J. Math. Kyoto Univ.
9-3 (1969) 439-448

Flatness of an extension of a commutative ring

Dedicated Professor K. Asano for his sixtieth birthday

By

Masayoshi NAGATA

(Received September 3, 1969)

Throughout the present paper, we mean by a ring a commutative ring with identity and by a module a unitary one. Let R be a ring and let A be a homomorphic image of the polynomial ring $R[X]$ of a set of variables X with kernel I. The main purpose of the present paper is to discuss some topics related to the following

Theorem 1. *Assume that I is the principal ideal generated by $f(X)=a_0X^{(0)}+a_1X^{(1)}+\cdots+a_nX^{(n)}$ ($a_i\in R$; $X^{(i)}$ monomials, $X^{(i)}\neq X^{(j)}$ if $i\neq j$). Let J be the ideal $\sum a_iR$ generated by the coefficients a_i of $f(X)$. Then A is R-flat if and only if J is a direct summand of R (i.e., $J=eR$ with an element $e\in R$ such that $e^2=e$).*

1. Preliminary results.

Besides very well known elementary facts on flatness, we use the following two results:

Lemma 1.1. *Assume that R and R^* are noetherian rings such that R^* is an R-module. Let ϕ be the homomorphism from R into R^* such that $\phi a=a\cdot 1$ (in R^*). Let $\mathfrak{M}^*$ be the set of maximal ideals of R^* and let $\mathfrak{M}$ be the set of prime ideals $\mathfrak{m}$ of R such that $\mathfrak{m}=\phi^{-1}(\mathfrak{m}^*)$ with $\mathfrak{m}^*\in\mathfrak{M}^*$. Then R^* is a flat R-module if* *and only if the following is true: If* $\mathfrak{q}$ *is a primary ideal with prime divisor* $\mathfrak{m}\in\mathfrak{M}$ *and if* b *is an element of* R *such that* $\mathfrak{q}:bR=\mathfrak{m}$, *then* $\mathfrak{q}R^*: bR^*=\mathfrak{m}R^*$.

Lemma 1.2. *Let* R *be a ring and let* M *be an* R*-module. Let* $\mathfrak{S}$ *be a set of multiplicatively closed subsets* S *of* R *such that* $0\notin S$. *Assume that for every maximal ideal* $\mathfrak{m}$ *of* R, *there is an* $S\in\mathfrak{S}$ *such that* $\mathfrak{m}\cap S$ *is empty. Then* M *is* R*-flat if and only if* $M\otimes R_S$ *is* R_S*-flat for every* $S\in\mathfrak{S}$.[1]

As for Lemma 1.1, see [L],[2] (18.7). Though Lemma 1.2 is also well known, we give an explicit proof: The only if part is obvious and we prove the if part. Assume that $\phi: A\to B$ is an injection with respect to R-modules A, B. Let K be the kernel of $\phi\otimes id.: A\otimes M\to B\otimes M$. By our assumption, $K\otimes R_S=0$ for every $S\in\mathfrak{S}$. Assume for a moment that $K\neq 0$ and let k be a non-zero element of K. We consider the natural injection $i: kR\to K$. By our assumption on $\mathfrak{S}$, there is an $S\in\mathfrak{S}$ such that $kR_S\neq 0$. Since R_S is R-flat, we see that $0\neq kR_S\subseteq K\otimes R_S=0$, which is a contradiction. Thus $K=0$ and M is R-flat.

2. The only if part of Theorem 1.

We prove first the following

Proposition 2.1. *Let* $(R,\mathfrak{m})$ *be a quasi-local ring and let* I *be an ideal of the polynomial ring* $R[X]$ *of a set of variables* X. *If* $B=R[X]/I$ *is* R*-flat and if* $I\subseteq\mathfrak{m}[X]$, *then* $I=0$.

Proof. Assume that $I\neq 0$. Let $f(X)=\sum c_{(i)}X^{(i)}$ ($X^{(i)}$ being monimials in X, $X^{(i)}\neq X^{(j)}$ if $i\neq j$) be a non-zero element of I. There is an ideal J^* of R such that $\sum c_{(i)}\mathfrak{m}\subseteq J^*\subset\sum c_{(i)}R$ and such that $\sum c_{(i)}R/J^*\cong R/\mathfrak{m}$. Then B/J^*B is R/J^*-flat. Therefore observing B/J^*B and R/J^* instead of B and R, we may assume that

1) M is a faithfully flat R-module if and only if $M\otimes R_S$ is a faithfully flat R_S-module for every S.

