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Characterization and algebraic deformations of projective space

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Let V and V' be complete algebraic varieties of dimension n defined over K and k respectively. Then V' is a specialization of V if there exists a discrete valuation ring $\mathfrak{o}$ with quotient field K and residue field k and a variety W proper and flat over $\mathfrak{o}$ such that $V \cong W \times_{\mathfrak{o}} K$ over K and $V' \cong W \times_{\mathfrak{o}} k$ over k. The above notations will be fixed throughout this paper. U and U^* are called deformations of each other if there exist $U_1, \cdots, U_m$ with $U \cong U_1$, $U^* \cong U_m$ and $U_i \cong U_{i+1}$ or U_i a specialization of U_{i+1} or the converse. The question considered here is, what are the possible deformations of $\boldsymbol{P}^n$? If $V \cong \boldsymbol{P}^n$ over K, then we will show that $V' \cong \boldsymbol{P}^n$ over k. Conversely, if $V' \cong \boldsymbol{P}^n$ over k and if V admits a K-rational divisor with self-intersection number 1, then $V \cong \boldsymbol{P}^n$ over K. I wish here to thank Professor Matsusaka for his advice, suggestions, and encouragement in this work.

1) The proofs of the above results are based on the following characterization of $\boldsymbol{P}^n$:

Let V be a complete variety of dimension n defined over K. Then $V \cong \boldsymbol{P}^n$ over K if and only if there exists a non-degenerate positive Cartier divisor X on V rational over K such that $X^{(n)}$, the self-intersection number of X on V, equals 1 and $l(X) \geq n+1$.

X non-degenerate means that a high multiple of X is ample, i.e., $H^0(V, \mathcal{L}^{\otimes m})$ defines a projective embedding of V for m sufficiently large where $\mathcal{L}$ is the invertible sheaf associated to X. $X^{(n)}=\chi(\mathcal{L}^n)_V$ is defined ([5], p. 296) to be the coefficient of the leading term in the Hilbert polynomial $\chi(V, \mathcal{L}^{\otimes m})=\sum_{i=0}^n(-1)^i \times \dim H^i(V, \mathcal{L}^{\otimes m})$. The above characterization of projective space is proved using the following sequence of lemmas:

Lemma 1. *Suppose X is a positive non-degenerate Cartier divisor on V and $X^{(n)}=1$. Then X is irreducible.*

Proof. Let $Y_1, \cdots, Y_r$ be the irreducible components of X and let $\mathcal{L}$ be the sheaf associated to X. Then

$$1=X^{(n)}=\chi(\mathcal{L}^n)_V=\chi(\mathcal{L}^{n-1}\cdot\mathcal{O}_X) \quad ([5], \text{ prop. } 4, \text{ p. } 297)$$
$$=\sum_i\chi(\mathcal{L}^{n-1}\cdot\mathcal{O}_X\otimes\mathcal{O}_{Y_i}) \quad ([5], \text{ cor. 1 of prop. 5, p. } 298)$$
$$\geq r \text{ since } \mathcal{L} \text{ is non-degenerate } ([5], \text{ thm. 1, p. } 317).$$

Lemma 2. *Under the hypotheses of lemma 1, $(X, \mathcal{O}_X)$ is a reduced subscheme of $(V, \mathcal{O}_V)$ and $\chi(\mathcal{L}_X^{n-1})_X=1$ where $\mathcal{L}_X=\mathcal{L}\otimes\mathcal{O}_X$.*

Proof. Let x be a generic point of X and let $f\colon X'\to X$ where $X'=X_{red}$. Then $\deg(f)=1/s$ where $s=\text{length}_{\mathcal{O}_{X,x}}\mathcal{O}_{X,x}$ ([5], p. 299, ex. 1). Let $\mathcal{L}_{X'}=\mathcal{L}\otimes\mathcal{O}_{X'}$. Then $\mathcal{L}_{X'}=\mathcal{L}_X\otimes_{\mathcal{O}_X}\mathcal{O}_{X'}=f^*\mathcal{L}_X$. Therefore, $\chi(\mathcal{L}_{X'}^{n-1})_{X'}=\deg(f)\cdot\chi(\mathcal{L}_X^{n-1})_X$ ([5], prop. 6, p. 299). But $\chi(\mathcal{L}_X^{n-1})_X=\chi(L^{n-1}\cdot X)_V=\chi(L^n)_V=1$ ([5], props. 4 and 5, p. 298). Therefore $\chi(\mathcal{L}_{X'}^{n-1})_{X'}=1/s$. But $\mathcal{L}_{X'}$ is a non-degenerate invertible sheaf on X' so $\chi(\mathcal{L}_{X'}^{n-1})_{X'}$ is an integer. Therefore $s=1$ and $\mathcal{O}_{X,x}$ is a field so $(X, \mathcal{O}_X)$ is reduced.

