant components of the unit vector having the same direction with the element of support b^i. It is obvious that this P_{ijk} does not satisfy (4. 11). From (4. 9) we obtain

$$F_j{}^i{}_k=\underset{c}{F}{}_j{}^i{}_k+\frac{1}{2}A_j{}^i{}_k-\frac{1}{2}(\delta^i_j l_k+\delta^i{}_k l_j-g_{jk}l^i). \tag{4. 13}$$

This connection satisfies the axiom I, but not II and III.

B. $$P_{ijk}=A_{ijk}.$$

It is clear that this P_{ijk} satisfies (4. 11). In this case, we have from (4. 12) that

$$F_j{}^i{}_k=\underset{\circ}{F}{}_j{}^i{}_h-\frac{1}{2}A_j{}^i{}_k. \tag{4. 14}$$

This connection satisfies both of axioms I and II, but not III.

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Note : In *1937*, A Kawaguchi gave the method of metrisation of a non-metrical linear connection (Ned. Akad. Wet. Proc. Ser. A 40, 596-601). By means of this method, we can obtain metrical connections from two non-metrical ones given in the final section of the present paper. It is easily shown that the connection obtained as thus from **B** coincides with the famous connection of Cartan, but not the one from **A**.