2) By [L], we refer to M. Nagata, Local rings, John Wiley, 1962.

$f(X)=cg(X)$ with a polynomial $g(X)$ one of whose coefficients is 1. Denoting the residue classes of X by x, we have $cg(x)=0$, whence $g(x)\in 0 : cB=(0 : cR)B=\mathfrak{m}B$. This shows that there is an element $h(X)$ of $\mathfrak{m}[X]$ such that $g(x)=h(x)$, that is, $g(X)-h(X)\in I$. Since 1 appears as a coefficient in $g(X)$ and since $h(X)\in\mathfrak{m}[X]$, we see that $I\ni g(X)-h(X)\in\mathfrak{m}[X]$, which is a contradiction, and Proposition 2.1 is proved.

Now, in view of Lemma 1.2, we have the following result which include the only if part of Theorem 1:

Theorem 2. *Let R be a ring and let I be an ideal of the polynomial ring $R[X]$ of a set of variables X. Let J be the ideal generated by coefficients of elements of I. If $R[X]/I$ is R-flat, then J is a direct summand of R.*

3. The if part of Theorem 1.

A proof of the part was given by D. Mumford,[3] and we are to give a generalization of it. For the purpose, we introduce a symbol ϕ and a modified notion of a regular sequence as follows:

1) When $\mathfrak{a}$ is an ideal of R, we denote by $\phi_{\mathfrak{a}}$ the natural homomorphisms $R[X]\rightarrow(R/\mathfrak{a})[X]$.

2) A regular sequence[4] in a ring S is a sequence $f_1,\cdots,f_n$ of elements of S such that $(\sum_{i\vee\alpha}f_iS) : f_\alpha S=\sum_{i<\alpha}f_iS$ for every $\alpha=1,2,\cdots,n$.

Now our generalization of the if part of Theorem 1 can be stated as follows:

Theorem 3. *Let $\mathfrak{M}'$ be the set of maximal ideals $\mathfrak{m}'$ of $R[X]$ such that $\mathfrak{m}'\supseteq I$, and let $\mathfrak{M}$ be the set of prime ideals $\mathfrak{p}$ for which there is an $\mathfrak{m}'\in\mathfrak{M}'$ such that $\mathfrak{p}=\mathfrak{m}'\cap R$. $A=R[X]/I$ is R-flat if there is a basis $f_1,\cdots,f_n$ for I such that a permutation of $\phi_{\mathfrak{p}}f_1,\cdots,$*

---

3) D. Mumford, Introduction to algebraic geometry, Harvard Univ. Lect. Notes, 1967.

4) Under usual definition, one requires one more condition that $\sum_{i\leq n}f_iS\neq S$.

$\phi_{\mathfrak{p}} f_n$ *from a regular sequence in* $(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})[X]$ *for each* $\mathfrak{p} \in \mathfrak{M}$.

In order to prove Theorem 3, we need the following preliminary results:

Lemma 3.1. *Let* $\mathfrak{a}$ *be an ideal of* R *and let* $f_1, \cdots, f_n$ *be elements of* $R[X]$. *Assume that* $h \in \mathfrak{a}[X] \cap \sum f_i R[X]$. *If* $\phi_{\mathfrak{a}} f_1, \cdots, \phi_{\mathfrak{a}} f_n$ *from a regular sequence in* $\phi_{\mathfrak{a}} R[X]$, *then* h *is expressed as* $\sum f_i g_i$ *with* $g_i \in \mathfrak{a}[X]$, *i.e.*, $\mathfrak{a}[X] \cap \sum f_i R[X] = \mathfrak{a}(\sum f_i R[X])$.

Lemma 3.2. *Assume that* R *is noetherian and that* $f_1, \cdots, f_n \in R[X]$. *If* $\phi_{\mathfrak{p}} f_1, \cdots, \phi_{\mathfrak{p}} f_n$ *from a regular sequence in* $\phi_{\mathfrak{p}} R[X]$ *for every prime ideal* $\mathfrak{p}$ *of* R, *then* $\phi_{\mathfrak{a}} f_1, \cdots, \phi_{\mathfrak{a}} f_n$ *form a regular sequence in* $\phi_{\mathfrak{a}} R[X]$ *for an arbitrary ideal* $\mathfrak{a}$ *of* R.