Lemma 3. *If in lemma 2 we assume $\dim H^0(V, \mathcal{L})\geq n+1$, then $\dim H^0(X, \mathcal{L}_X)\geq n$.*

Proof. $0\to\mathcal{O}_V(-X)\to\mathcal{O}_V\to\mathcal{O}_X\to 0$ is exact, so, since $\mathcal{L}=\mathcal{O}_V(X)$ is locally free, $0\to\mathcal{O}_V\to\mathcal{L}\to\mathcal{L}_X\to 0$ is exact and hence $0\to H^0(V, \mathcal{O}_V)\to H^0(V, \mathcal{L})\to H^0(V, \mathcal{L}_X)$ is exact. Therefore, $\dim H^0(X, \mathcal{L}_X)=\dim H^0(V, \mathcal{L}_X)\geq\dim H^0(V, \mathcal{L})-\dim H^0(V, \mathcal{O}_V)\geq n$.

Theorem 1. *Let V be a complete variety defined over a field K and X a non-degenerate positive Cartier divisor on V rational over K such that $X^{(n)}=1$ and $l(X)\geq n+1$. Then $V\cong \boldsymbol{P}^n$ over K and the isomorphism is determined by X.*

Proof. Let $\mathcal{L}$ be as above. By lemmas 1, 2, and 3 X is a complete variety defined over K with the non-degenerate positive Cartier divisor $\mathcal{L}_X$ on X rational over K and with $\chi(\mathcal{L}_X^{n-1})_X=1$ and dim $H^0(X, \mathcal{L}_X)\geq n$. Therefore, by induction on n, since the theorem is true for $n=1$, $(X, \mathcal{L}_X)\cong(\boldsymbol{P}^n, \mathcal{O}(1))$ over K and $\dim H^0(X, \mathcal{L}_X)=n$. Therefore $l(X)=n+1$ and $0\to H^0(V, \mathcal{O}_V)\to H^0(V, \mathcal{L})\to H^0(X, \mathcal{L}_X)\to 0$ is exact.

Now let x be any point of V and Y a positive Cartier divisor on V such that $\mathcal{O}_V(Y)\cong\mathcal{O}_V(X)$ and $x\in\operatorname{Supp} Y$. There exists such Y since $l(X)\geq 2$. Then, since Y is a Cartier divisor, there exists $t\in\mathcal{O}_{V,x}$ such that $\mathcal{O}_{Y,x}\cong\mathcal{O}_{V,x}/t\mathcal{O}_{V,x}$. But, as above, $Y\cong\boldsymbol{P}^{n-1}$ so the maximal ideal of $\mathcal{O}_{Y,x}$ is generated by $n-1$ elements. Therefore, by Nakayama, the maximal ideal of $\mathcal{O}_{V,x}$ is generated by n elements and hence x is a simple point of V. Thus V is non-singular.

Since $l(X)=n+1$, to show $V\cong\boldsymbol{P}^n$ over K with the isomorphism determined by X it will suffice to show that X is ample on V. By Weil's criterion for ampleness ([6], ch. IX, §5, thm. 12, p. 288) it will suffice, given $x\in V$, to find n divisors $Y_1, \cdots, Y_n$ on V in $\Lambda(X)$, the complete linear system associated to X, intersecting properly such that $Y_1\cdot Y_2\cdots Y_n=1\cdot x$. Let $Y\in\Lambda(X)$ such that $x\in Y$. As above, $Y\cong\boldsymbol{P}^{n-1}$. Then choose $n-1$ hyperplanes of Y whose intersection is just x and pull them back to V. This is possible since $H^0(V, \mathcal{L})\to H^0(Y, \mathcal{L}_Y)$ is onto. Their intersection with Y is now $1\cdot x$. Q.E.D.

