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On F-connections and associated non-linear connections

Dedicated to Professor Dr. W. Barthel,
wishing a quick recovery of his health

By

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In 1963, W. Barthel [1] developed an elegant theory of holonomy groups of homogeneous non-linear connections. He defined a homogeneous non-linear connection on a differentiable manifold M as a special distribution on the tangent bundle $T(M)$.

As is well-known (for example, see [9]), a *linear* connection on M, however, can be defined as a connection in the bundle of linear frames $L(M)$ over M, and then its holonomy group is a subgroup of $GL(n, R)$ acting on $L(M)$.

The purpose of the present paper is to give a concept of an F-connection, a collection of special distributions on $L(M)$, and to show that *any homogeneous non-linear connection in $T(M)$ is associated with an F-connection.* For this purpose, a concept of Finsler connections will be quite useful. The first section is devoted to summarize basic concepts of Finsler connections, which have been described in a series of our papers [2], $\cdots$, [8]. In the second section, some properties of homogeneous Finsler connections will be derived. Then, the main result will be given in Theorem 6 of the third section.

§1. Introduction

This section is an introductory summary of basic concepts of Finsler connections, needed for the later treatment. Throughout the present paper, we denote by P_p the tangent space to a differentiable manifold P at a point p, and by B^v the vertical distribution $b \in B \rightarrow B_b^v$ on the total space B of a fibre bundle, where B_b^v is the vertical subspace of the tangent space B_b, the kernel of the differential of the projection of B. It is further noted that the differential of a differentiable mapping μ will be denoted by μ itself.

[1] We shall consider a differentiable n-manifold M and the following fibre bundles.

The bundle of non-zero tangent vectors $T(M)(M, \tau, F, G)$:

M…… base space, τ …… projection $T(M) \rightarrow M$,
F …… standard fibre (real vector n-space),
$G = GL(n, R)$ …… structural group.

The principal bundle of linear frames $L(M)(M, \pi, G)$:

M…… base space, π…… projection $L(M) \rightarrow M$,
$G = GL(n, R)$ …… structural group.

The induced bundle $\tau^{-1}L(M) = F(M)(T(M), \pi_1, G)$:

$F(M) = \{(y, z) \in T(M) \times L(M) \mid \tau y = \pi z\}$ …… total space,
$T(M)$…… base space,
π_1 …… projection $F(M) \rightarrow T(M)$, $[(y, z) \rightarrow y]$,
$G = GL(n, R)$…… structural group.

The bundle $F(M)$ is called the *Finsler bundle* of M. The operation r of G on F is determined by

$$r: G \times F \rightarrow F, \ [(g = (g_b^a), \ f = f^a e_a) \rightarrow gf = g_b^a f^b e_a],$$

with respect to a fixed base (e_a), $a = 1, \cdots, n$, of F. Next, the operation t of G on the total space $L(M)$ is given by

$$t: L(M) \times G \rightarrow L(M), \ [(z = (z_a), \ g = (g_b^a)) \rightarrow zg = (z_b g_a^b)],$$

and then, the operation T of G on the total space $F(M)$ is induced from t as follows.

$$T\colon F(M)\times G\to F(M),\ [((y,z),g)\to(y,zg)].$$

Let us denote by R^+ the differentiable manifold composed of all the positive numbers, and let h be a mapping

$$h\colon R^+\times T(M)\to T(M),\ [(\alpha,y)\to\alpha y].$$

Then, the mapping

$$H\colon R^+\times F(M)\to F(M),\ [(\alpha,u=(y,z))\to\alpha u=(\alpha y,z)]$$

is induced from h. A transformation ${}_\alpha h$ of $T(M)$ is obtained from the above h by becoming $\alpha\in R^+$ fixed. Then, a distribution $D\colon y\in T(M)\to D_y\subset T(M)_y$ on $T(M)$ is called *h-invariant*, if ${}_\alpha hD_y=D_{\alpha y}$ holds good at any y and for any α. The notion of the *H-invariance* will be similarly defined for distributions on $F(M)$.