Lemma 3.3. *Assume that* R *is a (noetherian) local ring with maximal ideal* $\mathfrak{m}$. *If* I *is generated by elements* $f_1, \cdots, f_n$ *such that* $\phi_{\mathfrak{m}} f_1, \cdots, \phi_{\mathfrak{m}} f_n$ *form a regular sequence in* $\phi_{\mathfrak{m}} R[X]$, *then for every* $\mathfrak{m}$*-primary ideal* $\mathfrak{q}$, *we have* $\mathfrak{q}[X] \cap I = \mathfrak{q} I$.

Proof of Lemma 3.1. Since $h \in \sum f_i R[X]$, $h = \sum f_i g'_i$ with $g'_i \in R[X]$. Then $\sum \phi_{\mathfrak{a}} f_i g'_i = 0$ and therefore $\phi_{\mathfrak{a}} g'_n \in {}_{i<n} \phi_{\mathfrak{a}} f_i R[X] : \phi_{\mathfrak{a}} f_n = \sum_{i<n} \phi_{\mathfrak{a}} f_i R[X]$. Thus there are $k_i \in R[X]$ such that $g_n = g'_n - \sum_{i<n} f_i k_i \in \mathfrak{a}[X]$. Then $h = \sum_{i<n} f_i (g'_i + f_n k_i) + f_n g_n$. Since $h - f_n g_n \in \mathfrak{a}[X] \cap \sum_{i<n} f_i R[X]$, we have the required result by induction on n.

Proof of Lemma 3.2. Using induction argument on n, we assume that the assertion is true for such sequence consisting of $n-1$ elements. Consider the set $\mathfrak{B}$ of ideals $\mathfrak{b}$ of R such that $\phi_{\mathfrak{b}} f_1, \cdots, \phi_{\mathfrak{b}} f_n$ do not form a regular sequence. We want to show that $\mathfrak{B}$ is empty. Assume the contrary. Then, taking a maximal member $\mathfrak{b}_0$ of $\mathfrak{B}$ and considering $R/\mathfrak{b}_0$ instead of R, we may assume that $\mathfrak{B}$ consists only of the zero ideal. By our assumption, the zero ideal is not prime and there is a non-unit a of R such that $0 : aR$ is a prime ideal, say $\mathfrak{p}$. $hf_n = \sum_{i<n} f_i g_i$ with $g_i \in R[X]$. Since $\phi_{aR} f_1, \cdots, \phi_{aR} f_n$ form a regular sequence (by our assumption that $\mathfrak{B}$ consists only of the zero

ideal), we have that $\phi_{aR}h \in \sum_{i<n}\phi_{aR}f_iR[X]$, i.e., there is an $h_0 = \sum_{i<n}f_ig'_i$ ($g'_i \in R[X]$) such that $h-h_0=ah'$ with $h' \in R[X]$. We apply Lemma 3.1 to $ah'f_n$ and we see that $ah'f_n=\sum_{i<n}af_ig''_i$ with $g''_i \in R[X]$. Then $\phi_{\mathfrak{p}}(h'f_n-\sum_{i<n}f_ig''_i)=0$ because $\mathfrak{p}=0:aR$, and, since $\phi_{\mathfrak{p}}f_1, \cdots, \phi_{\mathfrak{p}}f_n$ form a regular sequence, we see that $\phi_{\mathfrak{p}}h' = \sum_{i<n}\phi_{\mathfrak{p}}f_ig^*_i$ with $g^*_i \in R[X]$. Then $ah'=a\sum f_ig^*_i$ and we see that $h=h_0+ah' \in \sum_{i<n}f_iR[X]$. This completes the proof of Lemma 3.2.

Proof of Lemma 3.3. By virtue of Lemma 3.2, we see that $\phi_{\mathfrak{q}}f_1, \cdots, \phi_{\mathfrak{q}}f_n$ form a regular sequence, hence the assertion follow from Lemma 3.1.

Now we go back to the proof of Theorem 3. At the first step, we consider the case where R is noetherian and then we shall observe the general case:

(1) Noetherian case.

