J. Math. Kyoto Univ.
6-2 (1966) 313-322

ε-entropy of subsets of the spaces of solutions of certain partial differential equations

by

Shun'ichi TANAKA

(Received and communicated by Prof. H. Yoshizawa, on December 10, 1966.)

§1. Introduction

In recent years ε-entropy of subsets of various function spaces were estimated by A. N. Kolmogorov and B. M. Tikhomirov [2] and by others. In the present paper we estimate ε-entropy of sets in the space of harmonic functions (published earlier in [3]) and the space of solutions of certain parabolic equation.

Our results are stated in §2 after the definition of ε-entropy is stated. We prove two lemmas in §3 and the conditions of these lemmas are examined separately for each case in §4 and §5.

The author expresses his hearty thanks to Professor H. Yoshizawa who suggested (in 1962) the problem of estimating the ε-entropy of sets in space of solutions of partial differential equations.

§2. Definitious and statement of the results

Following [2], we shall list definitions which are necessary to state our results.

Let R be a metric space and A a subset of R.

Difinition 1. A system γ of sets $U\subset R$ is called *ε-covering* of A, if $A\subset\bigcup_{U\in\gamma} U$ and the diameter of each $U\in\gamma$ does not exceed 2ε.

Definition 2. A set B in R is called *ε-separated* if the distance of any distinct points of B are greater than ε.

Now assume that the set A is totally bounded.

Definition 3. $N(\varepsilon, A)$ is the minimal number of elements of all possible ε-coverings of A. $H(\varepsilon, A)=\log N(\varepsilon, A)$ is called ε-*entropy* of the set A ($\log N$ will always mean the logarithm of the number N in the base 2).

Definition 4. $M(\varepsilon, A)$ is the maximal number of points in all possible ε-separated subsets of the set A. $C(\varepsilon, A)=\log M(\varepsilon, A)$ is called the ε-*capacity* of A.
Obviously

(1) $M(2\varepsilon, A)\leqslant N(\varepsilon, A)$

Let $f(\varepsilon)$ and $g(\varepsilon)$ be positive functions of ε defined for $0<\varepsilon<\varepsilon_0$. We write $f\sim g$ if $\lim_{\varepsilon\to 0} f(\varepsilon)/g(\varepsilon)=1$ and $f\lesssim g$ if $\lim_{\varepsilon\to 0} f(\varepsilon)/g(\varepsilon)\leqq 1$.

Let K be a continuum in finite dimensional space and G be an open set containing K. For bounded continuous function $u(x)$ in K, we define $\|u\|=\sup_{x\in K}|u(x)|$. We consider class $F_G(C)$ of continuous functions in G which satisfies some condition (F) in G and bounded in G by the constant C. We introduce the metric $\|u\|$ in $F_G(C)$ and denote it by $F_G^K(C)$.

Now we state our results.

i) Harmonic functions. In q-dimensional Euclid space, put $K_r=\{x, |x|\leqslant r\}$ and $G=\{x; |x|<R\}$, where $|x|^2=x_1^2+\cdots+x_q^2$ for $x=(x_1,\cdots,x_q)$. Condition (F) in this case means that $u(x)$ is harmonic in G and we write $H_R^r(C)$ instead of $F_{G_R}^{K_r}(C)$. Then

(2) $H(\varepsilon, H_R^r(C))\ \ (C(2\varepsilon, H_R^r(C)))$
$=\{4/q!\ (\log R/r)^{q-1}\}(\log 1/\varepsilon)^q+O((\log 1/\varepsilon)^{q-1}\log\log 1/\varepsilon)$.

ii) Solutions of certain parabolic equation. Let $K=I^s\times[0,\infty)$ and $G=I^s\times(-T,\infty)$, where $I=[0, 1]$ and T is a fixed positive number. Condition (F) in this case is that $u(x, t)$ satisfies

