J. Math. Kyoto Univ.
9-3 (1969) 339-361

On mixed problems for hyperbolic systems of first order with constant coefficients

By

Takashi SADAMATSU

(Received April 15, 1969)

§1. Introduction

The mixed problem for linear hyperbolic equations with constant coefficients in a quarter space has been treated by S. Agmon [1], R. Hersh [2] and L. Sarason [7]. S. Agmon treated single higher order equations and R. Hersh and L. Sarason the first order systems.

In the present paper we consider the mixed problem for hyperbolic systems of first order with principal part having constant coefficients:

$$(1.1) \qquad \begin{cases} \dfrac{\partial}{\partial t}u+A\dfrac{\partial}{\partial x}u+\sum_{j=1}^{n}B_j\dfrac{\partial}{\partial y_j}u=f(t;\ x,y) \\ u(0;\ x,y)=0 \\ Pu(t;\ 0,y)=0 \end{cases}$$

in a quarter space $\{(t,x,y);\ t>0,\ x>0,\ y\in R^n\}$, where u is a N-vector, A, B_j $(j=1,2,\cdots,n)$ are $N\times N$-constant matrices and A is non-singular, and P is $m\times N$-constant matrix and its rank m. Already L. Sarason [7] also gave a priori estimates for the system (1.1), our approach is slightly different from Sarason's one. Moreover we treat the problem (1.1) under less stringent conditions in some sense.

Our argument is based on Wiener-Hopf's method. Taking Laplace transformation in t and Fourier transformation in y, the ploblem (1.1) becomes to the problem of a system of ordinary difierential equations depending on parameters with a homogeneous boundary condition:

$$(1.2)\qquad \begin{cases} \left(A\dfrac{d}{dx}+\tau I+i\eta B\right)\hat{u}(\tau;\ x,\eta)=\hat{f}(\tau;\ x,\eta) \\ P\hat{u}(\tau;\ 0,\eta)=0 \end{cases}$$

where $\hat{u}(\tau;\ x,\eta)$ denotes the Fourier-Laplace image of $u(t;\ x,y)$. Using a compensating function $\hat{g}(\tau;\ x,\eta)$ which will be constructed later and setting $u=u_1+v$, we transform the problem (1.2) to the non-homogeneous equation:

$$(1.3)\qquad \left(A\frac{d}{dx}+\tau I+i\eta B\right)\hat{u}_1(\tau;\ x,\eta)=\hat{f}(\tau;\ x,\eta)+\hat{g}(\tau;\ x,\eta)\quad \text{in } x\in R^1$$

and to the homogeneous equation with non-hemogemeous boundary condition:

$$(1.4)\qquad \begin{cases} \left(A\dfrac{d}{dx}+\tau I+i\eta B\right)\hat{v}(\tau;\ x,\eta)=0 \qquad \text{in } x>0 \\ P\hat{v}(\tau;\ 0,\eta)=-P\hat{u}_1(\tau;\ 0,\eta). \end{cases}$$

In §3 we treate the problem (1.3) and $P\hat{u}_1(\tau;\ 0,\eta)$. Preparing some lemmas, we determine a compensating function $\hat{g}(\tau;\ x,\eta)$ such that $|P\hat{u}_1(\tau;\ 0,\eta)|$ ($\hat{u}_1(\tau;\ x,\eta)$ is the solution of the problem (1.3)) is bounded by L^2-norm of $\hat{f}(\tau;\ x,\eta)$. We construct a base of the subspace $E^+(\tau,\eta)$ of $\boldsymbol{C}^N$, solve the boundary value problem (1.4) and estimate the solution in L^2-sence in §4. In appendix we consider a Puiseux expansion in a neighbourhood of a point such that the characteristic equation has a real double root and characterize a property of hyperbolic systems in a generalized sense. It is shown that the ratios of the imaginary parts of two roots which approach a real double root are bounded below and above. This inequality is used for estimating $P\hat{u}_1(\tau;\ 0,\eta)$ in §3.

The author is deeply indebted to Professor S. Mizohata [5] who suggested the problem and contributed much invaluable encourage ment and advice. He wishes also to thank Professors M. Yamaguti and M. Matsumura for helpful conversations.

§2. Assumptions and result

We consider the mixed problem

$$(1.1)\qquad \begin{cases} L[u]=\dfrac{\partial}{\partial t}u+A\dfrac{\partial}{\partial x}u+\sum_{j=1}^{n}B_j\dfrac{\partial}{\partial y_j}u=f(t;\,x,y) \\ u(0;\,x,y)=0 \\ Pu(t;\,0,y)=0 \end{cases}$$

in a quarter space $\{(t;\,x,y);\ t>0,\ x>0,\ y\in R^n\}$.

We suppose the following:

Condition I

The operator L is hyperbolic in a generalized sense, that is, i) the matrix $\xi A+\sum_{j=1}^{n}\eta_j B_j$ (we write briefly $\xi A+\eta B$) has only real eigen values for any real $(\xi,\eta)\neq(0,0)$, ii) the matrix $\xi A+\eta B$ is diagonalizable and the multiplicity of any root of $\det(\tau I+i\xi A+i\eta B)=0$ in τ is invariant for any real $(\xi,\eta)\neq(0,0)$ where $\det A$ denotes the determinant of a matrix A, that is,

$$(2.1)\qquad \det(\tau I+i\xi A+i\eta B)=\prod_{j=1}^{s}(\tau-i\lambda_j(\xi,\eta))^{p_j}$$

for any real (ξ,η) where $\lambda_i(\xi,\eta)$ $(i=1,2,\cdots,s)$ are real and distinct for any real $(\xi,\eta)\neq(0,0)$, p_i $(i=1,\cdots,s)$ do not depent on ξ, η and $p_1+p_2+\cdots+p_s=N$.