We assume that R is noetherian. We use symbols $\mathfrak{M}^*$, $\mathfrak{M}$, ϕ and $\mathfrak{M}'$ as in Lemma 1.1 (for the case $A=R^*$) and in Theorem 3 (note that $\mathfrak{M}$ is common). A is R-flat if and only if $A_{\mathfrak{m}^*}$ is R-flat for every $\mathfrak{m}^* \in \mathfrak{M}^*$, as is obvious by the definition of flatness. Thus, in view of Lemma 1.2, we have only to show that if $\mathfrak{m}^* \in \mathfrak{M}^*$ and if $\mathfrak{m}=\phi^{-1}(\mathfrak{m}^*)$, then $A_{\mathfrak{m}^*}$ is $R_{\mathfrak{m}}$-flat. Considering $R_{\mathfrak{m}}$ instead of R, we may assume that R is a local ring with maximal ideal $\mathfrak{m}$. Let $\mathfrak{m}'$ be the maximal ideal of $R[X]$ such that $\mathfrak{m}^*=\mathfrak{m}'/I$, and we observe the triple R, $A_{\mathfrak{m}^*}$, $R[X]_{\mathfrak{m}'}$. Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal and let $b \in R$ be such that $\mathfrak{q}:b=\mathfrak{m}$. By Lemma 1.1, we have only to show that $\mathfrak{q}A_{\mathfrak{m}^*}:b=\mathfrak{m}A_{\mathfrak{m}^*}$. Then considering things modulo $\mathfrak{q}$, we may assume that $\mathfrak{q}=0$. Assume now that $\mathfrak{q}A_{\mathfrak{m}^*}:b \neq \mathfrak{m}A_{\mathfrak{m}_*}$. Then there is an element y of A which is not in $\mathfrak{m}A_{\mathfrak{m}^*}$ such that $by \in \mathfrak{q}A$. Let h be a representative of y in $R[X]$. Then $bh \in I$, whence $bh \in bR[X] \cap I=bI$ by virtue of Lemma 3.3. Thus $bh=bh_1$, $h_1 \in I$. Then $b(h-h_1)=0$, hence $h-h_1 \in \mathfrak{m}[X]$ and on the other hand $h-h_1$ represents y. This means that $y \in \mathfrak{m}A$, which is a contradiction. Thus $\mathfrak{q}A_{\mathfrak{m}^*}:b=\mathfrak{m}A_{\mathfrak{m}^*}$, and we settle the case.

(2) General case.

Let R_0 be a finitely generated subring of R containing all coefficients of $f_1, \cdots, f_n$. Then the condition in Theorem 3 holds good for $R_0[X]$ and $I_0 = \sum f_i R_0[X]$.[5] Then by the noetherian case proved above, $A_0 = R_0[X]/I_0$ is R_0-flat. Obviously A is identified with $A_0 \otimes_{R_0} R$ and therefore A is R-flat.

Thus the proof of Theorem 3 is completed.

4. Some remarks on generators of *I*.

We maintain the meanings of R, X, I, ϕ as before. But we are to treat the case where R is noetherian and X is a finite set.

Main remark we are to give here is the following

Theorem 4. *Let $\mathfrak{a}$ be an ideal of R and set $S = \{g \in R[X] \mid \phi_{\mathfrak{a}} g = 1\}$. Assume that (1) R is noetherian, (2) X is a finite set (3) $f_1, \cdots, f_n$ are elements of I such that $\phi_{\mathfrak{a}} f_1, \cdots, \phi_{\mathfrak{a}} f_n$ generates $\phi_{\mathfrak{a}} I$ and (4) $A = R[X]/I$ is R-flat. Then*

$$\sum_i f_i R[X]_S = IR[X]_S .$$

In other words, there is an element s of S such that $sI \subseteq \sum_i f_i R[X]$.