The *Finsler subbundle* $F(x)$ at a point $x\in M$ is by definition a subbundle of $F(M)$ over a fibre $\tau^{-1}x\subset T(M)$. It will be obvious that the tangent space $F(x)_u$ is the subspace of $F(M)_u$ given by $F(M)_u^q=\{X\in F(M)_u|\tau\cdot\pi_1X=0\}$, which is called the *quasi vertical subspace* of $F(M)_u$.

[2] We shall present here concepts of some connections in $T(M)$, $L(M)$ and $F(M)$.

Definition 1. A distribution $N\colon y\in T(M)\to N_y\subset T(M)_y$ on $T(M)$ is called a *non-linear connection* in $T(M)$, if N is a complement of the vertical distribution T^v, that is,

$$T(M)_y=N_y\oplus T_y^v,\qquad \text{(direct sum)},$$

at any point $y\in T(M)$. Further, N is called *homogeneous*, if N is h-invariant.

Definition 2. An *F-connection* Γ_F in $L(M)$ is a collection $\{\Gamma_{(f)}\}$ of distributions $\Gamma_{(f)}\colon z\in L(M)\to\Gamma_{(f)z}\subset L(M)_z$, corresponding to any $f\in F$, which satisfies

(1) $L(M)_z = \Gamma_{(f)z} \oplus L^v_z$, at any point $z \in L(M)$,

(2) $t_g \Gamma_{(f)z} = \Gamma_{(g^{-1}f)zg}$, at any point z and for any $g \in G$.

The above mapping t_g is a right translation of $L(M)$ by $g \in G$, which is obtained from t by becoming g fixed. It is remarked that each $\Gamma_{(f)}$ is *not* a connection in $L(M)$ in the ordinary sense, because (2) differs a little from the t-invariance of an ordinary connection.

As for a connection Γ in $L(M)$, the associated connection Γ^* will be obtained in $T(M)$. In fact, the total space $T(M)$ is identified with the quotient space $(L(M) \times F)/G$ by the operation $(z, f) \in L(M) \times F \rightarrow (zg, g^{-1}f)$, $g \in G$, and hence the canonical projection $L(M) \times F \rightarrow (L(M) \times F)/G$ gives

$$a: L(M) \times F \rightarrow T(M), \ [(z, f) \rightarrow zf],$$

where we denote by zf the equivalence class containing (z, f). The mapping a_f: $L(M) \rightarrow T(M)$ obtained from a by becoming $f \in F$ fixed is called the *associated mapping*. Then, the associated connection Γ^* is defined by $\Gamma^*_y = a_f \Gamma_z$, $y = zf$. In the same way, a non-linear connection N will be obtained from an F-connection Γ_F as follows.

Proposition 1. *Let $\Gamma_F = \{\Gamma_{(f)}\}$ be an F-connection in $L(M)$, and then by the equation*

$$N_y = a_f \Gamma_{(f)z}, \quad y = zf,$$

a distribution N: $y \in T(M) \rightarrow N_y$ is well defined. Then N is a non-linear connection in $T(M)$.

The proof is omitted, because it will be easily obtained. The non-linear connection N as above introduced is called the *associated non-linear connection* with Γ_F.

From now on, we shall treat the Finsler bundle $F(M)$ of M, and first give the following definition.

Definition 3. A *vertical connection* Γ^v in $F(M)$ is a distribution $u \in F(M) \rightarrow \Gamma_u^v \subset F(M)_u$, such that the restriction $\Gamma^v | F(x)$ of Γ^v to each Finsler subbundle $F(x)$ is a connection in $F(x)$.