Condition II

For any real η and any pure imaginary $\tau(=i\gamma;\ \gamma$: real) the real roots of $\det(\gamma I+\xi A+\eta B)=0$ with respect to ξ are at most double for any real $(\gamma,\eta)\neq(0,0)$.

Remark 1. Let $\tau=\tau^0=i\gamma^0(\gamma^0$: real), $\eta=\eta^0$ and ξ^0 be a real double root of $\det(\tau^0 I+i\xi A+i\eta^0 B)=0$. Then this means that $\dfrac{\partial}{\partial\xi}\lambda_i(\xi^0,\eta^0)=0$ and $\dfrac{\partial^2}{\partial\xi^2}\lambda_i(\xi^0,\eta^0)\neq0$ $(i=1,2,\cdots,s)$ and a real simple root means $\dfrac{\partial}{\partial\xi}\lambda_i(\xi^0,\eta^0)\neq0$. As shown in Appendix (see lemma 7.), the rank of $\tau I+i\xi A+i\eta B$ is $N-p_1$ in a small neighbourhood of $(\tau,\xi,\eta)=(i\lambda_1(\xi^0,\eta^0),\xi^0,\eta^0)$, when (τ,ξ,η) satisfies $\det(\tau I+i\xi A+i\eta B)=0$.

Let ξ_j^0 $(j=1,2,\cdots,q+s+s')$ be all real roots of $\det(\tau I+i\xi A+i\eta B)=0$ in ξ for pure imaginary $\tau=\tau^0(=i\gamma^0,\gamma^0;$ real) and real η^0, where ξ_j^0 $(j=1,2,\cdots,q)$ are real double roots and ξ_j^0 $(j=q+1,\cdots,q+s+s')$ are real simple roots. If ξ_1^0 is a real double root of $\det(\tau I+i\xi A+i\eta B)=0$ for $\tau=\lambda_1(\xi_1^0,\eta^0)$ and $\eta=\eta^0$, ξ_1^0 is a real root of order $2p_1$ of $\det(\tau I+i\xi A+i\eta B)=0$ as mentioned in Remark 1. Let V be a small neighbourhood of $(i\lambda_1(\xi_1^0,\eta^0),\eta^0)$ and let us consider the problem (1.2) in $V\cap\{\mathrm{Re}\,\tau>0\}$ after this section.

Since A is non-singular, the problem (1.4) can be written by the form:

$$(2.2)\qquad \begin{cases} \dfrac{d}{dx}\hat{v}(\tau;\, x,\eta)+M(\tau,\eta)\hat{v}(\tau,x,\eta)=0 \quad \text{in } x>0 \\ P\hat{v}(\tau;\, 0,\eta)=-P\hat{u}_1(\tau;\, 0,\eta) \end{cases}$$

where $M(\tau,\eta)=A^{-1}(\tau I+i\eta B)$. Let $E^+(\tau,\eta)$ and $E^-(\tau,\eta)$ be the subspaces of $\boldsymbol{C}^N$ generated by the ordinary and the generalized eigen vectors corresponding to the roots in ξ of $\det(i\xi I+M(\tau,\eta))=0$ with positive and negative imaginary parts respectively when $\mathrm{Re}\,\tau>0$. In order to solve the problem (2.2) in $L^2(R_+^1)$, we construct a system of vectors $\{h_j^+(\tau,\eta)\}_{j=1,\cdots,m}$ locally which is a base of $E^+(\tau,\eta)$ $(\mathrm{Re}\,\tau>0)$, and the vectors $h_1^+(\tau,\eta),\cdots,h_m^+(\tau,\eta)$ are linearly independent, continuous and homogeneous of degree 0 in τ and η $(\mathrm{Re}\,\tau\geqslant 0)$. This fact is shown in §4. Finally we suppose

Condition III

The absolute value of Lopatinski determinant is uniformly bounded away from 0 in $|\tau|^2+|\eta|^2=1$ $(\mathrm{Re}\,\tau\geqslant 0)$, that is, there exists a some positive constant δ such that

$$|\det P\mathcal{H}(\tau,\eta)|\geqslant\delta \qquad \text{for } |\tau|^2+|\eta|^2=1,\ \mathrm{Re}\,\tau\geqslant 0$$

holds. Where $\mathcal{H}(\tau,\eta)$ is a $N\times m$-matrix $(h_1^+(\tau,\eta),\cdots,h_m^+(\tau,\eta))$.

Our theorem is the following

Theorem *Under the Conditions I, II and III, we have the inequality*

$$\|\hat{u}(\tau;\ x,\eta)\|_{L^2(R^1_+)} \leqslant \frac{\text{const.}}{\operatorname{Re}\tau} \|\hat{f}(\tau;\ x,\eta)\|_{L^2(R^1_+)}$$

for any solution $\hat{u}(\tau;\ x,\eta)$ *of the problem* (1.2) *where the constant does not depend on* τ *and* η.

