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Note on formally projective modules*

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§1. Let R be a commutative ring with units and $\mathfrak{m}$ an ideal of R. Let M be an R-module. For the simplicity we assume that $\mathfrak{m}$ has a finite base. We consider the $\mathfrak{m}$-adic topology on both R and M. A. Grothendieck introduced the notion of formally projective modules which can simply be stated in our case as follows (19.2, Chap. 0_{IV} [1]).

Definition 1: M is called a formally projective module if $M/\mathfrak{m}^n M$ is a projective $R/\mathfrak{m}^n$-module for every $n=1, 2, 3, \cdots$.

On the other hand the authur introduced the notion of $\mathfrak{m}$-adic free modules (Def. 1, 2, Part I, [2]), i.e.

Definition 2: M is called an $\mathfrak{m}$-adic free module if M is a Hausdorff $\mathfrak{m}$-adic module and contains a set of elements $\{\alpha_i\}_{i\in I}$ such that $M/\mathfrak{m}^n M$ is a free $R/\mathfrak{m}^n$-module with a free basis {the residue clase of a_i mod. $\mathfrak{m}^n M\}_{i\in I}$ for every $n=1, 2, 3, \cdots$. In this case we call $\{\alpha_i\}_{i\in I}$ $\mathfrak{m}$-adic free basis of M.

We introduce here a generalized notion of $\mathfrak{m}$-adic free modules.

Definition 3: M is called a weakely $\mathfrak{m}$-adic free module if

(a) $M/\mathfrak{m}^n M$ is a free $R/\mathfrak{m}^n$-module for every $n=1, 2, 3, \cdots$, or equivalently

(b) the $\mathfrak{m}$-adic completion of M is isomorphic to the $\mathfrak{m}$-adic completion of a free R-module.

As for the equivalence of (a) and (b), we shall see it afterwards. $\mathfrak{m}$-adic free modules are weakly $\mathfrak{m}$-adic free modules.

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Conversely, in case where R is a local ring every Hausdorff weakly $\mathfrak{m}$-adic free R-module is an $\mathfrak{m}$-adic free module (see Th. 1, Part I, [2]).

We intend to show that there is some relationship between formally projective modules and $\mathfrak{m}$-adic free modules analogous to the one between projective modules and free modules, and to study some related problems.

§ 2. For the brevity we put $R_n = R/\mathfrak{m}^n$ and $M_n = M/\mathfrak{m}^n$ for every $n = 1, 2, \cdots$. Assume that M is a formally projective module. Then M_1 is a direct summand of a free R_1-module. Hence there exist a free R-module F and R_1-homorphisms $\varphi_1 : M_1 \to F_1$ and $\psi_1 : F_1 \to M_1$ such that $\psi_1 \circ \varphi_1 = id_{M_1}$. (We put $F_n = F/\mathfrak{m}^n F$ for every $n = 1, 2, 3, \cdots$). Then by induction, we can construct R_n-homomorphisms $\varphi_n : M_n \to F_n$ and $\psi_n : F_n \to M_n$ for every $n = 1, 2, 3, \cdots$ such that $\psi_n \circ \varphi_n = id_{M_n}$ and the following commutative diagram holds:

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
M_{n+1} & \xrightarrow{\varphi_{n+1}} & F_{n+1} & \xrightarrow{\psi_{n+1}} & M_{n+1} \\
\downarrow{\alpha_n} & & \downarrow{\beta_n} & & \downarrow{\alpha_n} \\
M_n & \xrightarrow{\varphi_n} & F_n & \xrightarrow{\psi_n} & M_n \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
M_1 & \xrightarrow{\varphi_1} & F_1 & \xrightarrow{\psi_1} & M_1
\end{array}
$$

where α_n and β_n are the natural homomorphisms of M_{n+1} and F_{n+1} onto M_n and F_n respectively. Actually, suppose that we have φ_n and ψ_n of the said properties. Then by the projectivity of M_{n+1} and F_{n+1} over R_{n+1} we see that there exist φ_{n+1} and ψ' which satisfy the commutative diagram:

$$
\begin{array}{ccccc}
M_{n+1} & \xrightarrow{\varphi_{n+1}} & F_{n+1} & \xrightarrow{\psi'} & M_{n+1} \\
\downarrow{\alpha_n} & & \downarrow{\varphi_n} & & \downarrow{\alpha_n} \\
M_n & \xrightarrow{\beta_n} & F_n & \xrightarrow{\psi_n} & M_n .
\end{array}
$$