Therefore, Γ^v is a vertical connection, if the following conditions be satisfied:

(1) $F(M)_u^o = \Gamma_u^v \oplus F_u^v$, at any point $u = F(M)$,

(2) T-invariant: $T_g \Gamma_u^v = \Gamma_{ug}^v$, for any $g \in G$ and at any $u \in F(M)$.

The above mapping T_g is a right translation of $F(M)$ by g, which is obtained from T by becoming $g \in G$ fixed. We shall give a differentiable base $(B^v(e_a))$, $a = 1, \cdots, n$, of the vertical connection Γ^v. For this purpose, we shall first introduce a *parallel vector field* $P(f)$ on F, corresponding to $f \in F$. $P(f)$ is induced from a 1-parameter (t) group of transformations $\{s_{tf}\}$ of F, where the mapping s_f, $f \in F$, is the summation $f_1 \in F \rightarrow f_1 + f$. Then, a *v-basic vector field* $B^v(f)$ on $F(M)$, corresponding to $f \in F$, is defined by

$$B^v(f)_u = l_u^v \cdot z(P(f)_{\gamma(u)}),$$

at a point $u = (y, z)$, where l_u^v is the lift to u with respect to Γ^v, z the differential of the admissible mapping ${}_z a \colon F \rightarrow T(M)$ obtained from the mapping a by becoming a frame z fixed, and γ is the characteristic field $u = (y, z) \in F(M) \rightarrow z^{-1}y = ({}_z a)^{-1}y$ [2, p. 3]. It will be obvious that n v-basic vector fields $B^v(e_a)$, $a = 1, \cdots, n$, give a base of Γ^v at every point of $F(M)$.

Next, we shall introduce a special vertical connection F^i. Since $F(M)$ is the induced bundle $\tau^{-1}L(M)$, there is the induced mapping $\pi_2 \colon F(M) \rightarrow L(M)$, $[(y, z) \rightarrow z]$. The characteristic field γ, together with the induced mapping π_2, gives the diffeomorphism

$$i = (\pi_2, \gamma) \colon F(M) \rightarrow L(M) \times F, \quad [(y, z) \rightarrow (z, z^{-1}y)],$$

and its inverse i^{-1} is

$$i^{-1} \colon L(M) \times F \rightarrow F(M), \quad [(z, f) \rightarrow (zf, z)].$$

By means of this identification i, a parallel vector field $P(f)$ on F, corresponding to $f \in F$, gives a vector field $Y(f) = i^{-1}(0, P(f))$ on $F(M)$, which is called the *induced fundamental vector field*, corresponding to f. It is obvious that any $Y(f)$ is contained in the induced vertical subspace $F^i_u = \{X \in F(M)_u \mid \pi_2 X = 0\}$ of $F(M)_u$.

Proposition 2. *The induced vertical distribution $F^i: u \in F(M) \to F^i_u$ on $F(M)$ is a vertical connection, and the v-basic vector field $B^v(f)$ with respect to F^i is nothing but the above $Y(f)$.*

The proof is omitted, because it will be easily obtained. It is remarked that $Y(f)$ is induced from the 1-parameter (t) group of transformations $\{S_{tf}\}$ of $F(M)$, where $S_f = i^{-1} \cdot (1, s_f) \cdot i$. Since the equation $[Y(f_1), Y(f_2)] = 0$, $f_1, f_2 \in F$, will be derived in virtue of the identification i, the vertical connection F^i as above obtained should be called *flat*.

[3] We are now in a position to introduce a concept of Finsler connections.

Definition 4. A *Finsler connection* (Γ, N) of M is a pair of a connection Γ in $F(M)$ and a non-linear connection N in $T(M)$.

Given a Finsler connection (Γ, N), we obtain the distribution Γ^v, defined by the equation

$$\Gamma^v_u = l_u T^v_y, \qquad \text{at a point } u,$$

where $y = \pi_1 u \in T(M)$, and l_u is the lift to u with respect to the connection Γ. It will be easy to show that the above Γ^v is a vertical connection, which is called the *subordinate vertical connection* to(Γ, N).