§3. Wiener-Hopf's method

In (1.1) we take Laplace transformation in t and Fourier transformation in y and denotes the Fourier-Laplace image of $u(t;\ x,y)$ by $\hat{u}(\tau;\ x,\eta)$, then the problem (1.1) becomes to

$$(3.1)\quad \begin{cases} A\dfrac{d}{dx}\hat{u}(\tau;\ x,\eta)+(\tau I+i\eta B)\hat{u}(\tau;\ x,\eta)=\hat{f}(\tau;\ x,\eta) & \text{in } x>0, \\ P\hat{u}(\tau;\ 0,\eta)=0. \end{cases}$$

From now on we treat the problem (3.1) in stead of (1.1).

Lemma 1. (Hersh [2]) *The roots of* $\det(\tau I+i\xi A+i\eta B)=0$ *with respect to* ξ *are never real for any* $\operatorname{Re}\tau>0$ *and real* η.

This lemma follows easily from Condition I.

Remark 2. This shows that the numbers of the roots in ξ of $\det(\tau I+i\xi A+i\eta B)=0$ with positive and negative imaginary part do not change for any τ ($\operatorname{Re}\tau>0$) and real η. The number m of rows in the boundary matrix P is equal to that of the roots with positive imaginary part.

Using the notation;

$$(3.2)\qquad \mathcal{A}\left(\frac{d}{dx};\ \tau,\eta\right)=A\frac{d}{dx}+\tau I+i\eta B$$

the problem (3.1) is written in the form

$$(3.3)\qquad \begin{cases} \mathcal{A}\left(\dfrac{d}{dx};\ \tau,\eta\right)\hat{u}(\tau;\ x,\eta)=\hat{f}(\tau;\ x,\eta) & \text{in } x>0 \\ P\hat{u}(\tau;\ 0,\eta)=0. \end{cases}$$

In order to estimate the solution of this problem (3.3), we use a compensating function $\hat{g}(\tau;\ x,\eta)\in L^2(R^1)$ with a support in R^1_- which is constructed later, where τ and η are parameters. Let us consider the ordinary differential equation

$$(3.4)\qquad \mathcal{A}\left(\frac{d}{dx};\ \tau,\eta\right)\hat{u}_1(\tau;\ x,\eta)=\hat{f}(\tau;\ x,\eta)+\hat{g}(\tau;\ x,\eta)\quad \text{in } R^1_x.$$

The solution in $L^2(R^1)$ of the equation (3.4) can be represented by

$$(3.5)\qquad \hat{u}_1(\tau;\ x,\eta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ix\xi}\mathcal{A}(i\xi;\ \tau,\eta)^{-1}\{\tilde{\hat{f}}(\tau;\ \xi,\eta)+\tilde{\hat{g}}(\tau;\ \xi,\eta)\}d\xi$$

where $\tilde{\hat{f}}(\tau;\ \xi,\eta)$ denotes Fourier image of $\hat{f}(\tau;\ x,\eta)$ in x. And

$$(3.6)\qquad P\hat{u}_1(\tau;\ 0,\eta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}P\mathcal{A}(i\xi;\ \tau,\eta)^{-1}\{\tilde{\hat{f}}(\tau;\ \xi,\eta)+\tilde{\hat{g}}(\tau;\ \xi,\eta)\}d\xi.$$

Lemma 2. *Under the Condition I, the inequality*

$$(3.7)\qquad |\mathcal{A}(i\xi;\ \tau,\eta)^{-1}|\leq\frac{\text{const.}}{\operatorname{Re}\tau}$$

holds for $\operatorname{Re}\tau>0$, *where the constant does not depend on* ξ, τ *and* η.

Proof. Let $\tau=\sigma+i\gamma$ (σ,γ; real) and by the Condition I there exists a non-singular matrix $Q(\xi,\eta)$ such that

$$Q(\xi,\eta)\mathcal{A}(i\xi;\ \tau,\eta)Q^{-1}(\xi,\eta)=\sigma I+i(\gamma I+Q(\xi,\eta)(\xi A+\eta B)Q^{-1}(\xi,\eta))$$
$$=\begin{pmatrix}\sigma+i(\gamma-\lambda_1(\xi,\eta)) & & & 0\\ & \ddots & & \\ & & \sigma+i(\gamma-\lambda_1(\xi,\eta)) & \\ & & & \ddots \\ 0 & & & \sigma+i(\gamma-\lambda_s(\xi,\eta))\end{pmatrix}.$$

As

$$Q(\xi,\eta)\mathcal{A}(i\xi;\ \tau,\eta)^{-1}Q^{-1}(\xi,\eta)=(Q(\xi,\eta)\mathcal{A}(i\xi;\ \tau,\eta)Q^{-1}(\xi,\eta))^{-1}$$
$$=\begin{pmatrix}\{\sigma+i(\gamma-\lambda_1(\xi,\eta))\}^{-1} & & & 0\\ & \ddots & & \\ & & \{\sigma+i(\gamma-\lambda_1(\xi,\eta))\}^{-1} & \\ & & & \ddots \\ 0 & & & \{\sigma+i(\gamma-\lambda_s(\xi,\eta))\}^{-1}\end{pmatrix}.$$

Hence

$$|\mathcal{A}(i\xi;\ \tau,\eta)^{-1}|\leqslant\delta|Q(\xi,\eta)\mathcal{A}(i\xi;\ \tau,\eta)^{-1}Q^{-1}(\xi,\eta)|$$
$$\leqslant\delta\{\inf_{j,\gamma,\xi,\eta}|\sigma+i(\gamma-\lambda_j(\xi,\eta))|\}^{-1}\leqslant\frac{\text{const.}}{\sigma}=\frac{\text{const.}}{\operatorname{Re}\tau}.$$

This completes the proof.

