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Some results on PL-cobordism

By

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Dedicated to Prof. Atuo Komatu on his sixtieth birthday

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1. Introduction

In this paper we will use the results of Sullivan [10], Stasheff [9], and Peterson and Toda [8] concerning the classifying spaces $BSPL$ and BSF to obtain some further results concerning $H^*(BSPL)$[2)] and the p-torsion in Ω_*^{PL}, the oriented PL-cobordism ring. Since some of these results are computational in nature, we will often only give an outline of the proof.

Let $J_{PL}: BSPL \to BSF$ be the natural map. Let $q_i \in H^{ir}(BSF)$ be the Wu class ($r=2p-2$ throughout). Our first main result is that $J_{PL}^*(\beta q_i) \neq 0$ if $i \geq p+1$. (It is easy to show that $J_{PL}^*(\beta q_i)=0$ if $i \leq p$). Our other main result is a computation of the p-torsion of Ω_*^{PL} in dimensions $\leq p^2 r$. In particular, we show that there is a PL-manifold M of dimension $(2p+1)r-1$ of order p^2 in Ω_*^{PL} such that pM is not detected by any ordinary characteristic numbers. Finally, we make a few conjectures concerning $H^*(MSPL)$.

2. $H^*(BSPL)$

Sullivan [10] has proved that $BSPL$ is of the same mod p homotopy type as $BSO \times B\mathrm{Coker}\, J$, where B Coker J is a space

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2) All cohomology groups in this paper have coefficients Z_p, p an odd prime, unless otherwise stated.

such that $\pi_*(B\mathrm{Coker}\ J)=p$-torsion of

$$\mathrm{Coker}(J:\pi_*(BSO)\to\pi_*(BSF)).$$

The proof of this has not appeared so we make a few remarks on the proof. Sullivan shows that there exists a map $BSPL\to BSO$ which is onto mod p. $B\mathrm{Coker}\ J$ is defined to be the fibre of this map. It is known (e.g. by Adams, Anderson, Peterson and Sullivan) that BSO is of the same mod p homotopy type as $Y\times Y'$, where $\pi_i(Y)=\begin{cases}Z & i\equiv 0(r)\\ 0 & \text{otherwise}\end{cases}$ and $Y'=\prod_{i=1}^{\frac{p-1}{2}-1}\Omega^{4i}(Y)$. Also, Sullivan [11] proves that $F/PL\tilde{_p}BSO$. Map $BSO\times B\mathrm{Coker}\ J\to BSPL$ as follows: $BSO\times B\mathrm{Coker}J\to Y\times Y'\times B\mathrm{Coker}\ J\to BSPL\times F/PL\times B\mathrm{Coker}\ J\to BSPL\times BSPL\times BSPL\to BSPL$ and the composition is a homotopy equivalence. Note that $BSO\times\mathrm{pt.}\to BSPL$ is not the usual map of $BSO\to BSPL$. Also, $H^*(B\mathrm{Coker}\ J)$ has a Z_p-basis, in dimensions $\leq 2pr-2$ consisting of $\bar{e}_1$, $\beta\bar{e}_1$, $\mathcal{P}^1\beta\bar{e}_1$ and $\beta\mathcal{P}^1\beta\bar{e}_1$, by direct computation, where $\bar{e}_1$ is the image under $B\mathrm{Coker}\ J\xrightarrow{j}BSPL\xrightarrow{J_{PL}}BSF$ of $e_1\in H^{pr-1}(BSF)$.

Theorem 2.1. *$j^*J_{PL}^*(q_p)=\mu\beta\bar{e}_1$, where $\mu\not\equiv 0(p)$.*

Proof. Let $f:S^{pr-1}\to B\mathrm{Coker}\ J$ be a map corresponding to β_1 in $\pi_{pr-2}(S)$. Let $g:S^{pr-1}\cup_p e^{pr}\to B\mathrm{Coker}\ J$ be an extension of f. Let $\Phi:\mathrm{Ker}(2\mathcal{P}^2\beta-\beta\mathcal{P}^1)\to\mathrm{Coker}\ \mathcal{P}^{p-2}$ be an unstable secondary operation defined on classes of dimenion r by the relation

$$\mathcal{P}^{p-2}(2\mathcal{P}^1\beta-\beta\mathcal{P}^1)=-2\mathcal{P}^{p-1}\beta-\beta\mathcal{P}^{p-1}-\binom{p-2}{p-3}\mathcal{P}^{p-1}\beta=-\beta\mathcal{P}^{p-1}$$