Definition 5. A *Finsler pair* (Γ^h, Γ^v) in $F(M)$ is a pair of two distributions $\Gamma^h: u \in F(M) \to \Gamma^h_u \subset F(M)_u$ and $\Gamma^v: u \in F(M) \to \Gamma^v_u \subset F(M)_u$, both on $F(M)$, which satisfies

(1) $F(M)_u = \Gamma^h_u \oplus \Gamma^v_u \oplus F^v_u$, for any $u \in F(M)$,

(2) both of Γ^h and Γ^v are T-invariant,

(3) $\pi_1 \Gamma^v_u = T^v_y$, $y = \pi_1 u$, for any $u \in F(M)$.

It is clear that the second distribution Γ^v of a Finsler pair (Γ^h, Γ^v) is a vertical connection in $F(M)$.

The following theorem means that a Finsler connection can be also defined as a Finsler pair.

Theorem 1. *There is a natural one-to-one correspondence between the set of Finsler connections of M and the set of Finsler pairs in $F(M)$.*

As will be easily verified, the correspondence $(\Gamma, N) \to (\Gamma^h, \Gamma^v)$ is given by

$\Gamma^h_u = l_u N_y$, $y = \pi_1 u$,

Γ^v ...... subordinate vertical connection,

while the inverse correspondence $(\Gamma^h, \Gamma^v) \to (\Gamma, N)$ is

$$\Gamma_u = \Gamma^h_u \oplus \Gamma^v_u,$$

$$N_y = \pi_1 \Gamma^h_u, \quad u \in \pi_1^{-1} y.$$

In the following, we shall often express $(\Gamma, N) = (\Gamma^h, \Gamma^v)$, when (Γ, N) and (Γ^h, Γ^v) correspond each other by the above rule.

We shall give a differentiable base $(B^h(e_a))$, $a = 1, \cdots, n$, of the distribution Γ^h. In order to do this, we first introduce an *h-basic vector field* $B^h(f)$, corresponding to $f \in F$, by the equation

$$B^h(f)_u = l_u \cdot l_y(zf),$$

at a point $u = (y, z)$, where l_u and l_y are the respective lifts with respect to Γ and N. It then follows that n h-basic vector fields $B^h(e_a)$, $a = 1, \cdots, n$, give a base of Γ^h. As a consequence, $2n$ vector fields $B^h(e_a)$, $B^v(e_a)$, $a = 1, \cdots, n$, give a base of the connection Γ.

Let us project a Finsler pair (Γ^h, Γ^v) on the bundle of linear frames $L(M)$ by means of the induced mapping π_2: $F(M) \to L(M)$.

Then, corresponding to any $f \in F$, we obtain two distributions $\Gamma_{(f)}$ and $\Gamma^v_{(f)}$ on $L(M)$, such that

$$\Gamma_{(f)z} = \pi_2 \Gamma^h_u, \qquad \Gamma^v_{(f)z} = \pi_2 \Gamma^v_u, \qquad u = i^{-1}(z, f).$$

We are not interested in the latter $\Gamma^v_{(f)}$, because it is vertical, that is, contained in the vertical distribution L^v. On the other hand, the former $\Gamma_{(f)}$ is very important, because it constitutes an F-connection $\Gamma_F = \{\Gamma_{(f)}\}$, as will be easily shown. This Γ_F is called the *subordinate F-connection* to the Finsler connection $(\Gamma, N) = (\Gamma^h, \Gamma^v)$.

Definition 6. A *Finsler triad* (Γ_F, N, Γ^v) of M is a triad of an F-connection Γ_F in $L(M)$, a non-linear connection N in $T(M)$, and a vertical connection Γ^v in $F(M)$.

Then, the following theorem means that a Finsler connection can be thought of as a Finsler triad.

