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PL-submanifolds and homology classes of a PL-manifold II

By

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In [1] we have proved the fundamental theorem of the realization problem of homology classes by submanifolds in the PL-case. In the present paper we shall give some consequences. One is based on the results of Browder-Liulevicius-Peterson [2] on the homotopy types of the PL Thom spectrum $\boldsymbol{MPL}$, and the other on the results of Kuiper-Lashof [5] on the homotopy groups of PL_1.

We use the notations and terminologies in [1].

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1. Statements of the results

Theorem 1. *Let V^n be a closed PL-manifold of dimension n. For $k \leqq n/2$, all homology classes of $H_k(V^n, Z_2)$ can be realized by PL-submanifolds which have normal PL-microbundles.*

Theorem 2. *Let V^n be a closed PL-manifold of dimension n. All homology classes of $H_{n-1}(V^n, Z_2)$ can be rerlized by PL-submanifolds which have normal PL-microbundles.*

These results are quite parallel to those of C^∞-case in Thom [8].

2. Study of the homotopy type of Thom complexes MPL_k

a) Preliminaries.

Let $\boldsymbol{MPL} = \{MPL_n, \mu_n; n \geqq 0\}$ be the PL Thom spectrum defined by Williamson [10]. In Browder-Liulevicius-Peterson [2], the following proposition is obtained:

Proposition 1. *There is a map*

$$h : \boldsymbol{MPL} \to \prod_i \boldsymbol{K}(Z_2, n_i),$$

such that

$$h^* : H^*(\prod_i \boldsymbol{K}(Z_2, n_i), Z_2) \to H^*(\boldsymbol{MPL}, Z_2)$$

is an isomorphism.

On the other hand the following proposition was proved by Williamson [10].

Proposition 2. *The PL cobordism group $\mathfrak{N}^n_{PL}$ is isomorphic to the homotopy group of the PL Thom spectrum $\pi_n(\boldsymbol{MPL})$.*

b) Stability theorem for MPL_k.

Haefliger-Wall [3] proved the following stability theorem.

Proposition 3. *Let*

$$i_n : PL_n \to PL_{n+1}$$

be the natural inclusion. Then

$$(i_n)_* : \pi_q(PL_n) \to \pi_q(PL_{n+1})$$

is an isomorphism for $q<n-1$ and an epimorphism for $q=n-1$.

The inclusion $i_n : PL_n \to PL_{n+1}$ induces naturaally the following map

$$\rho_n : BPL_n \to BPL_{n+1}.$$

By Propesition 3, the homomorphism

$$(\rho_n)_* : \pi_q(BPL_n) \to \pi_q(BPL_{n+1})$$

is an isomorphisms for $q<n$ and an epimorphism for $q=n$. Then by the theorem of J. H. C. Whitehead [9], the homomorphism

$$(\rho_n)^* : H^q(BPL_{n+1}, Z) \to H^q(BPL_n, Z)$$

is an isomorphism for $q<n$ and a monomorphism for $q=n$.

We shall denote by

$$s^* : H^{q+n+1}(SMPL_n, Z_p) \to H^{q+n}(MPL_n, Z_p)$$

the homomorphisms induced from suspension, where p is a prime, and by σ_n the compositipon of the following two homomorphisms:

$$\sigma_n : H^{q+n+1}(MPL_{n+1}, Z_p) \xrightarrow{(\mu_n)^*} H^{q+n+1}(SMPL_n, Z_p) \xrightarrow{s^*} H^{q+n}(MPL_n, Z_p).$$

Then we know the following commutative diagram:

$$\begin{array}{ccc} H^{q+n+1}(MPL_{n+1}, Z_2) & \xrightarrow{\sigma_n} & H^{q+n}(MPL_n, Z_2) \\ \varphi^*_{n+1}\uparrow & & \varphi^*_n\uparrow \\ H^q(BPL_{n+1}, Z_2) & \xrightarrow{(\rho_n)^*} & H^q(BPL_n, Z_2), \end{array}$$

where φ^*_n, φ^*_{n+1} are Thom isomorphisms (cf. Williamson [10]). Thus we have the following:

Proposition 4. *The homomorphism*

$$\sigma_n : H^{q+n+1}(MPL_{n+1}, Z_2) \to H^{q+n}(MPL_n, Z_2)$$

is an isomorphism for $q<n$ and a monomωrphism for $q=n$.