(3) $\dfrac{\partial}{\partial t}u(x, t)=-(-\Delta)^p u(x, t)$

in G and we denote $P_T(C)$ instead of $F_G^K(C)$. Then

(4) $H(\varepsilon, P_T(C)$ $(C(2\varepsilon, P_T(C))$
$\sim \{4p\Omega_{s-1}/s(2p+s)\,(2\pi)^s\,(T\log e)^{s/2p}\}\,(\log 1/\varepsilon)^{(s/2p)+1}$

where

$$\Omega_{s-1}=2\pi^{s/2}/\Gamma\left(\frac{s}{2}\right)$$

§3. Fundamental lemmas

Let R be a normed space with the system of elements $\{\varphi_k, k\in K\}$. Let further the following conditions be satisfied: Any $f\in R$ is expanded uniquely as

(5) $f=\sum_{k\in K}c_k(f)\varphi_k$.

Let the decomposition of K, $K=K_0+K_1+\cdots$, be given. Let $S_l=K_0+K_1\cdots+K_l$ and d_l, P_l be the number of elements in K_l, S_l respectively. $P_l=d_0+\cdots+d_l$.

Lemma 1. Assume in expansion (5) that

(6) $\|f\|\leqslant C'\sum_{k\in K}|c_k(f)|$.

Let $b_0, b_1, \cdots$ be positive constants such that $\sum_{l=0}^{\infty}b_l d_l<\infty$ and, for $f\in A\subset R$, $|c_k(f)|\leqslant b_l$ $(k\in K_l)$. Let $n(\varepsilon)$ be a number satisfying

(7) $\sum_{l>n(\varepsilon)}b_l d_l\leqslant\varepsilon/2C'$,

then

(8) $H(\varepsilon, A)\leqslant 2P_n\log(1/\varepsilon)+\sum_{l=0}^{n(\varepsilon)}d_l\log b_l+2P_{n(\varepsilon)}\log P_{n(\varepsilon)}$
$+O(P_{n(\varepsilon)})$.

Proof. Put $n=n(\varepsilon)$. Let T_n be the mapping from R to the finite dimensional subspace R_n, spanned by $\{\varphi_k, k\in S_n\}$, defined by $T_nf=\sum_{k\in S_n}c_k(f)\varphi_k$. If $f\in A$, then $\|f-Tf\|\leqslant\varepsilon/2$.

Let us further define the mapping S_n from R_n to $2\times P_n$ matrices. S_n is defined by

$$S_n(\textstyle\sum_{k\in S_n}c_k\varphi_k)=\begin{pmatrix}m_1^1\cdots m_{P_n}^1\\ m_1^2\cdots m_{P_n}^2\end{pmatrix}$$

where

$$m_k^i = [\sqrt{2}P_nC'\,c_k^i/\varepsilon] \qquad (c_k = c_k^1 + \sqrt{-1}c_k^2)$$

for $k \in S_n$.

Let, for $f, f' \in A$, $S_nT_nf = S_nT_nf'$. Then

$$\begin{aligned}\|f-f'\| &\leqslant \|f-T_nf\| + \|f'-T_nf'\| + \|T_nf - T_nf'\| \\ &\leqslant \varepsilon + \|\textstyle\sum_{k\in S_n}\{c_k(f) - c_k(f')\}\varphi_k\| \\ &\leqslant \varepsilon + C'\textstyle\sum_{k\in S_n}|c_k(f) - c_k(f')| \qquad \text{(by (6))} \\ &\leqslant \varepsilon + C'P_n(\varepsilon/P_nC') = 2\varepsilon.\end{aligned}$$

So $N(\varepsilon, A)$ is estimated from above by N, the number of elements in the set $S_nT_n(A)$.