Using this lemma and (3.5) we can estimate $\hat{u}_1(\tau;\ x,\eta)$;

$$\|\hat{u}_1(\tau;\ x,\eta)\|^2_{L^2(R^1_x)}=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathcal{A}(i\xi;\ \tau,\eta)^{-1}\{\widetilde{\hat{f}}(\tau;\ \xi,\eta)+\widetilde{\hat{g}}(\tau;\ \xi,\eta)\}|^2 d\xi$$
$$\leqslant\left(\frac{\text{const.}}{\operatorname{Re}\tau}\right)^2\|\hat{f}(\tau;\ x,\eta)+\hat{g}(\tau;\ x,\eta)\|^2_{L^2(R^1_x)}.$$

Thus we have the following

Proposition 1. *If we suppose the Condition I and let* $\hat{u}_1(\tau;\ x,\eta)$ *be the solution of* (3.4), *then the inequality*

$$\|\hat{u}_1(\tau;\ x,\eta)\|_{L^2(R^1_x)}\leqslant\frac{\text{const.}}{\operatorname{Re}\tau}\|\hat{f}(\tau;\ x,\eta)+\hat{g}(\tau;\ x,\eta)\|_{L^2(R^1_x)} \tag{3.8}$$

holds, where the constant does not depend ond τ *an* η.

A construction of $\hat{g}(\tau;\ x,\eta)$

In order to estimate $|P\hat{u}_1(\tau;\ 0,\eta)|$ by $\|\hat{f}(\tau;\ x,\eta)\|_{L^2(R^1_x)}$ only we construct a compensating function $\hat{g}(\tau;\ x,\eta)$. For simplicity let us put $\tilde{f}(\xi)=\widetilde{\hat{f}}(\tau;\ \xi,\eta)$ and $\tilde{g}(\xi)=\widetilde{\hat{g}}(\tau;\ \xi,\eta)$, then

$$P\hat{u}_1(\tau;\ 0,\eta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}P\mathcal{A}(i\xi;\ \tau,\eta)^{-1}\{\tilde{f}(\xi)+\tilde{g}(\xi)\}d\xi \tag{3.9}$$

and change the variables (τ,ξ,η) to (τ',ξ',η') where $(\tau',\xi',\eta')=\frac{1}{c}(\tau,\xi,\eta)$ and $c=(|\tau|^2+|\eta|^2)^{1/2}$, and write

$$P\hat{u}_1(\tau;\ 0,\eta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}P\mathcal{A}(i\xi';\ \tau',\eta')^{-1}\{\tilde{f}(c\xi')+\tilde{g}(c\xi')\}d\xi'$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}P\mathcal{A}(i\xi;\ \tau',\eta')^{-1}\{\tilde{f}(c\xi)+\tilde{g}(c\xi)\}d\xi.$$

Here we decompose $\det\mathcal{A}(i\xi;\ \tau',\eta')$ into factors:

$$\det\mathcal{A}(i\xi;\ \tau',\eta')=i^N\det A\cdot A^+(\xi;\ \tau',\eta')A^-(\xi;\ \tau',\eta') \tag{3.10}$$

$$A^+(\xi;\ \tau',\eta')=\prod_{j=1}^{m}(\xi-\xi_j^+(\tau',\eta')) \tag{3.11}$$

$$A^-(\xi;\ \tau',\eta')=\prod_{j=1}^{N-m}(\xi-\xi_j^-(\tau',\eta')) \tag{3.12}$$

where $\xi_j^+(\tau',\eta')$ and $\xi_j^-(\tau',\eta')$ are the roots of $\det \mathcal{A}(i\xi;\tau',\eta')=0$ in ξ with positive and negative imaginary part respectively ($\xi_1^+=\xi_2^+=\cdots\xi_{p_1}^+$, $\xi_{p_1+1}^+=\cdots=\xi_{p_1+p_2}^+,\cdots$). We define the matrix $\mathcal{P}(\xi;\tau',\eta')$ by

$$(3.13)\qquad \frac{\mathcal{P}(\xi;\tau',\eta')}{A^+(\xi;\tau',\eta')A^-(\xi;\tau',\eta')}=P\mathcal{A}(i\xi;\tau',\eta')^{-1}$$

and each component of $\mathcal{P}(\xi;\tau',\eta')$ has degree at most $N-1$ as a polynomial in ξ. As explained in Remark 1 if $A^+(\xi;\tau',\eta')$ and $A^-(\xi,\tau',\eta')$ has zeros of degree p_1 at $\xi=\xi_1^+(\tau',\eta')$ and $\xi=\xi_1^-(\tau',\eta')$ respectively, that is,