Kuiper-Lashof [4] proved that very PL-microbundle over a locally finite simplicial complex contains an R^n-bundle, and this bundle is unique up to equivalence. Therefore, the universal PL-microbundle

$$\gamma(PL_n) : BPL_n \xrightarrow{i_n} EPL_n \xrightarrow{j_n} BPL_n$$

contains an R^n-bondle $u=(E, p, BPL_n)$. For an odd prime p, we have the following commutative diagram:

$$\begin{array}{ccc} H^{q+n+1}(MPL_{n+1}, Z_p) & \xrightarrow{\sigma_n} & H^{q+n}(MPL_n, Z_p) \\ \varphi^*_{n+1}\uparrow & & \varphi^*_n\uparrow \\ H^q(BPL_{n+1}, Z_p \circ T_{n+1}) & \xrightarrow{(\rho'_n)^*} & H^q(BPL_n, Z_p \circ T_n), \end{array}$$

where T_n is the "faisceau tordu" associated with the orientation of R^n-bundles u, and $(\rho'_n)^*$ is the homomorphism induced from $\rho_n : BPL_n \to BPL_{n+1}$ (see R. Thom [7], Chapter I, §II). Since $(\rho_n)_* : \pi_1(BPL_n)$

$\rightarrow \pi_1(BPL_{n+1})$ is an isomorphim for $n \geqq 2$, we have that the homomorphism $(\rho_n')^*$ is an isomorphism for $q<n$, $n \geqq 2$ and a monomorphism for $q=n \geqq 2$. Thus we have

Proposition 4′. *Let $n \geqq 2$ and p be an odd prime. Then the homomorphism*

$$\sigma_n : H^{q+n+1}(MPL_{n+1}, Z_p) \rightarrow H^{q+n}(MPL_n, Z_p)$$

is an isomorphism for $q<n$ and a monorphism for $q=n$.

c) $2n$-type of the Thom complex MPL_n.

We know the following commutative diagram:

$$\begin{array}{ccc}
H^{q+n}(\prod_i K(Z_2, n+n_1), Z_2) & \xrightarrow{(h_n)^*} & H^{q+n}(MPL_n, Z_2) \\
\uparrow & & \uparrow \\
\varprojlim H^{q+n}(\prod_i K(Z_2, n+n_i), Z_2) & \xrightarrow{\varprojlim (h_n)^*} & \varprojlim H^{q+n}(MPL_n, Z_2) \\
\| & & \| \\
H^q(\prod_i \boldsymbol{K}(Z_2, n_i), Z_2) & \xrightarrow{\boldsymbol{h}^*} & H^q(\boldsymbol{MPL}, Z_2).
\end{array}$$

In the above diagram the bottom horizontal map $\boldsymbol{h}^*$ is an isomorphism by Proposition 1, the right vertical map is an isomorphism for $q<n$ and a monomorphism for $q=n$, and the left vertical map is an isomorphism for $q \leqq n$. Therefore,

$$(h_n)^* : H^j(\prod_i K(Z_2, n+n_i), Z_2) \rightarrow H^j(MPL_n, Z_2)$$

is an isomorphism for $j<2n$ and a monomorphism for $j=2n$.