Professor Mumford pointed out that for the above characterization of $\boldsymbol{P}^n$ it was unnecessary to assume V nonsingular.

2) **Theorem 2.** *Let W be a variety flat and proper over*

$\mathcal{O}$ and let $V \cong W \times_{\mathfrak{o}} K$, $V' \cong W \times_{\mathfrak{o}} k$. If *$V \cong \boldsymbol{P}^n$ over K then $V' \cong \boldsymbol{P}^n$ over k and $W \cong \boldsymbol{P}^n$ over $\mathfrak{o}$. Furthermore, if the divisor X on V corresponds to a hyperplane of $\boldsymbol{P}^n$, then $X' = \bar{X} \cdot V'$ and $\bar{X}$ respectively determine these last two isomorphisms where $\bar{X}$ is the closure of X on W.*

Assuming V' nonsingular a proof of this result was given in [3], lemma 1.7, while for $k = \boldsymbol{C}$ and V' a compact Kähler manifold this result was proved in [4]. The proof of the general result will use the following proposition and lemma.

Proposition. *Let W be flat and proper over $\mathcal{O}$, $V \cong W \times_{\mathfrak{o}} K$, $V' \cong W \times_{\mathfrak{o}} k$. Assume W is nonsingular in codimension* 1, $\mathrm{Pic}^0(V) = 0$, *and the rank of $G(V)/G_a(V)$ is one. Then W is projective over $\mathfrak{o}$ and, if X is a non-degenerate Weil divisor on V, the $\bar{X}$ is a non-degenerate Weil divisor on W.*

Proof. We may write $W = \bigcup_{i=1}^{t} U_i$, U_i open affines over $\mathfrak{o}$ such that $U_i \times_{\mathfrak{o}} k$ is non-empty. Assume all $U_i \subset \boldsymbol{A}^N$ as closed subsets and let h_i be the composition of the inclusions $U_i \subset \boldsymbol{A}^N \subset \boldsymbol{P}^N$. Let Γ_{h_i} be the graph of h_i, Γ_i be the closure of Γ_{h_i} in $W \times P^N$, and $X_i = \mathrm{pr}_W(\Gamma_i \cdot W \times H)$ where H is a generic hyperplane of $\boldsymbol{P}^N$. Let $Y_i = \mathrm{pr}_{U_i}(\Gamma_{h_i} \cdot U_i \times H)$ and $\bar{Y}_i$ be the closure of Y_i in W. Then $X_i \succ Y_i$ so $X_i \succ \bar{Y}_i$.

Claim: If $Z_i = X_i - \bar{Y}_i$, $|Z_i|$ does not meet U_i.

Proof of claim. Let x be a generic point of a component of $U_i \cap |Z_i|$. Since U_i is open in W, x is a generic point of a component of $|Z_i|$ and hence is simple on W. $\Gamma_{h_i} = \Gamma_i \cap (U_i \times P^N)$ so $\Gamma_{h_i} \cdot (U_i \times H) = \Gamma_i \cdot (U_i \times H)$. Therefore, Γ_{h_i} regular at x implies Γ_i regular at x so the unique component T of $|X_i|$ containing x appears with multiplicity 1 in X_i and $T \cap U_i$ appears with multiplicity 1 in Y_i. Hence T does not appear in $|Z_i|$.