Theorem 2. *There is a natural one-to-one correspondence between the set of Finsler connections of M and the set of Finsler triads on M.*

The correspondence $(\Gamma, N) = (\Gamma^h, \Gamma^v) \rightarrow (\Gamma_F, N, \Gamma^v)$ is given by

Γ_F ······ subordinate F-connection,

while the inverse correspondence $(\Gamma_F, N, \Gamma^v) \rightarrow (\Gamma, N) = (\Gamma^h, \Gamma^v)$ is

$$\Gamma^h_u = \{X \in F(M)_u \mid \pi_1 X \in N_y, \ \pi_2 X \in \Gamma_{(f)}, \ \ y = \pi_1 u, \ f = \gamma(u)\}.$$

[4] We shall give a modern definition of tensor field appearing in the classical theory of Finsler spaces, whose components are functions not only of point, but also of element of support. Let V be a vector space and ρ: $G \rightarrow GL(V)$ be a representation of $G = GL(n, R)$ on V. Then, a *Finsler tensor field K of ρ-type* is by definition a V-valued function on $F(M)$, satisfying the equation $K \cdot T_g = \rho(g^{-1})K$ for any $g \in G$. If V is the tensorial product

$F^r_s = \underbrace{F\otimes\cdots\otimes F}_{r}\otimes\underbrace{F^*\otimes\cdots\otimes F^*}_{s}$ (space of linear mappings $\underbrace{F^*\times\cdots\times F^*}_{r}\times\underbrace{F\times\cdots\times F}_{s}\to R$) and ρ is the usual representation, then K is called *of* (r, s)*-type.*

For a typical example, the characteristic field γ is a Finsler tensor field of $(1, 0)$-type. In order to show another example, we shall consider the difference between a general vertical connection Γ^v and the vertical flat connection F^i. Then, a Finsler tensor field C of the adjoint-type is introduced by the equation

$$(1.1) \qquad Y(f) = B^v(f) + Z(C(f)),$$

where $Z(A)$, corresponding to $A\in L(n, R)$ (the Lie algebra of $GL(n, R)$), is a well-known fundamental vector field, defined by $Z(A)_u = {}_uTA$, ${}_uT$ being the differential of the mapping obtained from T by becoming $u\in F(M)$ fixed. C as thus defined is called *Cartan tensor field* of $(\Gamma, N) = (\Gamma^h, \Gamma^v)$ under consideration. In the case of famous Finsler connection due to E. Cartan, C is nothing but the well-known tensor, components of which are C^i_{jk}.

While the equation

$$(1.2) \qquad B^v(f)\gamma = f + C(\gamma, f), \qquad C(\gamma, f) = C(f)\gamma,$$

will be easily verified in virtue of (1.1), the equation

$$(1.3) \qquad B^h(f)\gamma = D(f)$$

introduces a new Finsler tensor field D of $(1, 1)$-type, which is called the *deflection tensor field* of (Γ, N). It will be observed that the deflection tensor D vanishes identically in the case of almost all of classical Finsler connections.

Finally, let us consider two Finsler connections (Γ, N) and (Γ', N'), and let $B^h(f)$, $B^v(f)$ and $B'^h(f)$, $B'^v(f)$ be respective h- and v-basic vector fields. Then, the equations

$$(1.4) \qquad B'^h(f) = B^h(f) + B^v(D^h(f)) + Z(A^h(f)),$$

$$(1.5) \qquad B'^v(f) = \qquad B^v(f) \qquad + Z(A^v(f)),$$

will be easily derived, and thus we obtain three Finsler tensor fields D^h, A^h and A^v; D^h being of $(1,1)$-type, A^h, A^v being of the adjoint type.