which is zero on classes of dimension $\leq r$. In the first lemma of §6 of [9], Stasheff shows the following relation in $H^{pr}(BSF)$: one can choose Φ such that $q_p=\lambda\Phi(q_1)+\mu\beta e_1$, where $\lambda\not\equiv 0(p)$. Clearly, $g^*j^*J_{PL}^*(\Phi(q_1))=0$ with zero indeterminacy. To show $\mu\not\equiv 0(p)$, it is enough to show $g^*j^*J_{PL}^*(q_p)\neq 0$. Let X be the Thom space of $J_{PL}jg:S^{pr-1}\cup_p e^{pr}\to BSF(N)$. Then $X=S^N\cup_h e^{N+pr-1}\cup e^{N+pr}$, and $[h]=\beta_1$ (see proposition 4.5 of [13]). By the main result of [3], on $H^N(Y)$, $\mathcal{P}^p=\Sigma a_i\Phi_i+\mu\beta\Psi$, where Φ_i and Ψ are secondary operations, $\mu\not\equiv 0(p)$, $\dim a_i>1$, and Ψ detects β_1. Thus, in $H^*(X)$, we have $\mathcal{P}^p(U)=g^*j^*J_{PL}^*(q_p)\cdot U=\mu\beta\Psi(U)=\mu g^*(\beta\bar{e}_1)\cdot U$ which proves the theorem.

Corollary 2.2. $J_{PL}^*(\beta q_{p+1})\neq 0$.

Proof. By theorem 2.1, $j^*J_{PL}^*(q_p)=\bar{e}_1$. But $\beta\mathcal{P}^1q_p=\beta q_{p+1}-\beta q_1\cdot q_p-q_1\cdot\beta q_p$, hence $j^*J_{PL}^*(\beta q_{p+1})=\mu\beta\mathcal{P}^1\beta\bar{e}_1+0=\mu\beta\mathcal{P}^1\beta\bar{e}_1\neq 0$ as $J_{PL}^*(\beta q_1)=0$ and so $J_{PL}^*(\beta q_i)=0$, $i\leq p$ by the relation $\mathcal{P}^1\beta q_i=i\beta q_{i+1}-q_1\beta q_i$ (see [9]).

Corollary 2.3. $J_{PL}^*(\beta q_i)\neq 0$ if $i\geq p+1$.

Proof. Let $\psi: H^*(BSPL)\to H^*(BSPL)\otimes H^*(BSPL)$ be the diagonal map. Since $\psi(q_i)=\Sigma q_j\otimes q_{i-j}$, the corollary follows by induction using corollary 2.2.

In preparation for the next section, we note the following corollary of 2.2. Let $\theta:\mathcal{A}\to H^*(MSPL)$ be defined by $\theta(a)=a(U)$. One might conjecture that $\theta(Q_i)=0$, where Q_i are the Milnor elements [6]. $\theta(Q_0)=Q_0(U)=0$ and $\theta(Q_1)=J_{PL}^*(\beta q_1)\cdot U=0$.

Corollary 2.4. $\theta(Q_2)\neq 0$.

Proof. By proposition 3.1 of [8], $\theta(Q_2)=J_{PL}^*(\lambda\beta q_{p+1})\cdot U$ with $\lambda\not\equiv 0$. Now apply corollary 2.2.

3. Ω_*^{PL}

In this section we state our results on Ω_*^{PL}. We first note that corollary 2.2 shows that the first lemma on p. 32 of [12] is incorrect, so the calculations of the 3-torsion of Ω_*^{PL} in [12] are incorrect. However, the answers in [12] are correct.

Peterson and Toda [8] have proved $H^*(BSF)\approx Z_p[q_i]\otimes E(\beta q_i)\otimes C$, where C is $(pr-2)$-connected. In the range we will work in it is not difficult to show that $H^*(B\text{Coker } J)\approx C$ as an algebra over $\mathcal{A}$. In fact, one conjectures that this is true in all dimensions.