Proof of proposition. Since $\Lambda(Y_i)$ is ample on U_i, $\Lambda(X_i)$ is ample on U_i because $\Lambda(X_i)\cap U_i=\Lambda(Y_i)$. Now let r be the order of the torsion part of $G(V)/G_a(V)$. Then $r\cdot G(V)/G_l(V)=r\cdot G(V)/G_a(V)\cong \boldsymbol{Z}$ by hypothesis. Since $\Lambda(X_i)\cap U_i$ is ample on U_i, $\Lambda(rX_i)\cap U_i$ is ample on U_i and hence $\Lambda(rX_i\times_{\mathfrak{o}}K)\cap U_i\times_{\mathfrak{o}}K$ is ample on $U_i\times_{\mathfrak{o}}K$. Therefore $rX_i\times_{\mathfrak{o}}K$ corresponds to a positive element r_i in $\boldsymbol{Z}$ under the above isomorphism. If X is a non-degenerate divisor on V, let rX correspond to $r_0>0$ in $\boldsymbol{Z}$. Let s be a common multiple of the r_i and let $a_i=rs/r_i$. Then all $a_iX_i\times_{\mathfrak{o}}K$ correspond to the same element of $\boldsymbol{Z}$ so $a_iX_i\times_{\mathfrak{o}}K\sim a_tX_t\times_{\mathfrak{o}}K(\sim a_0X)$. Let f_i be a function on V such that $(f_i)=a_iX_i\times_{\mathfrak{o}}K-a_tX_t\times_{\mathfrak{o}}K$. Modifying f_i by a constant if necessary we can assume f_i extends to a function $\overline{f_i}$ such that $(\overline{f_i})=a_iX_i-a_tX_t$. This is possible since the only subvariety of W of codim 1 wholly contained in V' is V' itself. Similarly, $a_0\overline{X}\sim a_tX_t$. Thus $\Lambda(a_tX_t)=\Lambda(a_iX_i)(=\Lambda(a_0\overline{X}))$ is ample on U_i for all i and hence on $\bigcup_{i=1}^{t} U_i=W$.

Lemma 4. *Let the hypotheses be as in theorem* 2. *Then* $\overline{X}$ *is a positive Cartier divisor on* W *not containing* V' *in its support.*

Proof (suggested by Dr. W. Fulton of Brandeis University). Let (U_i) be an open affine cover of W such that all U_i meet V'. Suppose $U_i=\operatorname{Spec} B_i$. Let (f_{ij}) be a collection of non-units of B_i generating B_i over $\mathfrak{o}$. Then the $U_{ij}=(\operatorname{Spec} B_i)_{f_{ij}}$ which intersect V' form a collection of open affine subsets of W covering V' and hence all of W. The $U_{ij}\times_{\mathfrak{o}}K$ form an open affine cover of V. f_{ij} can be viewed as a function on $U_i\times_{\mathfrak{o}}K$ and, by taking the closure in V of the divisor of zeroes of f_{ij}, we get a hypersurface H_{ij} of V not containing any point of $U_{ij}\times_{\mathfrak{o}}K$. If we can show that in $U_{ij}\times_{\mathfrak{o}}K$ X is given as the divisor of a single function $g_{ij}\in(B_i\times_{\mathfrak{o}}K)_{f_{ij}}$ then, modifying g_{ij} by an element of K and extending g_{ij} to U_{ij}, we can assume that $g_{ij}\in(B_i)_{f_{ij}}$ and $U_{ij}\times_{\mathfrak{o}}k$

is not a component of (g_{ij}). Thus, $\overline{X}=(U_{ij}, g_{ij})$ will be a positive Cartier divisor on W not containg V' in its support. Therefore, to complete the proof, it will suffice to show that $B=(B_i \otimes K)_{f_{ij}}$ is a unique factorization domain. $f(U_{ij}\times_{\mathfrak{o}}K)\subset \boldsymbol{P}^n-f(H_{ij})\cong \boldsymbol{A}^n$ so $\operatorname{Spec} B$ may be viewed as an open affine subset of $\boldsymbol{A}^n$. Then $K[X_1, \cdots, X_n]\subset B\subset K(X_1, \cdots, X_n)$ and B is a noetherian integral domain.

Let Q be a minimal prime of B. Then Q induces an irreducible subvariety of codim 1 of $\operatorname{Spec} B$ and hence of $\boldsymbol{A}^n$. But $K[X_1, \cdots, X_n]=K[X]$ is a UFD so there exists $h\in K[X]$ such that $Q\cap K[X]=h\cdot K[X]$. Thus it remains only to show that $Q=h\cdot B$. Let $r(X)/s(X)\in Q$ in lowest terms, $r(X)=\Pi r_i(X)$, $s(X)=\Pi s_j(X)$. Then $r=s(r/s)\in Q\cap K[X]$ so $r/s=hr^*/s$. Suppose $1/s_j\notin B$. Then $\{x: s_j(x)=0\}\cap \operatorname{Spec} B$ is an open non-empty subset of $\{x: s_j(x)=0\}$. Since no $r_i=s_j$, $r(x)$ can not vanish on this set so $r/s\notin B$. Contradiction. Thus $1/s\in B$ so $r/s=\mathrm{hr}^*/s\in h\cdot B$.