§2. Homogeneous Finsler connections

Given a Finsler connection (Γ, N), its torsions T, C, R^1, P^1, S^1 and its curvatures R^2, P^2, S^2 are introduced by the equations

$$(2.1)\qquad [B^h(f_1), B^h(f_2)] = B^h(T(f_1, f_2)) + B^v(R^1(f_1, f_2)) + Z(R^2(f_1, f_2)),$$

$$(2.2)\qquad [B^h(f_1), B^v(f_2)] = B^h(C(f_1, f_2)) + B^v(P^1(f_1, f_2)) + Z(P^2(f_1, f_2)),$$

$$(2.3)\qquad [B^v(f_1), B^v(f_2)] = B^v(S^1(f_1, f_2)) + Z(S^2(f_1, f_2)).$$

C as appearing in (2.2) is nothing but the Cartan tensor. S^1 and S^2 are the torsion and curvature of the subordinate vertical connection Γ^v respectively, and expressed by C as follows.

$$S^1(f_1, f_2) = C(f_1, f_2) - C(f_2, f_1),$$
$$S^2(f_1, f_2) = \Delta^0 C(f_1, f_2) - \Delta^0 C(f_2, f_1) - [C(f_1), C(f_2)],$$

where the covariant differential operator Δ^0 is the differentiation by $Y(f)$, that is, $\Delta^0 C(f_1, f_2) = Y(f_2)C(f_1)$.

Next, it follows from (1.1) and (2.2) that

$$(2.4)\qquad [B^h(f_1), Y(f_2)] = Y(P^1(f_1, f_2)) + Z(P^2(f_1, f_2) + \Delta^h C(f_2, f_1) - C(P^1(f_1, f_2))),$$

where the h-covariant differential operator Δ^h is the differentiation by $B^h(f)$, that is, $\Delta^h C(f_2, f_1) = B^h(f_1)C(f_2)$. Further, it follows from (2.4) and (1.3) that

$$(2.5)\qquad [B^h(f), Y(\gamma)] = B^v(P^1(f, \gamma) + D(f)) + Z(P^2(f, \gamma) + (\Delta^h C(\gamma))f).$$

Now, we shall be concerned with the homogeneous property of some geometrical objects. A function μ on $F(M)$ is called *homogeneous of degree r*, if the equation $\mu\cdot{}_{\alpha}H=\alpha^{r}\cdot\mu$ holds good for any $\alpha\in R^{+}$. Next, a tangent vector field X on $F(M)$ is called homogeneous of degree r, if the equation ${}_{\alpha}HX=\alpha^{r}\cdot X$ holds good. Finally, a distribution D on $F(M)$ is called homogeneous, if D is H-invariant.

Definition 7. A Finsler connection (Γ, N) is called *homogeneous*, if Γ and N be homogeneous in the respective sense of $F(M)$ and $T(M)$.

Proposition 3. *A necessary and sufficient condition for a Finsler connection (Γ, N) to be homogeneous is that $B^{h}(f)$ and $B^{v}(f)$ be homogeneous of degree* 0 *and* 1 *respectively.*

The proof will be easily obtained.

Proposition 4. *If a function μ on $F(M)$ is homogeneous of degree r, then $B^{h}(f)\mu$ is homogeneous of the same degree, provided that the Finsler connection under consideration be homogeneous.*

The proof will be easily obtained from Proposition 3. The following is the well-known Euler's theorem on homogeneous functions.

Proposition 5. *If a function μ on $F(M)$ is homogeneous of degree r, then the equation $Y(\gamma)\mu=r\cdot\mu$ holds good.*

Proof. Since the induced fundamental vector field $Y(f)$ is induced from the 1-parameter group of transformations $\{S_{tf}\}$, it is seen that, at a point $u=(y, z)$,

$$\begin{aligned}Y(\gamma)_{u}\mu&=\lim_{t\to 0}\frac{1}{t}\{\mu(y+ty, z)-\mu(y, z)\}\\&=\lim_{t\to 0}\frac{1}{t}\{\mu\cdot(1+t)H(u)-\mu(u)\}\end{aligned}$$

$$=\lim_{t\to 0}\frac{1}{t}\{(1+t)^r\mu(u)-\mu(u)\}=r\cdot\mu(u).$$

The following theorem gives the interesting properties of a homogeneous Finsler connection, although (2) will not need in future.