$$A^+(\xi;\tau',\eta')=(\xi-\xi_1^+(\tau',\eta'))^{p_1}(\xi-\xi_{p_1+1}^+(\tau',\eta'))\cdots(\xi-\xi_m^+(\tau',\eta'))$$

and

$$A^-(\xi;\tau',\eta')=(\xi-\xi_1^-(\tau',\eta'))^{p_1}(\xi-\xi_{p_1+1}^-(\tau',\eta'))\cdots(\xi-\xi_{N-m}^-(\tau',\eta')),$$

then each component of $\mathcal{P}(\xi;\tau',\eta')$ has zeros of degree p_1-1 at $\xi=\xi_1^+(\tau',\eta')$ and $\xi_1^-(\tau',\eta')$ when (τ',η') lies in $V'\cap\{\mathrm{Re}\,\tau'>0\}$ and $V'=\frac{1}{c}V$ is small, where $\xi_1^+(\tau',\eta')$ and $\xi_1^-(\tau',\eta')$ approach a real double root $\xi_1^0(i\gamma^{0\prime},\eta^{0\prime})$ as (τ',η') tends to $(i\gamma^{0\prime},\eta^{0\prime})$. Hence we can rewrite

$$(3.14)\qquad \frac{\mathcal{P}(\xi;\tau',\eta')}{A^+(\xi;\tau',\eta')A^-(\xi;\tau',\eta')}=\frac{\mathcal{P}_0(\xi;\tau',\eta')}{A_0^+(\xi;\tau',\eta')A_0^-(\xi;\tau',\eta')}$$

where $A_0^+(\xi;\tau',\eta')=\prod_{j=1}^{m'}(\xi-\xi_j^+(\tau',\eta'))$, $A_0^-(\xi;\tau',\eta')=\prod_{j=1}^{m''}(\xi-\xi_j^-(\tau',\eta'))$. Here we changed the notation in the following way: we denotes $\xi_1^+=\cdots=\xi_{p_1}^+$ simply by ξ_1^+, $\xi_{p_1+1}^+=\cdots=\xi_{p_1+p_2}^+$ by $\xi_2^+,\cdots$, and $\xi_1^+,\xi_2^+,\cdots,\xi_p^+$ are all distinct roots of $\det\mathcal{A}(i\xi,\tau',\eta')=0$ with positive imaginary part.

Thus we can decompose

$$(3.15)\qquad P\mathcal{A}(i\xi;\tau',\eta')^{-1}=\frac{\mathcal{P}_0^+(\xi;\tau',\eta')}{A_0^+(\xi;\tau',\eta')}+\frac{\mathcal{P}_0^-(\xi;\tau',\eta')}{A_0^-(\xi;\tau',\eta')}$$

where $\frac{\mathcal{P}_0^+(\xi;\tau',\eta')}{A_0^+(\xi;\tau',\eta')}$ is holomorphic in $\mathrm{Im}[\xi]<0$ and its inverse

Fourier image has a support in $x \geqslant 0$ and $\frac{\mathcal{P}_0^-(\xi;\tau',\eta')}{A_0^-(\xi;\tau',\eta')}$ is homorphic in $\mathrm{Im}[\xi]>0$ and its inverse Fourier has a support in $x \leqslant 0$.

Hence

$$(3.16)\quad P\hat{u}_1(\tau;0,\eta)$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{\mathcal{P}_0^+(\xi;\tau',\eta')}{A_0^+(\xi;\tau',\eta')}+\frac{\mathcal{P}_0^-(\xi;\tau',\xi')}{A_0^-(\xi;\tau',\eta')}\right\}\{\tilde{f}(c\xi)+\tilde{g}(c\xi)\}d\xi$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{\mathcal{P}_0^+(\xi;\tau',\eta')}{A_0^+(\xi;\tau',\eta')}\tilde{g}(c\xi)+\frac{\mathcal{P}_0^-(\xi;\tau',\eta')}{A_0^-(\xi;\tau',\eta')}\tilde{f}(c\xi)\right\}d\xi.$$

Concerning $\frac{\mathcal{P}_0^+(\xi;\tau',\eta')}{A_0^+(\xi;\tau',\eta')}$ and $\frac{\mathcal{P}_0^-(\xi;\tau',\eta')}{A_0^-(\xi;\tau',\eta')}$ the following decompositions hold:

$$(3.17)\quad \frac{\mathcal{P}_0^+(\xi;\tau',\eta')}{A_0^+(\xi;\tau',\eta')}=\sum_{j=1}^{q+s}\frac{c_j^+(\tau',\eta')\mathcal{P}_0(\xi_j^+;\tau',\eta')}{\xi-\xi_j^+}+R^+(\xi;\tau',\eta')$$

$$(3.18)\quad \frac{\mathcal{P}_0^-(\xi;\tau',\eta')}{A_0^-(\xi;\tau',\eta')}=\sum_{j=1}^{q+s'}\frac{c_j^-(\tau',\eta')\mathcal{P}_0(\xi_j^-;\tau',\eta')}{\xi-\xi_j^-}+R^-(\xi;\tau',\eta')$$

where $\xi_j^{\pm}(\tau',\eta')$ denote the roots which approach the real double root $\xi_j^0(i\gamma^{0'},\eta^{0'})$ $(j=1,2,\cdots,q)$ and $\xi_j^+(\tau',\eta')$ $(j=q+1,\cdots,q+s=p)$ and $\xi_j^-(\tau',\eta')$ $(j=q+1,\cdots,q+s')$ denote the roots which approach the real simple roots $\xi_j^0(i\gamma^{0'},\eta^{0'})$ and $\xi_{j+s}^0(i\gamma^{0'},\eta^{0'})$ respectively when (τ',η') tends to $(i\gamma^{0'},\eta^{0'})$.