If we put $(m_k^i) = (ST)(f)$, for $f \in A$, we have

$$|m_k^i| \leqslant (\sqrt{2}P_nC'\,b_l/\epsilon) + 1 \qquad (k \in K_l).$$

So $N \leqslant \Pi_{k\in S_n} N_k^2$, where

$$N_k = (2\sqrt{2}P_nC'\,b_l/\varepsilon) + 3 \qquad (k \in K_l).$$

Consequently

$$\begin{aligned}H(\varepsilon, A) &\leqslant \log N \leqslant 2\textstyle\sum_{k\in S_n}\log N_k \\ &\leqslant 2\textstyle\sum_{k\in S_n}\{\log(1/\varepsilon) + \log b_l + \log P_n + O(1)\} \\ &= 2P_n\log(1/\varepsilon) + 2\textstyle\sum_{l=0}^{n} d_l \log b_l + 2P_n \log P_n + O(P_n).\end{aligned}$$

The Lemma 1 is thus proved.

Lemma 2. We assume that in expansion (5),

$$(9) \qquad |c_k(f)| \leqslant a_l\|f\| \qquad (k \in K_l)$$

and $|c_k(f)| \leqslant b_l'$ $(k \in K_l)$ imply $f = \sum_{k\in K} c_k\varphi_k \in A$.

Then,

$$(10) \qquad C(2\varepsilon, A) \geqslant 2P_{m(\varepsilon)}\log(1/\varepsilon) + \textstyle\sum_{l=0}^{m(\varepsilon)} d_l \log(b_l'/a_l) + O(P_{m(\varepsilon)}),$$

where $m(\varepsilon)$ is a number such that $b_l'/2\sqrt{2}\varepsilon a_l \geqslant 1$ $(1 \leqslant l \leqslant m(\varepsilon))$.

Proof. Put $m = m(\varepsilon)$. The set

$$\begin{aligned}B = \{&f = \textstyle\sum_{k\in S_m} c_k\varphi_k; \\ &c_k = (s_k^1 + is_k^2)2\varepsilon a_l (k \in K_l),\ s_k^i \in \boldsymbol{Z}\}\end{aligned}$$

is 2ε-separated. The subset of B obtained by restricting s_k^i to $|s_k^i| \leqslant [b_l'/2\sqrt{2}\varepsilon a_l]$ $(k \in K_l)$ is contained in A. So we have $M(2\varepsilon, A) \geqslant \Pi_{k \in S_m} M_k^2$, $M_k = 2[b_l'/2\sqrt{2}\varepsilon a_l]+1$. Consequently,

$$
\begin{aligned}
C(2\varepsilon, A) &\geqslant 2\textstyle\sum_{k \in S_m} \log M_k \\
&\geqslant \textstyle\sum_{k \in S_m} \{\log(1/\varepsilon) + \log(b_l'/a_l) + O(1)\} \\
&\geqslant 2P_m \log(1/\varepsilon) + \textstyle\sum_{l=0}^m d_l \log(b_l'/a_l) + O(P_m).
\end{aligned}
$$

The Lemma 2 is thus proved.

§4. The ε-entropy of a set in space of solutions of certain parabolic equation.

Let R be the space of bounded continuous functions $u(x,t)$ satisfying (3) in $(x,t) \in I^s \times [0,\infty)$ with norm $\|u\| = \sup_{(x,t) \in I^s \times [0,\infty)} |u(x,t)|$. Any element of R are expanded uniquely as

$$
(11) \quad u(x,t) = \textstyle\sum_k \exp\{-(2\pi|k|)^{2p} t\} v_k \exp(2\pi i k \cdot x) \qquad (k = (k_1 \cdots, k_s) \in \mathbf{Z}^s),
$$

where v_k are Fourier coefficients $\int_{I^s} u(x,0) \exp(-2\pi i k \cdot x)\, dx$ of $u(x, 0)$. Put $K = \{k = (k_1 \cdots, k_s) \in \mathbf{Z}^s\}$ and $K_l = \{k = (k_1 \cdots, k_s)\,;\ l \leqslant |k| < l+1\}$ $(l = 0, 1, 2, \cdots)$. We take $\varphi_k = \exp\{-(2\pi|k|)^{2p} t\} \exp(2\pi i k \cdot x)$ and $c_k(u) = v_k$, then (11) is written as (5) and