(Note however that theorem 2.1 shows that the map $J_{PL}^*: Z_p[q_i] \otimes E(\beta q_i) \otimes C \to H^*(BSO) \times H^*(B\text{Coker } J)$ is not as simple as one might expect.) Furthermore, Stasheff [9] has computed C explicitly in dimensions $<(p^2+1)r-1$ and has shown the following result.[3)]

Theorem. 3.1. *In dim.$<(p^2+1)r-1$, C is a free commutative algebra with truncation of height p on generators $\{a(e_1)\}$, where $\{a\}$ runs through an additive base of $\mathcal{A}/\mathcal{A}(\mathcal{P}^1, 2\mathcal{P}^p\mathcal{P}^1\beta-\mathcal{P}^{p+1}\beta)$, and e_2, βe_2, and $\beta_2(e_1\cdot(\beta e_1)^2)$, where $e_2 \in C^{p^2r-1}$ and $e^r \in C^{pr-1}$.*

Sullivan [10] shows that the spliting $BSPL \tilde{_p} BSO \times B\text{Coker } J$ respects the universal bundle and hence $MSPL \tilde{_p} MSO \wedge M\text{Coker } J$. (This is not hard when $p=3$, but more difficult if $p>3$.) Hence, $H^*(MSPL) \approx H^*(MSO) \otimes H^*(M\text{Coker } J)$. To compute $\pi_*(MSPL) \approx \Omega_*^{PL}$ (by Williamson [12]), we wish to compute $H^*(MSPL)$ as a module over $\mathcal{A}$ and apply the Adams spectral sequence. Now $H^*(MSO) = \Sigma' \mathcal{A}$ (see [1]), where $'\mathcal{A} = \mathcal{A}/\mathcal{A}\bar{E}$, and $E=E(Q_0, Q_1, Q_2, \cdots)$ is the exterior algebra on the Milnor elements $Q_1 \in \mathcal{A}^{(p^{i-1}+\cdots+p+1)r+1}$. The results of [1] show that the $\mathcal{A}$-module structure of $'\mathcal{A} \otimes N$ depends only on the E-module structure of N and further that $\text{Ext}_{\mathcal{A}}('\mathcal{A} \otimes N, Z_p) \approx \text{Ext}_E(N, Z_p)$. Hence we must compute $H^*(M\text{Coker } J)$ as an E-module. Corollary 2.4 shows that $Q_2(U) = \mu\beta\mathcal{P}^1\beta\bar{e}_1 \cdot U$, with $\mu \not\equiv 0(p)$. Using this result, theorem 3.1, and direct computation, we obtain the following theorem.

Theorem 3.2. *In dimensions $<(p^2+1)r-1$, as module over $\mathcal{A}$, $H^*(MSPL)$ is isomorphic to a direct sum of copies of a module M plus copies of $\mathcal{A}/\mathcal{A}Q_0$ plus a free module, where M has four generators,* $\dim X_0 = 0$, $\dim X_1 = pr-1$, $\dim X_2 = 2pr-1$ $\dim X_3 = (2p+1)r-1$, *with relations* $Q_0(X_0)=0$, $Q_1(X_0)=0$, $Q_2(X_0) = Q_0Q_1(X_1)$, *and* $Q_0(X_3)+Q_1(X_2)+Q_2(X_1)=0$. *Let $p=3$ for convenience in stating specific results. Let $\{y_\alpha \cdot U\}$ be an $'\mathcal{A}$-basis for $H^*(MSO)$. Then the generators for the modules M are*

3) Recently, May [4] and Milgram [5] have made great progress towards determining $H^*(BSF)$ and one hopes that the results in this section can be generalized.

quadruples $(X_0=y_\alpha\cdot U,\ X_1=y_\alpha\cdot e_1\cdot U,\ X_2=y_\alpha\cdot(e_1\cdot\beta e^r-\mathcal{P}^3e_1)\cdot U,\ X_3=y_\alpha\cdot(\mathcal{P}^4e_1-e_1\cdot\mathcal{P}^1\beta e_1)\cdot U)$, *the generators for copies of* $\mathcal{A}/\mathcal{A}Q_0$ *are* $e_1(\beta e_1)^2\cdot U$ *and* $\beta_2(e_1\cdot(\beta e_1)^2\cdot U)$, *and generators for copies of* $\mathcal{A}$ *are* $\mathcal{P}^3e_1\cdot U$, $\mathcal{P}^4e_1\cdot U$, $e_1\cdot\mathcal{P}^3e_1\cdot U$, $e_2\cdot U$, $e_1\cdot\beta\mathcal{P}^3e_1\cdot U$, *and* $e_1\cdot\mathcal{P}^4e_1\cdot U$.