If we know that $\psi' \circ \varphi_{n+1}$ is an isomorphism, φ_{n+1} and $\psi_{n+1} = (\psi' \circ \varphi_{n+1})^{-1} \circ \psi'$ satisfy the required properties. The surjectivity of

$\psi' \circ \varphi_{n+1}$ follows from the fact that M_{n+1} is discrete in the $\mathfrak{m}$-adic topology and $M_{n+1} = \psi' \circ \varphi_{n-1}(M_{n-1}) + \mathfrak{m}^h M_{n+1}$. Again by the projectivity of M_{n+1}, we see that there exists an R_{n+1}-homomorphism $\gamma : M_{n+1} \rightarrow M_{n+1}$ such that $\psi' \circ \varphi_{n+1} \circ \gamma = id_{M_{n+1}}$. The surjectivity of γ can be proved by the similar reasoning as above. Hence $\psi' \circ \varphi_{n+1}$ is injective.

REMARK: In the above description, if we assume that φ_n is surjective, we can show the surjectivity of φ_{n+1}, using the same reasoning as above again.

§ 3. We denote by $\hat{M}$ and $\hat{F}$ the $\mathfrak{m}$-adic completions of M and F. They are projective limits of the systems $\{M_n\}$ and $\{F_n\}$ respectively.

Proposition: *The conditions (a) and (b) in the definition* 3 *are equivalent to each other.*

Proof: (b) $\Longrightarrow$ (a) is trivial. (a) $\Longrightarrow$ (b) follows follows from the remark at the end of § 2.

Theorem 1: *M is a formally projective R-module if and only if M is a direct summand of a weakly $\mathfrak{m}$-adic free module.*

Proof. The if part is obvious. Conversely, assume that M is a formally projective R-module. Then taking the projective limits of $\{\varphi_n\}$ and $\{\psi_n\}$ in § 2, we see that there exists an R-module N such that $\hat{M} \oplus N = \hat{F}$. Put $F' = M \oplus N$. Then the $\mathfrak{m}$-adic completion of F' is $\hat{M} \oplus N$. Hence F' is a weakly $\mathfrak{m}$-adic free R-module.

Theorem 2: *M is a weakly $\mathfrak{m}$-adic free R-module if and only if M is a formally projective R-module and $M/\mathfrak{m}M$ is a free $R/\mathfrak{m}$-module.*

Proof. The only if part is trivial. The if part follows from the remark in § 2, for by our assumption we see that all the φ_n in § 2 are surjective, which shows that $\hat{M}$ is isomorphic to a completion of a free module.

Corollary 1: *Assume that R is a local ring and $\mathfrak{m}$ is its* *maximal ideal. Then the following three conditions are equivalent to each other:*

(1) *M is a formally projective R-module,*

(2) *M is a weakly $\mathfrak{m}$-adic free R-module*

and

(3) *the Hausdorffization $M/\bigcap_{n=1}^{\infty}\mathfrak{m}^n M$ of M is an $\mathfrak{m}$-adic free module.*

Proof: This follows directly from Th. 2.

Corollary 2: *Assume that R is a semi-local ring and $\mathfrak{m}$ is its Jacobson radical. Let $\hat{R}$ be the completion of R. Then M is formally projective if and only if the completion $\hat{M}$ of M is $\hat{R}$-isomorphic to the completion of a projective $\hat{R}$-module.*

Proof: This follows directly from Corollary 1, because $\hat{R}$ is a direct sum of a finite number of complete local rings.

Remark: In Corollary 2, it is impossible to replace our statement "$\hat{M}$ is $\hat{R}$-isomorphic to the completion of a projetive $\hat{R}$-module" by the statement "$\hat{M}$ is $\hat{R}$-isomorophic to the completion of a projective R-module", except in the case where R is a local ring. This situation will be shown by the following example.

Example: Let R be a semi-local domain which is not a local ring. Let $\mathfrak{P}$ be one of its maximal ideals. $R_{\mathfrak{P}}$ is a formally projective R-module, because of Corollary 2. On the other hand every projective R-module is a free R-module. Hence the completion of $R_{\mathfrak{P}}$ can not be expressed as a completion of a projective R-module.

REFERENCES

[1] A. Grothendieck, Éléments de Géométrie Algébrique IV, Publications Math., No. 20, 1964.

[2] S. Suzuki, Some results on Hausdorff $\mathfrak{m}$-adic modules and $\mathfrak{m}$-adic differentials, J. Math. Kyoto Univ. vol. 2, no. 2, 1963.