Theorem 3. *The torsion* P^1 *and the curvature* P^2 *of a homogeneous Finsler connection satisfy*

(1) $P^1(f,\gamma)=-D(f)$,

(2) $P^2(f,\gamma)=-\varDelta^h C(\gamma,f)-C(D(f))$.

Proof. We first obtain from (2.4) one of the Ricci's identities

$$\varDelta^h(\varDelta^0 K)(f_2,f_1)-\varDelta({}^0\varDelta^h K)(f_1,f_2)$$
$$=\varDelta^0 K(P^1(f_1,f_2))-P^2(f_1,f_2)K-(\varDelta^h C(f_2,f_1))K+C(P^1(f_1,f_2))K$$

If we put $f_1=f$ and $f_2=\gamma$ in the above, it follows that

$$\begin{aligned}\varDelta^0 K(P^1(f,\gamma))&-P^2(f,\gamma)K-(\varDelta^h C(\gamma,f))K+C(P^1(f,\gamma))K\\ &=\varDelta^h(\varDelta^0 K)(\gamma,f)-\varDelta^0(\varDelta^h K)(f,\gamma)\\ &=\varDelta^h(\varDelta^0 K(\gamma))(f)-\varDelta^0 K(\varDelta^h\gamma(f))-\varDelta^0(\varDelta^h K(f))(\gamma).\end{aligned}$$

If K is supposed to be homogeneous of degree 1, it follows from Propositions 4 and 5 that

$$\varDelta^h(\varDelta^0 K(\gamma))(f)=(\varDelta^h K)(f),\quad \varDelta^0(\varDelta^h K(f))(\gamma)=\varDelta^0 K(f),$$

and hence the above equation leads us to

$$\begin{aligned}\varDelta^0 K(P^1(f,\gamma)+D(f))&-(\varDelta^h C(\gamma,f))K+C(P^1(f,\gamma))K\\ &=P^2(f,\gamma)K.\end{aligned}$$

Therefore, the equation (1), together with the above equation, gives (2). In order to prove (1), it is sufficient to show that $[B^h(f), Y(\gamma)]$ is vertical, because of (2.5). Let μ be any homogeneous function of degree 1 on $T(M)$, and then $\mu\cdot\pi_1$ is obviously homogeneous of degree 1 on $F(M)$. It then follows from Propositions 4 and 5 that

$$(\pi_1[B^h(f),\ Y(\gamma)])\mu$$
$$=B^h(f)(Y(\gamma)(\mu\cdot\pi_1))-Y(\gamma)(B^h(f)(\mu\cdot\pi_1))=0,$$

from which the equation $\pi_1[B^h(f),\ Y(\gamma)]=0$ is derived.

Definition 8. The *D-simple Finsler connection* (Γ', N') of a Finsler connection (Γ, N) is defined by (1.4) and (1.5), where $D^h = A^v = 0$ and $A^h(f) = -P^1(f,\ )$.

The following proposition will be immediately shown from (1.3), (1.4) and (1.5).

Proposition 6. *The D-simple Finsler connection (Γ', N') of a Finsler connection (Γ, N) is such that*

(1) $N'=N$, (2) $D'(f)=D(f)+P^1(f,\ )$.

Theorem 4. *The deflection tensor D' of the D-simple Finsler connection (Γ', N) of any homogeneous Finsler connection (Γ, N) vanishes identically.*

This important theorem is a direct result of Theorem 3-(1) and Proposition 6-(2).

§3. Homogeneous non-linear connections

First of all, we shall consider the differential of the characteristic field γ, the mapping $F(M)\to F$, $[(y,z)\to z^{-1}y]$.