Remark 3. One of the roots which approach a real double root $\xi_j^0(i\gamma^{0'},\eta^{0'})$ has a positive imaginary part and the other has a negative imaginary part in $V'\cap\{\mathrm{Re}\,\tau'>0\}$ where $V'=\frac{1}{c}V$ and V is a sufficiently small neighbourhood of $(i\gamma^0,\eta^0)$. This will be shown in Appendix. We have denoted the roots with a positive and a negative imaginary part by $\xi_j^+(\tau',\eta')$ and $\xi_j^-(\tau',\eta')$ respectively.

We prepare some lemmas in order to estimate $P\hat{u}_1(\tau;0,\eta)$. The following lemma 3 is concerned about the above decompositions (3.17) and (3.18).

Lemma 3. *Under the Condition II, we have*

i) $|c_j^{\pm}(\tau', \eta')| = 0\left(\frac{1}{\xi_j^+ - \xi_j^-}\right)$ *for* $j=1, 2, \cdots, q$

ii) $\left|\frac{c_j^-(\tau', \eta')}{c_j^+(\tau', \eta')}\right| \leqslant \text{const.}$ *for* $j=1, 2, \cdots, q$

iii) $|c_j^+(\tau', \eta')| \leqslant \text{const.}$ *for* $j=q+1, \cdots, q+s$

$|c_j^-(\tau', \eta')| \leqslant \text{const.}$ *for* $j=q+1, \cdots, q+s'$

iv) $|R^{\pm}(\xi; \tau', \eta')| \leqslant \frac{\text{const.}}{1+|\xi|}$ *for real* ξ

for any (τ', η') *in* $V' \cap \{\mathrm{Re}\,\tau' > 0\}$, *where constants do not depend on* (τ', η').

Proof. Multiplying $A_0^+(\xi; \tau', \eta')A_0^-(\xi; \tau', \eta')$ to (3. 14) and using (3. 15), (3. 17) and (3. 18), (3. 14) becomes to

$$(3.19)\quad \begin{cases} \mathcal{P}_0(\xi; \tau', \eta') \\ \quad = \sum_{j=1}^{q+s} A_0^+(\xi; \tau', \eta')A_0^-(\xi; \tau', \eta') \frac{c_j^+(\tau', \eta')\mathcal{P}_0(\xi_j^+; \tau', \eta')}{\xi - \xi_j^+} \\ \quad + \sum_{j=1}^{q+s'} A_0^+(\xi; \tau', \eta')A_0^-(\xi; \tau', \eta') \frac{c_j^-(\tau', \eta')\mathcal{P}_0(\xi^-_j; \tau', \eta')}{\xi - \xi_j^-} \\ \quad + A_0^+(\xi; \tau', \eta')A_0^-(\xi; \tau', \eta')\{R^+(\xi; \tau', \eta') + R^-(\xi; \tau', \eta')\} \end{cases}$$

and let us set $\xi = \xi_j^+$, then

$$(3.20)\quad \mathcal{P}_0(\xi_i^+; \tau', \eta') = \mathcal{P}_0(\xi_i^+) = \prod_{k \neq i}(\xi_i^+ - \xi_k^+)A_0^-(\xi_i^+; \tau', \eta')c_i^+(\tau', \eta')\mathcal{P}_0(\xi_i^+).$$

Hence

$$(3.21)\quad c_i^+(\tau', \eta') = \frac{1}{\prod_{k \neq i}(\xi_i^+ - \xi_k^+)\prod_k(\xi_i^+ - \xi_k^-)}.$$

This shows that $c_i^+(\tau', \eta')$ depends only on $A_0^+(\xi; \tau', \eta')A_0^-(\xi; \tau', \eta')$. i), ii) and iii) of lemma 3 are obvious by (3. 21). Next we show iv) of lemma 3. $\xi_1^+(\tau', \eta'), \cdots, \xi_{q+s}^+(\tau', \eta')$ are simple roots and approach real roots $\xi_1^0(i\gamma^{0'}, \eta^{0'}), \cdots, (\xi_{q+s}^0(i\gamma^{0'}, \eta^{0'})$ respectively when (τ', η') tends to $(i\gamma^{0'}, \eta^{0'})$ and other roots move in a fixed closed curve c_0' which has a strictly positive distance from the real axis. Here we use a lemma to show iv) of lemma 3.