$$
(12) \quad |c_k(u)| \leqslant \|u\| \leqslant \textstyle\sum_k |c_k(u)|.
$$

If $u \in P_T(C)$, we have

$$
|c_k(u)| \leqslant C \exp\{-(2\pi l)^{2p} T\} \qquad (k \in K_l),
$$

So Lemma 1 is applicable with $b_l = C\exp\{-(2\pi l)^{2p} T\}$. In this case

$$
(13) \quad P_n = (\Omega_{s-1} n^s/s)\,\{1 + O(n^{-1})\},
$$

where $\Omega_{s-1} = 2\pi^{s/2}/\Gamma(s/2)$. And we have

$$
\begin{aligned}
(14) \quad \textstyle\sum_{l=0}^n d_l l^{2p} &= \textstyle\sum_{k \in S_n} |k|^{2p} \\
&= (\Omega_{s-1} n^{2p+s}/2p+s)\,\{1 + O(n^{-1})\}. \\
&\qquad \text{(see [2, P. 59]))}
\end{aligned}
$$

We now give an estimate for $n(\varepsilon)$ satisfying (7).

$$\sum_{l>n} b_l d_l \leqslant C \sum_{|k| \geqq n+1} \exp\{-(2\pi|k|)^{2p} T\}$$
$$\leqslant C_1 \int_n^\infty r^{s-1} \exp\{-(2\pi r)^{2p} T\}\, dr$$
$$\leqslant C_2 n^{s'} \exp\{-(2\pi n)^{2p} T\} \qquad (s' \text{ positive number})$$

(henceforce C_i are constants, not affecting our final results). So we can take $n(\varepsilon)$ satisfying

$$\{2\pi n(\varepsilon)\}^{2p} T = \log(1/\varepsilon)/\log e + O(\log\log 1/\varepsilon)$$

(see [2], P46),

therefore

$$(15) \quad n(\varepsilon) = (2\pi)^{-1} \{\log(1/\varepsilon)/T \log e\}^{1/2p} + O(\log\log 1/\varepsilon)$$

By Lemma 1 with estimates (13), (14) and (15), we have

$$(16) \quad \begin{aligned} &H(\varepsilon, P(C)) \lesssim \\ &\{4p\Omega_{s-1}/s(2p+s)(2\pi)^s (T \log e)^{s/2p}\} (\log 1/\varepsilon)^{(s/2p)+1}. \end{aligned}$$

Given $\Delta > 0$, define $C(\Delta) > 0$ such that

$$C(\Delta) \sum_k \exp\{-(2\pi|k|)^{2p} \Delta\} \leqslant C.$$

If

$$|c_k(u)| \leqslant C(\Delta) \exp\{-(2\pi|k|)^{2p}(T+\Delta)\},$$

we have $u(x,t) = \sum_k c_k(u) \cdot \varphi_k \in P_T(C)$. So Lemma 2 is applicable to $b'_l = C(\Delta) \exp\{-(2\pi l)^{2p}(T+\Delta)\}$.
We can take $m(\varepsilon)$ such that

$$(17) \quad m(\varepsilon) = (2\pi)^{-1} \{\log(1/\varepsilon)/(T+\Delta) \log e)\}^{1/2p} + O(1).$$

By Lemma 2 with estimates (13), (14) and (17), we have

$$(18) \quad \begin{aligned} &C(2\varepsilon, P_T(C)) \gtrsim \\ &[4p\Omega_{s-1}/s(2p+s)(2\pi)^s \{(T+\Delta) \log e\}^{s/2p}] (\log 1/\varepsilon)^{(s/2p)+1} \end{aligned}$$

Combining (16) and (18) by (1), we have the desired result (4).

§5. The ε-entropy of set in some classes of harmonic functions.

We denote a point $x = (x, \cdots, x)$ in q-dimensional Euclid space with (ρ, s) where $\rho=|x|$ and s is the intersection of the unit sphere S and the line $\overrightarrow{Ox}$.

