J. Math. Kyoto Univ.
4-3 (1965) 471-475.

On torsion of the module of differentials of a locality which is a complete intersection

By

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(Received Feb. 16, 1965)

In his paper [1], E. Kunz introduced the concept of differential constant fields of algebraic varieties or localities over a field of prime characteristics. According to his definition, it does not depend on the choice of differential constant field whether the module of differentials of locality is free or not. He proved that the freeness of the module of differentials is equivalent to the regularity of the locality. (I also proved it in a rather analytical way in [3]). It is also independ of the choice of a differential constant field whether the module of differentials of a locality has torsion or not. In the present paper, we prove mainly that the torsionfreeness of the module of differentials is equivalent to the normality of the locality, in case where it is a complete intersection. This result can be extended easily to the same assertion on an affine ring. The case where the field of definition is of characteristic zero is treated at the same time.

The general case, where a locality or an affine ring is not necessarily a complete intersection, is more complicated. We give an example of a normal affine ring whose module of differential has torsion. It seems to me very likely to be true that the module of differentials of a locality which is not normal has torsion. However, I have no proof of it in this general case.

We refer mainly to Kunz [1]. So we use the same notations and terminology apeared in it for the convenience of the readers, although we used other notations in my former papers than those used by him.

I would like to express my heartfelt thanks to Professor M. Nagata for his suggestions on this work.

1. Preliminaries: Let $A=k[x_1, \cdots, x_n]$ be an affine ring over a field k of characteristic p, with quotient field K, which is a residue class ring of a polynomial ring $B=k[X_1, \cdots, X_n]$ with respect to a prime ideal $\mathfrak{S}$ of height r. If $\mathfrak{S}$ is generated by r elements, we call A a complete intersection. Let k^* be either k itself in case $p=0$ or a differential constant field of A in case $p>0$. Let $z_1, \cdots, z_t$ be a p-independent base of k over k^* in the latter case. We denote by $\bar{d}$ the natural derivation map of B into $K\otimes_B M\left(\frac{B}{k^*}\right)$ $\left(=K\otimes_A\left(A\otimes_B M\left(\frac{B}{k^*}\right)\right)\right)$. If $\mathfrak{p}$ is a prime ideal of B which contains $\mathfrak{S}$ and $\bar{\mathfrak{p}}=\mathfrak{p}/\mathfrak{S}$, then we know that $A_{\bar{\mathfrak{p}}}\otimes M\left(\frac{B}{k^*}\right)$ is a free $A_{\bar{\mathfrak{p}}}$-module with a free base $\bar{d}X_1, \cdots, \bar{d}X_n$ in case $p=0$ or $\bar{d}X_1, \cdots, \bar{d}X_n, \bar{d}z_1, \cdots, \bar{d}z_t$ in case $p>0$.

If $f_1(X), \cdots, f_m(X)$ is a set of polynomials in B which generates the ideal $\mathfrak{S}B_{\mathfrak{p}}$ in $B_{\mathfrak{p}}$, then

$$M\left(\frac{A_{\bar{\mathfrak{p}}}}{k^*}\right)=A_{\bar{\mathfrak{p}}}\otimes_B M\left(\frac{B}{k^*}\right)\Big/\sum_{i=1}^{m} A_{\bar{\mathfrak{p}}}\bar{d}f_i$$

Hereafter we assume that k^* is a special type of differential consant field such that k^* is linearly disjoint over k^p with the field which is obtained by adjoining all the coefficients of the f_i to k^p in case $p>0$. Then if we express the $\bar{d}f_i$ as linear combinations of $\bar{d}X_1, \cdots, \bar{d}X_n$ or $\bar{d}X_1, \cdots, \bar{d}X_n, \bar{d}z_1, \cdots, \bar{d}z_t$. The matrix of these coefficients $\bar{J}(f_1, \cdots, f_m\colon k^*)$ is the Jacobian matrix or the mixed Jacobian matrix $J(f_1, \cdots, f_m\colon k^*)$ modulo $\mathfrak{S}$, hence we see that $A_{\bar{\mathfrak{p}}}$ is a regular local ring if and only if rank $(\bar{J}(f_1, \cdots, f_m\colon k^*)$ modulo $\bar{\mathfrak{p}})=r$.

Let R be a locality over the field k. We call R a complete intersection if the following conditions are satisfied. B, A, $\mathfrak{S}$, $\mathfrak{p}$ and $\bar{\mathfrak{p}}$

as above can be chosen such as $R=A_{\bar{\mathfrak{p}}}$ and $\mathfrak{S}B_{\mathfrak{p}}$ is generated by r polynomials of B. This is equivalent to say that R is a residue class ring of a regular locality over k with respect to a prime ideal of height r and generated by r elements.