Proof. Let $\mathfrak{B}$ be the set of ideals $\mathfrak{b}$ of R such that $\mathfrak{b} \subseteq \mathfrak{a}$ and $\phi_{\mathfrak{b}}(\sum f_i R[X]_S) \neq \phi_{\mathfrak{b}}(IR[X]_S)$ (here $\phi_{\mathfrak{b}}$ is naturally entended to $R[X]_S \rightarrow \phi_{\mathfrak{b}} R[X]_{\phi_{\mathfrak{b}} S}$). We want to show that $\mathfrak{B}$ is empty. Assume the contrary, and let $\mathfrak{c}$ be a maximal member of $\mathfrak{B}$. Then considering $\phi_{\mathfrak{c}}$, we may assume that $\mathfrak{B}$ consists only of $\{0\}$. Since $\mathfrak{a} \notin \mathfrak{B}$, $\mathfrak{a} \neq 0$. Let d be a non-zero element of $\mathfrak{a}$. Since $\phi_{dR}(IR[X]_S) = \phi_{dR}(\sum f_i R[X]_S)$, we see that for an arbitrary element h of I, there is an element s of S such that $sh \in \sum f_i R[X] + dR[X]$, i.e., $sh = f' + dg$ with $f' \in \sum f_i R[X]$ and $g \in R[X]$. Then $dg \in I$, and $d(g \text{ modulo } I) = 0$. Therefore $(g \text{ modulo } I) \in (0 : dR)A$ (by the flatness). This means

5) Note the following obvious fact: Let $g_1, \cdots, g_n$ be elements of a polynomial is $K[X]$ over a field K and let K' be an extension field of K. Then $g_1, \cdots, g_n$ form a regular sequence in $K[X]$ if and only if they do in $K'[X]$.

that there is an element g' of $(0:dR)[X]$ which represents (g modulo I). That is, $g-g'\in I$ and $g'\in(0:dR)[X]$. Then $dg=d(g-g')\in dI$. Thus we have $IR[X]_S\subseteq\sum f_iR[X]_S+dIR[X]_S$. Since d is in the Jacobson radical of $R[X]_S$, we have the required equality. Thus $\mathfrak{B}$ must be empty, and our proof is completed.

Corollary 4.1. *Under the assumptions* (1)$\sim$(4) *in Theorem* 4, *if* $\mathfrak{a}$ *is nilpotent, then* $\sum f_iR[X]=I$.

Corollary 4.2. *Under the assumptions* (1)$\sim$(4) *in Theorem* 4, *if R is a local ring with maximal ideal* $\mathfrak{m}$ *and if the radical of* $\sum f_iR[X]$ *contains* $\mathfrak{m}$, *then* $\sum f_iR[X]=I$.

At the rest of the present article, we consider the case where X consists only of one element x. In the case, if $A=R[x]/I$ is R-flat, then I is "nearly" principal as we can state as follows:

Corollary 4.3. *Assume that* (1) *R is a* (*noetherian*) *local ring with maximal ideal* $\mathfrak{m}$, (2) $X=\{x\}$ *and* (3) $A=R[x]/I$ *is R-flat. Then*:

(i) *There is an element f of I such that, for a suitable element* $s\in R[x]$ *such that* $\phi_{\mathfrak{m}}s=1$, $sI\subseteq fR[x]$.

(ii) *If M is a maximal ideal of $R[x]$ containing I, then* $IR[x]_M$ *is principal.*

(iii) *If I contains a monic polynomial f, such that $\phi_{\mathfrak{m}}f$ generates $\phi_{\mathfrak{m}}I$, then* $I=fR[X]$.

(iv)[6] *Consider the radical* $\sqrt{0}$. *If* $R/\sqrt{0}$ *is normal, then I is principal.*

Proof. Except for (iv), the assertions follows from Theorem 4 and Corollary 4.2. As for (iv), by virtue of Corollary 4.1, we may assume that R is normal. In this case, if $s\in R[X]$ and if $\phi_{\mathfrak{m}}s=1$, then s is a product of prime elements (for, if $s=a_0x^n+\cdots+a_{n-1}x+1$, then factorization of s corresponds to factorization of the monic poly-

6) The writer owes the main part of this result to Professor Paul Monsky.

nomial $x^n+a_{n-1}x^{n-1}+\cdots+a_0$). Therefore we see that I is principal by (i). This completes the proof.

We add two examples. Example 1 shows that in (iv) it is important that R is local.[7] Example 2 shows that normality is important in (iv).