Proposition 7. *The differential of the characteristic field γ is given by*

$${}_z a\cdot\gamma X=\pi_1 X-a_f\cdot\pi_2 X,$$

where $X\in F(M)_u$ and $u=i^{-1}(z,f)$.

Proof. It follows from the identification i: $F(M)\to L(M)\times F$ and the mapping a: $L(M)\times F\to T(M)$ that

$$\pi_1 X=a\cdot iX=a(\pi_2 X,\gamma X)=a_f\cdot\pi_2 X+{}_z a\cdot\gamma X,$$

which proves the proposition.

Let us remember the definition of a Finsler triad (Γ_F, N, Γ^v), where there is not any interrelationship among Γ_F, N and Γ^v. Now, a special Finsler triad is required for our purpose.

Definition 9. A Finsler connection (Γ, N) is called *N-simple*, if N is the associated non-linear connection with the subordinate F-connection Γ_F.

A geometrical meaning of the deflection tensor D will be given by the following.

Theorem 5. *A necessary and sufficient condition for a Finsler connection to be N-simple is that the deflection tensor D vanishes identically.*

Proof. It follows from (1.3) that ${}_\Gamma B^h(f) = P(D(f))$, and hence Proposition 7 leads us to

$${}_z aP(D(f_1)) = \pi_1 B^h(f_1) - a_f \cdot \pi_2 B^h(f_1),$$

for any $f_1 \in F$. The proof follows then immediately from the definition of the subordinate F-connection.

The main result of the present paper is now stated as the following theorem on a homogeneous non-linear connection.

Theorem 6. *Any homogeneous non-linear connection in the tangent bundle $T(M)$ is the associated one with an F-connection in the bundle of linear frames $L(M)$.*

Proof. Let N be a given homogeneous non-linear connection in $T(M)$, and construct a homogeneous Finsler connection (Γ, N), combining with N an arbitrary homogeneous connection Γ in $F(M)$. In order to do so, it is enough to observe that the induced connection Γ from a linear connection $\underline{\Gamma}$ in $L(M)$ by the induced mapping π_2 is surely homogeneous, where Γ is given by

$$\Gamma_u = \{X \in F(M)_u \mid \pi_2 X \in \underline{\Gamma}_z,\ z = \pi_2 u\}.$$

Next, let us construct the D-simple Finsler connection (Γ', N') of the above (Γ, N). It then follows from Proposition 6 that $N' = N$, and from Theorem 4 that $D' = 0$. Therefore, Theorem 5 leads us to the conclusion that the connection (Γ', N') is N-simple, that is, the original non-linear connection N is the associated one with the subordinate F-connection Γ'_F of (Γ', N').

It should be remarked that a homogeneous non-linear connection N may be associated with two different F-connections. It is, however, observed that the above F-connection Γ'_F satisfies $\Gamma'_{(\alpha f)} = \Gamma'_{(f)}$ for any $\alpha \in R^+$. In general, we have

Proposition 8. *The subordinate F-connection $\Gamma_F = \{\Gamma_{(f)}\}$ of a homogeneous Finsler connection (Γ, N) satisfies $\Gamma_{(\alpha f)} = \Gamma_{(f)}$ for any $f \in F$ and any $\alpha \in R^+$.*

Proof. The distribution $\Gamma_{(\alpha f)}$ is defined by $\Gamma_{(\alpha f)z} = \pi_2 \Gamma^h_{u'}$, where

$$u' = i^{-1}(z, \alpha f) = (z\alpha f, z) = {}_\alpha H(zf, z) = \alpha u, \qquad u = i^{-1}(z, f).$$

Therefore we see

$$\Gamma_{(\alpha f)z} = \pi_2 \Gamma^h_{\alpha u} = \pi_2 \cdot {}_\alpha H \Gamma^h_u = \pi_2 \Gamma^h_u = \Gamma_{(f)z}.$$

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