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# Corrections to

*On the characters $\nu^*$ and $\tau^*$ of singularities*

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The proof of Lemma (1. 3), p. 26, is incomplete. The gap may be fixed up by adding a further analysis of the situation in which Hasse derivatives $d(g')=0$ for all choices of $\alpha$. Here, instead, we give an alternative proof which follows the same idea but improves the conclusion slightly.

Let the assumptions and the notation be the same as in Lemma (1. 3), loco cito. We may assume $e'=\{x\in e\,|\,x_0^{-1}x\in Q\}$. Let us pick and fix a free base $y=(y_1, \cdots, y_r)$ of $e'$, and then extend $(y, x_0)$ to a free base $(y, x)$ of $e$, where $x=(x_0, x_1, \cdots, x_s)$. Then we claim:

(1. 3. a) There exist forms $b_i$ of degrees $\nu_m-\nu_i$ in $k[e]$, $1\leq i\leq m-1$, such that if $h=h_m-\sum_{i=1}^{m-1}b_ih_i$, then $h$ is *normalized by* $(h_1, \cdots, h_{m-1})$ *with respect to* $(k(x); y)$. (For the definition of normalizedness, one should refer to a paragraph preceeding Lemma (1. 10), p. 35.)

A proof of (1. 3. a) is obtained by writing out $h_m=\sum_A \psi_A x^A$ with $\psi_A\in k[y]$ and $A\in Z_0^{s+1}$, and then applying Lemma (1. 11), p. 37, to the system $(h_1, \cdots, h_{m-1}, \psi_A)$ for each $A$.

Let us write $h=\sum_{B\in E}f_By^B$ where $E$ is a finite subset of $Z_0^r$ and $f_B\in k[x]$ for all $B\in E$. Let $\varphi_B=f_B/x_0^b$ for each $B\in E$, where $b=\nu_m-|B|$. We then claim:

(1. 3. b) $\varphi_B\in Q^b$ for all $B\in E$.

In fact, let $\lambda_i=y_i/x_0$, $\omega_j=x_j/x_0$ and $g=h/x_0^{\nu_m}$. Since $\lambda=(\lambda_1, \cdots,$

$\lambda_r$) extends to a regular system of parameters of $R$ with additional elements belonging to a localization of $k[\omega]$ with $\omega=(\omega_1, \cdots, \omega_s)$, (1, 3, b) is equivalent to saying that $g\in Q^{\nu_m}$. Suppose $g\notin Q^{\nu_m}$ and let $\beta=\nu_Q(g)$. Then the assumption (3) implies the following:

(3*) There exists $a_i\in Q^{\alpha(i)}$ with $\alpha(i)=\beta-\nu_i$, $1\leq i\leq m-1$, such that $g-\sum_{i=1}^{m-1}a_i g_i\in Q^{\beta+1}$.

To prove this, we can, word by word, follow the early portion (up to line 6, p. 24) of the proof of Lemma (1. 1), p. 22. Then (3*) implies that $g$ is not normalized by $(g_1, \cdots, g_{m-1})$ with respect to $(k(\omega); \lambda)$. But, since $h_i\in k[y]$ for all $i$, this is contradictory to the normalizedness of $h$ of (1. 3. a). We shall next prove

$$\text{(1. 3. c)} \quad h\in k[e'].$$

This clearly suffices for Lemma (1. 3) by (1. 3. a). We shall prove that $\varphi_B\in k$ for all $B\in E$. Note that (1. 3. c) follows from this. In fact, we then have $g\in k[\lambda]\cap Q^{\nu_m}$ by (1. 3. b). Hence $g$ is a form of degree $\nu_m$ in $\lambda$ and $h=gx_0^{\nu_m}$ is also such in $y$. Now, pick any $B\in E$. To prove $\varphi_B\in k$, let $S=R/(\lambda)R$, $P=Q/(\lambda)R$ and $\sigma=$the image of $\varphi_B$ in $S$. We have a natural monomorphism $k[\omega]\to S$, which induces an isomorphism of $S$ with a localization of $k[\omega]$. The following lemma clearly completes the proof of Lemma (1. 3).

*Lemma.* Let $k[\omega]$ be a polynomial ring of $s$ variables over a field $k$. Let $S$ be a localization of $k[\omega]$, and $P$ the maximal ideal of $S$. Let $b$ be a positive integer and $\sigma\in k[\omega]$ with $\deg\sigma\leq b$. Suppose $P$ contains no linear polynomial in $k[\omega]$. Then $\sigma\in P^b$ implies $\sigma=0$.

*Proof.* It is trivially true for $b=1$. By induction on $b$, we assume that $b>1$ and $\deg\sigma=b$. This will lead to a conclusion $\sigma=0$. Let $c\omega^A$ be any term of $\sigma$ with $A\neq(0)$. Let $(d^{(1)}, \cdots, d^{(s)})$ be the system of Hasse differentiations with respect to the variables $\omega$. Let $A=(a(1), \cdots, a(s))$, and say $a(j)\neq 0$. Let $d=d^{(j)}_{a(j)}$. If $a(j)<b$, then $d(\sigma)\in P^{b-a(j)}$ and $d(\sigma)=0$ by induction assumption. This means $c=0$. In other words, $\sigma$ must be of the form $c_0+c_1\omega_1^b+\cdots+c_s\omega_s^b$ with

$c_i \in k$. If $b$ is not a power of the characteristic of $k$, then we have an integer $a$, $1 \leq a < b$, such that $\deg(d_a^{(j)}(\sigma)) = b - a$. By induction assumption, this shows that the characteristic $p$ of $k$ is positive and $b$ must be a power of $p$. Now, let $\pi$ be any derivation of $k$, which extends to $k[\omega]$, trivially on $\omega$, and then to $S$. We have $\pi(\sigma) \in P^{b-1}$. But $\pi(\sigma)$ is clearly a $k$-linear combination of the $\omega_i^b$, so that $\nu_P(\pi(\sigma))$ must be a power of $p$. (In fact, $b$ is a power of $p$ and, by a suitable purely inseparable extension of the base field $k$, $\sigma$ becomes a $b$-th power of an irreducible element.) Therefore*), $\pi(\sigma) \in P^b$. Thus, for the given $c_i$, we can find a suitable iteration of derivations in $k$, say $\pi'$, such that $\pi'(c_j) = 1$ for some $j$ and $\pi'(c_i) \in k^p$ for all i. Then $\pi'(\sigma)$ must be of the form $(\sigma')^p$ with a polynomial $\sigma'$ of degree $b/p$ in $k[\omega]$. Moreover, $\pi'(\sigma) \in P^b$, and $\sigma' \in P^{b/p}$. By induction assumption, $\sigma' = 0$ and hence $\sigma = 0$.

*) This is clear except for $b = 2 = p$. In this case, however, if $\sigma$ is not a square in $k[\omega]$ then there exists a derivation $\pi$ of $k$ such that $\pi(\sigma) = (\sigma')^2$ with a linear polynomial $\sigma' \neq 0$ in $k[\omega]$. But $(\sigma')^2 \in P$ implies $\sigma' \in P$.