Example 1. Let D be a Dedekind domain with ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that i) there are non-zero elements c and d such that $c\mathfrak{a}=d\mathfrak{b}$ and ii) $\mathfrak{a}+\mathfrak{b}=D$. Then the ideal I of $D[x]$ generated by $\left\{ax+\frac{ca}{d}\,\middle|\,a\in\mathfrak{a}\right\}$ is not principal while $A=D[x]/I$ is R-flat.

Proof. That I is not pincipal is obvious. Flatness of A follows from Theorem 1 applied to $D_{\mathfrak{m}}$ for an arbitrary maximal ideal $\mathfrak{m}$.

Example 2. Let K be a field and let z be a transcendental element over K. Set $R=K[z^2, z^3]_P$ with maximal ideal P generated by z^2 and z^3. Let ψ be the homomorphism $R[x]\to R[1/z]$ such that $\psi f(x)=f(1/z)$. Then the kernel I of ψ is not principal, while $R[1/z]$ is R-flat.

Proof is easy observing that $R[1/z]$ is the field of quotients of R.

5. Supplementary remarks on regular sequences.

We give at first a remark that what we really proved at Lemma 3.1 is the following fact:[8]

Proposition 5.1. *Let $\mathfrak{a}$ be an ideal of R and let $f_1, \cdots, f_n$ be elements of R. If $\phi_{\mathfrak{a}}f_1, \cdots, \phi_{\mathfrak{a}}f_n$ form a regular sequence in $\phi_{\mathfrak{a}}R$, then $\mathfrak{a}\cap\sum f_iR=\mathfrak{a}(\sum f_iR)$.*

The following fact is obvious because of our definition of regularity:

Proposition 5.2. *If $f_1, \cdots, f_n$ form a regular sequence in R, then they do the same in any over-ring which is a flat R-module.*

7) A generalization to semi-local case is easy.

8) This and Lemma 3.1 are equivalent to each other.

Now we observe relationship between regularity of $f_1, \cdots, f_n$ and that of $\phi_{\mathfrak{a}} f_1, \cdots, \phi_{\mathfrak{a}} f_n$ in some sense.

Remark 5.3. Any one of regularity of $f_1, \cdots, f_n$ in R and regularity of $\phi_{\mathfrak{a}} f_1, \cdots, \phi_{\mathfrak{a}} f_n$ in $\phi_{\mathfrak{a}} R$ does not imply the other.

This is shown easily by examples.

Observe Lemma 3.2 as a result of contrary direction to this remark. We are to add some more remarks of similar direction.

Proposition 5.4. *Assume that R is the direct sum of subrings $R_1, \cdots, R_s$ with identities $e_1, \cdots, e_s$ respectively. Then a sequence $f_1, \cdots, f_n$ is a regular sequence in R if and only if $e_\alpha f_1, \cdots, e_\alpha f_n$ form a regular sequence in $R_\alpha = e_\alpha R$ for every $\alpha = 1, \cdots, s$.*

Proof. Assume that $f_1, \cdots, f_n$ form a regular sequence. If he_α is an element of $(\sum_{i<t} f_i e_\alpha R_\alpha) : f_t e_\alpha$, then he_α is in $(\sum_{i<t} f_i R) : f_t = \sum_{i<t} f_i R$. Thus $he_\alpha \in (\sum_{i<t} f_i R) \cap R_\alpha = \sum_{i<t} f_i e_\alpha R_\alpha$ and we see that $f_1 e_\alpha, \cdots, f_n e_\alpha$ from a regular sequence in R_α for every $\alpha = 1, \cdots, s$. Conversely, assume that $f_1 e_\alpha, \cdots, f_n e_\alpha$ form a regular sequence in R_α for every α. Consider an arbitrary element h of $\sum_{i<t} f_i R : f_t$. $h = \sum_\alpha he_\alpha$ and obviously he_α is in $(\sum_{i<t} f_i e_\alpha R_\alpha) : f_t e_\alpha$ which is equal to $\sum_{i<t} f_i e_\alpha R_\alpha$. Therefore $h \in \sum_\alpha (\sum_{i<t} f_i e_\alpha R_\alpha) = \sum_{i<t} f_i R$. This completes our proof.