Function $u(x)$ which is harmonic in $|x|<r$ and continuous in $|x|\leqslant r$ can be expanded into hyperspherical harmonics in $|x|<r$ (see [1])

$$(19)\quad \begin{aligned} &u(x)=u(\rho, s)=p^{-1}\textstyle\sum_{l=0}^{\infty}(2l+p)(\rho/r)^l u_l(s)\\ &u_l(s)=\int_S u(r, s')\,V_l^{(p)}(\cos\angle sOs')\,ds', \end{aligned}$$

where $q=p+2$, ds$'$ is the normalized uniform measure on S and $V_l^{(p)}(x)$ is defined by

$$(1-2ax+a^2)^{-p/2}=\textstyle\sum_{l=0}^{\infty}a^l\,V_l^{(p)}(x).$$

We list here some properties of the above expansion for later use (see [1]).

(A) $|V_l^{(p)}(\cos\gamma)|\leqslant c_l$, where $c_l=V_l^{(p)}(1)=(l,p)/(1,p)$,
$(\lambda,k)=\Gamma(\lambda+k)/\Gamma(\lambda)=\lambda(\lambda+1)\cdots(\lambda+k-1)$.

(B) Hyperspherical functions of order l form a d_l —dimensional vector space H_l, where $d_l=(2l+p)(l+1,p-1)/(1,p)$.

(C) $\int_S V_l^{(p)}(\cos\angle NOs)^2 ds=pc_l/(2+p)$,

where $N=(0,\cdots,0,1)$.
Henceforce by $P_i(l)$ we mean polynomials with positive coefficients. We choose $P_1(l)$ such that $\{pc_l/(2l+p)\}^{1/2}\leqslant P_1(l)$.

(D) If $y(s)\in H_l$, then

$$(20)\quad |y(s)|^2\leqslant p^{-1}(2l+p)c_l\int_S |y(s)|^2 ds.$$

In fact, putting $u(x)=\rho^l\,y(s)$ in (19), we have

$$y(s)=p^{-1}(2l+p)\int_S y(s)\,V_l^{(p)}(\cos\angle sOs')\,ds'.$$

So by Schwarz' inequality and (C), (20) follows. We choose $P_2(l)$ such that $\{p^{-1}(2l+p)c_l\}^{1/2}\leqslant P_2(l)$.

Now define a norm for bounded functions u on S by $\|u\|'=\sup_{s\in S}|u(s)|$. Then, for expansion (19),

(21) $\|u\| \leqslant p^{-1}\sum_{l=0}(2l+p)\|u_l\|'$

and

(22) $\|u_l\|' \leqslant P_1(l)\|u\|$.

Let $\{y_k^l(s), 1 \leqslant k \leqslant d_l\}$ be a complete orthonormal system in H_l and $u_l(s) \in H_l$ be expanded in $\{y_l^k(s)\}$: $u_l(s) = \sum_{k=1}^{d_l} b_k^l y_k(s)$, where $b_k^l = \int_S u_l(s)\overline{y_k^l(s)}ds$. We have

(23) $\|u_l\|' \leqslant P_2(l)\sum_{k=1}^{d_l}|b_k^l|$ (by (D))

and

(24) $|b_k^l| \leqslant \|u_l\|'$

Therefore we have, combining (21) and (23),

(25) $\|u\| \leqslant \sum_{l=0}^{\infty}\sum_{k=1}^{d_l} P_3(l)|b_k^l|$ $(P_3(l) = p^{-1}(2l+p)P_2(l))$

and, combining (22) and (24),

(26) $|b_k^l| \leqslant P_1(l)\|u\|$.

Now let $u \in H_R^r(C)$. It has an expansion of the form (19) in G_R, where r in (19) is substituted by R. By equating coefficients in this expansion and in (19), we get

$$u_l(s) = (r/R)^l \int_S u(R, s)\, V_l^{(p)}(\cos\angle sOs')ds'.$$

So if $u \in H_R^r(C)$,

(27) $\|u_l(s)\|' \leqslant CP_1(l)(r/R)^l$.