For an odd prime p, we have the following commutative diagram:

$$\begin{array}{ccc}
H^{q+2}(\prod_i K(Z_2, n+n_i), Z_p) & \xrightarrow{(h_n)^*} & H^{q+n}(MPL_n, Z_p) \\
\uparrow & & \uparrow \\
\varprojlim H^{q+n}(\prod_i K(Z_2, n+n_i), Z_p) & \xrightarrow{\varprojlim (h_n)^*} & \varprojlim H^{q+n}(MPL_n, Z_p) \\
\| & & \| \\
H^q(\prod_i \boldsymbol{K}(Z_2, n_i), Z_p) & \xrightarrow{h^*} & H^q(\boldsymbol{MPL}, Z_p).
\end{array}$$

We know that $H^q(\prod_i K(Z_2, n_i), Z_q)$ is zero. Moreover, by Proposition 2 and Serre's $\mathcal{C}$-theory [6], we know that $H^q(\boldsymbol{MPL}, Z_p)$ is also zero.

Therefore, in the above diagram the bottom horizontal map h^* is an isomorphism. Moreover, by Proposition 4', the right vertical map is an isomorphism for $q<n$ and a monomorphism for $q=n$, $n\geqq 2$, and the left vertical map is an isomorphism for $q\geqq n$. Therefore, for any odd prime p,

$$(n_n)^* : H^j(\prod_j K(Z_2, n+n_i), Z_p)\to H^j(MPL_n, Z_p)$$

is an isomorphism for $j<2n$ and a monomorphism for $j=2n$, $n\geqq 2$. Consequently, by the theorem of J. H. C. Whitehead [9], we have the following proposition.

Proposition 5. *Let $n\geqq 2$. There exists a mapping g of the $2n$-skeleton of $\prod_i K(Z_2, n+n_i)$ to MPL_n such that $h_n\circ g$ and $g\circ h_n$ (restricted to the $2n$-skeleton of MPL_n) are homotopic to the identities.*

d) Thom complex MPL_1.

Let $\lambda: O_1\to PL_1$ be the natural monomorphism. Kuiper-Lashof [5] have shown that for all i

$$\lambda_* : \pi_i(O_1)\to \pi_i(PL_1)$$

is an isomorphism. However, we know that $MO(1)$ has the homotopy type of $K(Z_2, 1)$ (see Thom [8]). Therefore, MPL_1 has the homotopy type of $K(Z_2, 1)$.

3. Proof of theorems

a) Proof of Theorem 1.

Let $k\leqq n/2$, $2\leqq n-k$, and z be a homology class of $H_k(V^n, Z_2)$. We shall denote by $u\in H^{n-k}(V^n, Z_2)$ the cohomology class corresponding to z by the Poincaré duality. There exists a map

$$f : V^n \to K(Z_2, n-k)$$

such that $f^*(\iota_{n-k})=u$, where ι_{n-k} is the fundamental class of $H^*(Z_2, n-k; Z_2)$. We shall denote by $K^{(q)}$ the q-skeleton of a CW-complex K. By Proposition 5, we have the map

$$g : (\prod_i K(Z_2, n-k+n_i))^{(n)} \to MPL_{n-k}$$

such that there exists the number i with $n_i=0$, and $g^*(U_{n-k})=\iota_{n-k}$, where $U_{n-k}\in H^{n-k}(MPL_{n-k}, Z_2)$ is the fundamental class of the Thom complex MPL_{n-k}. Thus we have the map $\varphi=g\circ i_n\circ f'$:

$$\begin{array}{ccc} V^n & \xrightarrow{\varphi} & MPL_{n-k} \\ f'\uparrow & & g\uparrow \\ (K(Z_2, n-k))^{(n)} & \xrightarrow{i_n} & (\prod_i K(Z_2, n-k+n))^{(n)}, \end{array}$$

where f' is the cellular approximation to f, and i_n is the restriction of inclusion to n-skeleton, and $\varphi^*(U_{n-k})=u$. By the fundamental theorem in [1], we obtain Theorem 1.

Next we consider the case $n=2$, $k=1$. PL-2-manifold V^2 has a smoothing α. For C^∞-manifold (V^2, α), any element of $H_1(V^2, Z_2)$ is realisable by C^∞-submanifold (see Thom [8]). Therefore, in this case, any element of $H_1(V^2, Z_2)$ is realisable by PL-submanifold with normal PL-microbundle.

b) Theorem 2 can be easily obtained by the fundamental theorem in [1] and the fact in §2, d).

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