Lemma *Let* $(\xi_1^+, \cdots, \xi_{q+s}^+)$ *and* $(\xi_{q+s+1}^+, \cdots, \xi_{m'}^+)$ *be variables and move in compact sets whose distance is positive and in addition let*

$$(3.22)\quad \mathcal{P}_{0ij}(\xi) = a_{ij}\prod_{k=1}^{r}(\xi-\zeta_k) \quad (a_{ij}\neq 0,\ r \text{ depends on } i \text{ and } j;\ r<m')$$

where $\mathcal{P}_{0ij}(\xi)$ *denotes the* (i,j)*-component of* $\mathcal{P}_0(\xi;\tau',\eta')$*. Define*

$$(3.23)\qquad A_1(\xi) = \prod_{k=1}^{q+s}(\xi-\xi_k^+),\quad A_2(\xi) = \prod_{k=q+s+1}^{m'}(\xi-\xi_k^+)$$

and set

$$(3.24)\qquad \frac{\mathcal{P}_{0ij}^+(\xi)}{A_0^+(\xi)} = \frac{\mathcal{P}_{0ij}^+(\xi;\tau',\eta')}{A_0^+(\xi;\tau',\eta')} = \frac{\mathcal{P}_{0ij}^{(1)}(\xi)}{A_1(\xi)} + \frac{\mathcal{P}_{0ij}^{(2)}(\xi)}{A_2(\xi)}$$

where the degree of $A_k(\xi)$ *in* ξ *is higher than that of* $\mathcal{P}_{0ij}^{(k)}(\xi)$ $(k=1,2)$*. Then the coefficients of* $\mathcal{P}_{0ij}^{(k)}(\xi)$ $(k=1,2)$ *are continuous functions of* (ξ_i^+, ζ_j) $(i=1,2,\cdots,m',\ j=1,2,\cdots,r)$*.*

Proof. Write $\mathcal{P}_{0ij}^+(\xi)$ by $\mathcal{P}_0^+(\xi)$ briefly. The proof is due to the induction with respect to $(q+s,\ m'-q-s)$. We assume the lemma is correct for (k_0, m_0), then the lemma is also correct for $(k_0+1,\ m_0)$. In fact let us set $A_1(\xi) = (\xi-\xi_{k_0+1}^+)A_{k_0}(\xi)$ and $\mathcal{P}_0(\xi) = (\xi-\xi_{k_0+1}^+)Q(\xi)+R(\xi)$, then

$$(3.25)\qquad \begin{aligned} \frac{\mathcal{P}_0^+(\xi)}{A_0^+(\xi)} &= \frac{(\xi-\xi_{k_0+1}^+)Q(\xi)+R(\xi)}{(\xi-\xi_{k_0+1}^+)A_{k_0}(\xi)A_2(\xi)} \\ &= \frac{Q(\xi)}{A_{k_0}(\xi)A_2(\xi)} + \frac{R(\xi)}{(\xi-\xi_{k_0+1}^+)A_{k0}(\xi)A_2(\xi)} \\ &= \frac{Q_1(\xi)}{A_{k_0}(\xi)} + \frac{Q_2(\xi)}{A_2(\xi)} + \frac{R(\xi)}{(\xi-\xi_{k_0+1}^+)}\left\{\frac{\widetilde{Q}_1(\xi)}{A_{k_0}(\xi)} + \frac{\widetilde{Q}_2(\xi)}{A_2(\xi)}\right\} \end{aligned}$$

where the coefficients of $Q_1(\xi)$, $Q_2(\xi)$, $\widetilde{Q}_1(\xi)$ and $\widetilde{Q}_2(\xi)$ are continuous in $\xi_1^+, \cdots, \xi_{k_0}^+, \zeta_1, \cdots, \zeta_r$ by the assumption of the induction. The last term of (3.25) is written by

$$\frac{R(\xi)}{\xi-\xi_{k_0+1}^+}\cdot\frac{\widetilde{Q}_2(\xi)}{A_2(\xi)} = R(\xi)\left\{\frac{Q}{\xi-\xi_{k_0+1}^+} + \frac{\approx{Q}_2(\xi)}{A_2(\xi)}\right\}$$

and we obtain

$$Q = \frac{\widetilde{Q}_2(\xi_{k_0+1}^+)}{A_2(\xi_{k_0+1}^+)} \quad\text{and}\quad \approx{Q}_2(\xi) = -\frac{QA_2(\xi)-\widetilde{Q}_2(\xi)}{\xi-\xi_{k_0+1}^+},$$

then $QA_2(\xi)-\widetilde{Q}_2(\xi)$ is devided by $\xi-\xi^+_{k_0+1}$. Hence the coefficients of $\mathcal{P}^{(k)}_0(\xi)$ $(k=1,2)$ are continuous in (ξ^+_i, ζ_j). Similary the lemma is correct for (k_0, m_0+1). Thus the lemma is proved.

Now let us return to the proof of lemma 3. We set

$$R^+(\xi)=\frac{\widetilde{R}(\xi)}{\prod\limits_{i=q+s+1}^{m'}(\xi-\xi^+_j)}.$$

Then the coefficients of $\widetilde{R}(\xi)$ are continuous in $\xi^+_1, \cdots, \xi^+_m, \zeta_1, \cdots, \zeta_r$ by the above lemma. When (τ', η') varies in $\overline{V}'\cap\{\mathrm{Re}\,\tau'\geqslant 0\}$, $\xi^+_1(\tau', \eta')$, $\cdots, \zeta_r(\tau', \eta')$ move in a bounded set and the coefficients of $\widetilde{R}(\xi)$ are bounded, and further $\xi^+_{q+s+1}(\tau', \eta'), \cdots, \xi^+_{m'}(\tau', \eta')$ do not approach the real axis for (τ', η') in $\overline{V}'\cap\{\mathrm{Re}\,\tau'\geqslant 0\}$. Therefore let ξ be restricted in the real axis we have

$$|R^+(\xi;\tau', \eta')|\leqslant\frac{\text{const.}}{1+|\xi|}. \tag{3.26}$$

This completes the proof of lemma 3.