Taking $u_k^l(\rho,s) = (\rho/r)^l y_k^l(s)/P_2(l)$ and $\widetilde{b}_k^l = b_k^l P_3(l)$, we have

(28) $u(\rho, s) = \sum_{l=0}^{\infty}\sum_{k=1}^{d_l}\widetilde{b}_k^l u_k^l(\rho, s)$

and

$$\|u\| \leqslant \sum_{l=0}^{\infty}\sum_{k=1}^{d_l}|\widetilde{b}_k^l|$$

and for $u(\rho, s) \in H_R^r(C)$, we have, by (27),

$$|\widetilde{b}_k^l| \leqslant CP_1(l)P_3(l)(r/R)^l = b_l.$$

For large l and n, we have

$d_l = 2l^p/p! + O(l^{p-1})$,

(29) $P_n = d_0 + \cdots + d_n = 2n^{p+1}/(p+1)! + O(n^p)$

and

(30) $\sum_{l=0}^{n} d_l \log b_l = 2n^{p+2} \log(R/r)/(p+2)p! + O(n^{p+1} \log n)$

We can take $n = n(\varepsilon)$ satisfying $\sum_{l>n} b_l d_l \leqslant \varepsilon/2C$ and

(31) $n(\varepsilon) = \log(1/\varepsilon)/\log(R/r) + O(\log\log 1/\varepsilon)$

Applying Lemma 1 with estimates (29), (30) and (31), we have

(32) $H(\varepsilon, H_R^r(C)) \leqslant \{4/q!(\log R/r)^{q-1}\}(\log 1/\varepsilon)^q$
$+ O((\log 1/\varepsilon)^{q-1} \log\log 1/\varepsilon)$.

Put $h = \log R/r$. Let, for $\varDelta > 0$,

(33) $|\tilde{b}_k^l| \leqslant C'\varDelta \exp\{-(h+\varDelta)l\}/d_l$

i. e. $|b_k^l| \leqslant C'\varDelta \exp\{-(h+\varDelta)l\}/d_l P_3(l)$

and define u by

(34) $u(\rho, s) = p^{-1} \sum_{l=0}^{\infty} (2l+p)(\rho/r)^l u_l(s)$,
$u_l(s) = \sum_{k=1}^{d_l} b_k^l y_k^l(s)$.

We have $\|u_l\|' \leqslant p(2l+p)^{-1} C'\varDelta \exp\{-(h+\varDelta)l\}$, so

$|u(x)| \leqslant \sum_{l=0}^{\infty} (R/r)^l C'\varDelta \exp\{-(h+\varDelta)l\} = C'\varDelta/(1-e^{-\varDelta})$

in G_R.

So we can choose C', independent of $\varDelta$ in $(0, \varDelta_0)$, such that u defined ay (34) with b_k^l satisfying (33) are in $H_R^r(C)$. So Lemma 2 is applicable with $b_l' = C'\varDelta \exp\{-(h+\varDelta)l\}/d_l$. Putting $\varDelta = h/\log(1/\text{-}\varepsilon)$ and estimating $m(\varepsilon)$ defined there, we may take

$m(\varepsilon) = \log(1/\varepsilon)/\log(R/r) + O(\log\log 1/\varepsilon)$.

So we have

(35) $C(2\varepsilon, H_R^r(C)) \geqslant \{4/q!(\log R/r)^{q-1}\}(\log 1/\varepsilon)^q$
$+ O((\log 1/\varepsilon)^{q-1} \log\log 1/\varepsilon)$.

Combining (32) and (35) by (1), we have the desired result (2).

BIOGRAPHY

[1] P. Appell-J. Kempe de Feriet: Foncttions Hypergeometriques et Hyperspheriques, Polynomes d'Hermite, Paris (1926).

[2] A. N. Kolmogorov and B. M. Tikhomirov: *ε-Entropy and ε-capacity of sets in function spaces* (in Russian), Uspekhi Mat. Nauk, **14**, No. 2 (86), 3–59 (1959).

[3] S. Tanaka: *The ε-entropy of some classes of harmonic functions*, Proc. Jap. Acad., Vol. **39**, No.2 (1963), 85–88.