Proof of theorem 2. Let $d: V'\to W$ be the closed immersion. Then $X'=d^*(\overline{X})$ is a positive non-degenerate Cartier divisor on V' by the proposition and lemma. Therefore, to show X' induces an isomorphism of V' with $\boldsymbol{P}^n$ over k, it suffices to show that $l(X')\geq n+1$ and $X'^{(n)}=1$. But these are immediate consequences of upper semicontinuity and invariance of Euler-Poincaré characteristic ([1], III. 7.7.5 and III. 7.9.4). Thus $V'\cong \boldsymbol{P}^n$ over k and the map is just the isomorphism on V extended. Q.E.D.

3) **Theorem 3.** *Let W be proper and flat over $\mathfrak{o}$, $V\cong W\times_{\mathfrak{o}}K$, $V'\cong W\times_{\mathfrak{o}}k\cong \boldsymbol{P}^n$ over k. Then V is projective nonsingular and*

a) *there exists a finite separable field extension K_0 of K such that $V\cong \boldsymbol{P}^n$ over K_0. (Such varieties are classified by the set of isomorphism classes of central simple algebras of dimension $(n+1)^2$ over K. They are isomorphic to $\boldsymbol{P}^n$ over K if and only if they carry a K-rational point* (F. Châtelet, [7]).) *If $\mathfrak{o}$ is complete we can take $K_0=K$.*

b) *if* V *carries a divisor rational over* K *of self-intersection number* 1, *then* $V \cong \boldsymbol{P}^n$ *over* K.

Proof. V' nonsingular implies V nonsingular. Let Y be any positive divisor on V and let $Y' = \bar{Y} \cdot V'$. Then Y' is a positive divisor on V' and so is non-degenerate. Hence $\bar{Y}$ and Y are non-degenerate ([1], III. 4. 7. 1.). (a) is proved in ch. 0 in [4] and in general in [2] exposé III, pp. 19–20. (b): Let $X = Y_1 - Y_2$, $Y_i > 0$ be a K-rational divisor on V such that $X^{(n)} = 1$. Suppose $Y'_i = \bar{Y}_i \cdot V'$, $X' = Y'_1 - Y'_2$. Then $X'^{(n)} = 1$ so $l(X') = n+1$ and $H^i(V', X') = 0$ if $i > 0$. Therefore $H^i(V, X) = 0$ for $i > 0$ and $l(X) = \chi(V, X) = \chi(V', X') = l(X') = n+1$. Therefore there exists $X^* \sim X$, positive and rational over K, hence non-degenerate. Also, $X^{*(n)} = 1$ and $l(X^*) = n+1$ so $V \cong \boldsymbol{P}^n$ over K. An alternate proof of (b) is given in [3], lemma 1. 6.

Bibliography

[1] A. Grothendieck, Eléments de Géometrie Algébriques, III. Publications Mathématiques Institut des Hautes Etudes Scientifiques, Paris, nos. 11 and 17.

[2] A. Grothendieck, Séminaire de Géométrie Algébrique, Institut des Hautes Etudes Scientifiques, 1960–61 exposé III.

[3] H. Hironaka, Smoothing of Algebraic Cycles of Lower Dimensions, to apper.

[4] F. Hirzebruch and K. Kodaira, "On the Complex Projective Spaces," Journal de Mathématiques Pures et Appliquées. ser. 9, vol. **36** (1957), pp. 201–216.

[5] S. Kleiman, "Toward a numerical theory of ampleness, "Ann. of Math. **84** (1966), pp. 293–344.

[6] A. Weil, Foundations of Algebraic Geometry, AMS Colloquium Publications no. 29 (1962).

[7] F. Chatelet, Variations sur un thème de H. Poincaré. Annales E. N. S. **61**, 1944, pp. 249-300.

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