Proposition 5.5. *Assume that R is noetherian and that $\mathfrak{a} (\neq R)$ is an ideal whose radical is the intersection of a finite number of maximal ideal, say $\mathfrak{m}_1, \cdots, \mathfrak{m}_s$. Let $f_1, \cdots, f_n$ be elements of $R[X]$. Then the following three conditions are equivalent to each other.*

(1) *$\phi_{\mathfrak{a}} f_1, \cdots, \phi_{\mathfrak{a}} f_n$ form a regular requence in $\phi_{\mathfrak{a}} R[X]$.*

(2) *$\phi_{\mathfrak{m}_\alpha} f_1, \cdots, \phi_{\mathfrak{m}_\alpha} f_n$ form a regular requence in $\phi_{\mathfrak{m}_\alpha} R[X]$ for every $\alpha = 1, \cdots, s$.*

(3) *For any ideal $\mathfrak{b} (\neq R)$ which contans a power of $\bigcap_\alpha \mathfrak{m}_\alpha$, $\phi_{\mathfrak{b}} f_1, \cdots, \phi_{\mathfrak{b}} f_n$ form a regular requence in $\phi_{\mathfrak{b}} R[X]$.*

Proof. By virtue of Lemma 3.2, we have only to show that (1) implies (2). For the purpose, considering $R/\mathfrak{a}$ instead of R, we may assume that $\mathfrak{a}=0$. Then R is an Artin ring, whence by virtue of of Proposition 5.4, we may assume that R is an Artin local ring with maximal ideal $\mathfrak{m}=\mathfrak{m}_1$. Reduction to the case where X is a finite set can be done easily. Set $T_t=\sum_{i<t} f_i R[X]$. $\phi_{\mathfrak{m}} T_t=\phi_{\mathfrak{m}} R[X]$ if and only if $T_t=R[X]$, and therefore we have only to observe the case where $T_n \neq R[X]$. Thus, that $f_1, \cdots, f_n$ form a regular requence implies that height $T_t=t$ for every $t=1, 2, \cdots, n$. Therefore height $\phi_{\mathfrak{m}} T_t=t$ for every t. Since $\phi_{\mathfrak{m}} R[X]$ is a polynomial ring over a field in a finite number of variables, $\phi_{\mathfrak{m}} R[X]$ is a Macaulay ring, and therefore we have that T_t is unmixed for every t. Thus $\phi_{\mathfrak{m}} f_1, \cdots, \phi_{\mathfrak{m}} f_n$ form a regular requence. This completes the proof of Proposition 5.5.

Proposition 5.6. *Let $\mathfrak{m}$ be a maximal ideal of R and let $f_1, \cdots, f_n$ be elements of $R[X]$. Set $S=\{f \in R[X] \mid \phi_{\mathfrak{m}} f=1)\}$. If $\phi_{\mathfrak{m}} f_1, \cdots, \phi_{\mathfrak{m}} f_n$ form a regular sequence in $\phi_{\mathfrak{m}} R[X]$ and if $\mathfrak{p}$ is a prime ideal contained in $\mathfrak{m}$, then $\phi_{\mathfrak{p}} f_1, \cdots, \phi_{\mathfrak{p}} f_n$ form a regular sequence in $\phi_{\mathfrak{p}} R_{\mathfrak{p}}[X]_S$.*

Proof. We can reduce easily to the case where R is a ring of quotients of a finitely generated ring. Thus we may assume that R is a (noetherian) local ring. We may assume also that $\mathfrak{p}=0$, and that X is a finite set. Set $T_t=\sum_{i<t} f_i R[X]$. Therefore we consider the case where $T_n R[X]_S \neq R[x]_S$. That $\phi_{\mathfrak{m}} f_1, \cdots, \phi_{\mathfrak{m}} f_n$ form a regular sequence implies that height $\phi_{\mathfrak{m}} T_t=t$ for every t. This implies that height $T_t R_{\mathfrak{p}}[X]_S \geq t$.[9] Since T_t is generated by t elements and since $R_{\mathfrak{p}}[X]_S$ is (locally) Macaulay ring, we see that $T_t R_{\mathfrak{p}}[X]_S$ is unmixed and therefore $f_1, \cdots, f_n$ from a regular sequence in $R_{\mathfrak{p}}[X]_S$. Thus the proof of Proposition 5.6 is completed.

Department of Mathematics
Kyoto University

---

9) See Theorem 1 in Nagata, *Finitely generated rings over a valuation ring*, J. Math. Kyoto Univ. vol. 5 no. 2 (1966), pp. 163-169.