2. Main results: Let R be a locality over the field k which is a complete intersection and take k^*, B, $\mathfrak{S}$, $\mathfrak{p}$, A, $\bar{\mathfrak{p}}$ and $f_1, \cdots, f_r$ ($r=$ height $\mathfrak{S}$) as above.

Theorem: *R is normal if and only if $M\left(\frac{R}{k^*}\right)$ is torsionfree.*

Proof. We prove here the assertion only in case $p>0$, because the case $p=0$ can be proved in a quite analogous way.

First we shall prove the only if part and we assume that R is normal. We have only to prove that if there is a relation

$$h\left(\sum_{i=1}^{n} a_i \bar{d}X_i + \sum_{j=1}^{t} b_j \bar{d}z_j\right) = \sum_{k=1}^{r} l_k \bar{d}f_k \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

where the h, a_i, b_j and l_k are elements of R and $h\neq 0$, then it holds that $\frac{l_k}{h}\in R$ for every $k=1, 2, \cdots, r$. Since R is the intersection of all quotient rings of R with respect to its prime ideals of height 1, we have only to prove that if $\mathfrak{q}$ is a prime ideal of R of height 1, then every $\frac{l_k}{h}$ belongs to $R_{\mathfrak{q}}$. Sinec $\bar{d}f_k=\sum_{i=1}^{n}\frac{\partial f_k(x)}{\partial X_i}\bar{d}X_i + \sum_{j=1}^{t}\frac{\partial f_k(x)}{\partial z_j}\bar{d}z_j$, (1) means a set of linear relations

$$\left.\begin{aligned} &\sum_{k=1}^{r}\frac{l_k}{h}\,\frac{\partial f_k(x)}{\partial X_i}=a_i \quad (i=1, 2, \cdots, n) \\ \text{and}\quad & \\ &\sum_{k=1}^{r}\frac{l_k}{h}\,\frac{\partial f_k(x)}{\partial z_j}=b_j \quad (j=1, 2, \cdots, t)\end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots (2)$$

Since $R_{\mathfrak{q}}$ is a regular locality, we have rank ($\bar{J}(f_1, \cdots, f_r\colon k^*)$ modulo $\mathfrak{q}$)$=r$. Hence (2) implies that $\frac{l_k}{h}\in R_{\mathfrak{q}}$. Conversely, assume that R is not normal. Let $\mathfrak{c}$ be the conductor of the derived normal ring $\bar{R}$ of R and let $\mathfrak{q}$ be an associated prime ideal of $\mathfrak{c}$. Then $R_{\mathfrak{q}}$ is not normal, hence it is not regular and rank $(J(f_1, \cdots, f_r\colon k^*)$ modulo $\mathfrak{q})<r$. Therefore there exist r elements $a_1, \cdots, a_r$ *of* R, some of which are not contained in $\mathfrak{q}$, say $a_1 \notin \mathfrak{q}$, such that $\sum_{k=1}^{r} a_k \bar{d} f_k \equiv 0$ modulo $\mathfrak{q}R \otimes_B M\left(\frac{B}{k^*}\right)$. We put $\mathfrak{c}' = \mathfrak{c}\colon \mathfrak{q}$. Then there exists an element g of $\bar{R}$, satisfying the conditions: $g \in \mathfrak{c}'\bar{R}$ and $g \notin R_{\mathfrak{q}}$ (the existence is obvious, because $\mathfrak{c}\bar{R}_{(R-\mathfrak{q})}$ is the conductor of $\bar{R}_{(R-\mathfrak{q})}$ with respect to $R_{\mathfrak{q}}$). We express g in the form $\frac{e}{h}$ with h, $e \in R$. Then

$$ga_1\bar{d}f_1 + ga_2\bar{d}f_2 + \cdots + ga_r\bar{d}f_r \in g\mathfrak{q}\left(R \otimes_B M\left(\frac{B}{k^*}\right)\right)$$
$$\subset \mathfrak{c}\bar{R}\left(R \otimes_B M\left(\frac{B}{k^*}\right)\right) \subset R \otimes_B M\left(\frac{B}{k^*}\right).$$