It is not difficult to construct $\mathrm{Ext}_{\mathcal{A}}(H^*(MSPL), Z_p)$ and to note that all differentials must be zero for dimensional reasons in the range of dimensions under discussion except a differential from $t-s=36$ to $t-s=35$. From this we obtain the following theorem.

Theorem 3.3. *In dimensions* $<(p^2+1)r-1$, *we have that* $(\Omega_*^{PL}/torsion)\otimes Z_p$ *is a polynomial ring. Let* $p=3$ *as above. In dimensions* <39, *the* 3-*torsion of* Ω_*^{PL} *is given by the following table*:

Generators	*Dimension*	*Order*	*Detected by*
$M_\alpha\times M^{11}$	$11+\dim M_\alpha$	3	$y_\alpha\cdot e_1$
$M_\alpha\times M_1^{23}$	$23+\dim M_\alpha$	3	$y_\alpha\cdot\mathcal{P}^3e_1$
$M_\alpha\times M_2^{23}$	$23+\dim M_\alpha$	3	$y_\alpha\cdot e_1\cdot\beta e_1$
$M_\alpha\times M_1^{27}$	$27+\dim M_\alpha$	9	$y_\alpha\cdot(\mathcal{P}^4e_1-e_1\cdot\mathcal{P}^1\beta e_1)$
$M_\alpha\times M_2^{27}$	$27+\dim M_\alpha$	3	$y_\alpha\cdot\mathcal{P}^4e_1$
M^{34}	34	3	$e_1\cdot\mathcal{P}^3e_1$
M_1^{35}	35	3	e_2
M_2^{35}	35	3	$e_1\cdot\beta\mathcal{P}^3e_1$
M_3^{35}	35	9	$e_1\cdot(\beta e_1)^2$
M^{38}	38	3	$e_1\cdot\mathcal{P}^4e_1$

Here M_α *are elements in* Ω_*^{SO} *detected by* y_α. *The dimensions such* M_α *appear in are* 0, 8, 12, 16, 20, 24, 24, 24 *in the range under discussion.*

Easy computation shows that $e_1\cdot(\beta e_1)^2$ gives an element of order 9 in $H^{35}(MSPL;\ Z_q)$ which detects all multiples of M_3^{35}. However $\mathcal{P}^4e_1-e_1\cdot\mathcal{P}^1\beta e_1$ is only of order 3 and we have the following corollary.

Corollary 3.4. *In dimensions* ≤ 26, *all elements in* Ω_*^{PL} *are detected by ordinary characteristic classes. There is an* M_1^{27} *of* *order* 9 *such that* $3M_1^{27}$ *is not detected by an ordinary characteristic class.*

Proof. Since all elements of 2-torsion are detected by ordinary characteristic classes [2], just look at the above table.

Bibliography

[1] D. W. Anderson, E. H. Brown, Jr. and F. P. Peterson, "Pin Cobordism and Related Topics", to appear.
[2] W. Browder, A. Liulevicius and F. P. Peterson, "Cobordism Theories", Ann. of Math., **84** (1966), 91-101.
[3] A. Liulevicius, "The Factorization of Cyclic Reduced Powers by Secondary Cohomology Operations", Memoirs of the A.M.S., No. 42, 1962.
[4] J. P. May, to appear.
[5] R. J. Milgram, to appear.
[6] J. W. Milnor, "The Steenrod Algebra and Its Dual", Ann. of Math., **67** (1958), 150-171.
[7] J, W. Milnor, "On the Cobordism Ring Ω_* and a Complex Analogue", Amer. J. Math., **82** (1960), 505-521.
[8] F. P. Peterson and H. Toda, "On the Structure of $H^*(BSF; Z_p)$", J. of Math. of Kyoto Univ., **7** (1967), 113-121.
[9] J. Stasheff, "More Characteristic Classes for Spherical Fibre Spaces", Comm. Math. Helv., **43** (1968), 78-86.
[10] D. Sullivan, to appear.
[11] D. Sullivan, to appear.
[12] R. E. Williamson "Cobordism of Combinatorial Manifolds"' Ann. of Math., **83** (1966), 1-33.
[13] S. Gitler and J. Stasheff, "The First Exotic Class of BF", Topology, **4** (1965), 257-266.

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