Hence we have an expression

$$ga_1\bar{d}f_1 + ga_2\bar{d}f_2 + \cdots + ga_r\bar{d}f_r = \sum_{i=1}^{n} c_i \bar{d}X_i + \sum_{j=1}^{t} d_j \bar{d}z_j \quad \text{where } c_i,\ d_j \in R.$$

Hence

$$h\left(\sum_{i=1}^{n} c_i \bar{d}X_i + \sum_{j=1}^{t} d_j \bar{d}z_j\right) = e\left(\sum_{k=1}^{r} a_k \bar{d}f_k\right) \equiv 0 \text{ modulo } R\bar{d}f_1 + \cdots + R\bar{d}f_r.$$

Hence if $\sum_{i=1}^{n} c_i \bar{d}X_i + \sum_{j=1}^{t} d_j \bar{d}z_j \not\equiv 0$ modulo $R\bar{d}f_1 + \cdots + R\bar{d}f_r$, then this is a torsion element of $M\left(\frac{R}{k^*}\right)$. Otherwise, there are r elements $l_1, \cdots, l_r \in R$ such that $\sum_{i=1}^{n} c_i \bar{d}X_i + \sum_{j=1}^{t} d_j \bar{d}z_j = \sum_{k=1}^{r} l_k \bar{d}f_k$. Then $\sum_{k=1}^{r}(ea_k - hl_k)\bar{d}f_k = 0$. Since the $\bar{d}f_k$ are linearly independent over R, we see that $ea_1 - hl_1 = 0$. Hence $g = \frac{e}{h} = \frac{l_1}{a_1} \in R_{\mathfrak{q}}$ because $a_1 \notin \mathfrak{q}$. This is a contradiction.

The following Corollaries 1 and 2 are direct consequences of Theorem, seeing that $M\left(\frac{R}{k_0}\right)$ is a direct summand of $M(R)$ for an arbitrary differential constant field k_0 of R.

Corollary 1. *In case* $p>0$, R *is normal if and only if* $M(R)$ *is torsionfree.*

Corollary 2. *In case* $p>0$, R *is normal if and only if* $M\left(\frac{R}{k_0}\right)$ *is torsionfree for an arbitrary differential constant field* k_0 *of* R.

Corollary 3. *Let A be an affine ring over k, such as every local ring in Spec (A) is a complete intersection (especially, it is so if A itself is a complete intersection). Then in case $p=0$, A is normal if and only if $M\left(\frac{A}{k}\right)$ is torsionfree, and in case $p>0$, A is normal if and only if $M(A)$ is torsionfree or, equivalently, if and only if $M\left(\frac{A}{k_0}\right)$ is torsionfree for an arbitary differential constant field k_0 of A.*

This follows from the fact that $M\left(\frac{A_{\mathfrak{p}}}{k}\right)=A_{\mathfrak{p}}\otimes_A M\left(\frac{A}{k}\right)$, $M(A_{\mathfrak{p}})=A_{\mathfrak{p}}\otimes_A M(A)$ and $M\left(\frac{A_{\mathfrak{p}}}{k_0}\right)=A_{\mathfrak{p}}\otimes_A M\left(\frac{A}{k_0}\right)$ for any prime ideal $\mathfrak{p}$ of A.

3. Example: Let k be a perfect field and let x and y be independent variables. Let A be an affine ring $k[x^3, x^2y, xy^2, y^3]$. This is the residue class ring of the polynomial ring $k[X, Y, Z, U]$ with respect to the prime ideal $\mathfrak{S}$ which is generated by three polynomials

$$f_1=Y^2-XZ,\quad f_2=Z^2-UY,\quad f_3=YZ-XU.$$

It is easily seen that A is a normal ring. We have

$$\bar{d}f_1=-xy^2\bar{d}X+2x^2y\bar{d}Y-x^3\bar{d}Z,\quad \bar{d}f_2=-y^3\bar{d}y+2xy^2\bar{d}Z-x^2y\bar{d}U,$$
$$\bar{d}f_3=-y^3\bar{d}X+xy^2\bar{d}Y+x^2y\bar{d}Z-x^3\bar{d}U.$$

Put $m=-y^3\bar{d}X+2xy^2\bar{d}Y-x^2y\bar{d}Z$. m is not contained in $A\bar{d}f_1+A\bar{d}f_2+A\bar{d}f_3$ but $x^3m=x^2\bar{d}f_1$.

Hence

$$M\left(\frac{A}{k}\right)=A\bar{d}X+(A\bar{d}Y+A\bar{d}Z+A\bar{d}U)/(A\bar{d}f_1+A\bar{d}f_2+A\bar{d}f_3)$$

has torsion.

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