Lemma 4. *Let α and β be not real, then the inequality*

$$\int_{-\infty}^{\infty}\frac{1}{\xi-\alpha}\cdot\overline{\frac{1}{\xi-\beta}}\,d\xi=\begin{cases}2\pi i\dfrac{1}{\alpha-\bar{\beta}} & \textit{for } \mathrm{Im}[\alpha]>0,\ \mathrm{Im}[\beta]>0\\ -2\pi i\dfrac{1}{\alpha-\bar{\beta}} & \textit{for } \mathrm{Im}[\alpha]<0,\ \mathrm{Im}[\beta]<0\\ 0 & \textit{for } \mathrm{Im}[\alpha]\cdot\mathrm{Im}[\beta]<0\end{cases} \tag{3.27}$$

holds. In particular

$$\int_{-\infty}^{\infty}\left|\frac{1}{\xi-\alpha}\right|^2 d\xi=\frac{\pi}{\mathrm{Im}[\alpha]}\,\mathrm{sgn}(\mathrm{Im}[\alpha]) \tag{3.28}$$

holds when $\alpha=\beta$.

We omit this proof.

Lemma 5. *Under the Condition I, we have*

$$|\mathrm{Im}\,\xi(\tau', \eta')|\geqslant\text{const. }\mathrm{Re}\,\tau' \tag{3.29}$$

where $\xi(\tau', \eta')$ is a root of $\det A(i\xi;\tau', \eta')=0$ in ξ and the constant does not depend on τ' and η'.

This lemma dues to P. D. Lax (see, P. D. Lax [3]).

Lemma 6. *If we assume the Condition I and II, then*

$$\left|\frac{\operatorname{Im}[\xi_j^+(\tau',\eta')]}{\operatorname{Im}[\xi_j^-(\tau',\eta')]}\right| \leqq \text{const.} \qquad (j=1,2,\cdots,q) \tag{3.30}$$

hold for $(\tau',\eta')\in V'\cap\{\operatorname{Re}\tau'>0\}$ *where the constants do not depend on* (τ',η').

This lemma is proved in the appendix.

Again we treat $P\hat{u}_1(\tau;0,\eta)$. By the decompositions (3.17) and (3.18)

$$P\hat{u}_1(\tau;0,\eta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\sum_{j=1}^{q+s}\frac{c_j^+(\tau',\eta')\mathcal{P}_0(\xi_j^+)}{\xi-\xi_j^+}+R^+(\xi;\tau',\eta')\right\}\tilde{g}(c\xi)d\xi$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\sum_{j=1}^{q+s'}\frac{c_j^-(\tau',\eta')\mathcal{P}_0(\xi_j^-)}{\xi-\xi_j^-}+R^-(\xi;\tau',\eta')\right\}\tilde{f}(c\xi)d\xi$$

$$(3.31)\quad\left\{\begin{aligned}&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{q}\left\{\frac{c_j^+(\tau',\eta')\mathcal{P}_0(\xi_j^+)}{\xi-\xi_j^+}\tilde{g}(c\xi)+\frac{c_j^-(\tau',\eta')\mathcal{P}_0(\xi_j^+)}{\xi-\xi_j^-}\tilde{f}(c\xi)\right\}d\xi\\&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{q}\frac{c_j^-(\tau',\eta')\{\mathcal{P}_0(\xi_j^-)-\mathcal{P}_0(\xi_j^+)\}}{\xi-\xi_j^-}\tilde{f}(c\xi)d\xi\\&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\sum_{j=q+1}^{q+s}\frac{c_j^+(\tau',\eta')\mathcal{P}_0(\xi_j^+)}{\xi+\xi_j^+}+R^+(\xi;\tau',\eta')\right\}\tilde{g}(c\xi)d\xi\\&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\sum_{j=q+1}^{q+s'}\frac{c_j^-(\tau',\eta')\mathcal{P}_0(\xi_j^-)}{\xi-\xi_j^-}+R^-(\xi;\tau',\eta')\right\}\tilde{f}(c\xi)d\xi\end{aligned}\right.$$

Using above lemmas, we shall estimate (3.31). By the Schwarz inequality

$$(3.32)\quad |\text{the 3-rd term of (3.31)}|\leqq\text{const.}\left\{\sum_{j=q+1}^{q+s}\left(\int_{-\infty}^{\infty}\left|\frac{1}{\xi-\xi_j^+}\right|^2 d\xi\right)^{1/2}\right.$$
$$\left.+\left(\int_{-\infty}^{\infty}\frac{1}{(1+|\xi|)^2}d\xi\right)^{1/2}\right\}\left(\int_{-\infty}^{\infty}|\tilde{g}(c\xi)|^2d\xi\right)^{1/2}$$
$$\leqq\text{const.}\left\{\sum_{j=q+1}^{q+s}\frac{1}{\sqrt{\operatorname{Im}[\xi_j^+]}}+1\right\}\left(\int_{-\infty}^{\infty}|\tilde{g}(c\xi)|^2d\xi\right)^{1/2}$$
$$\leqq\frac{\text{const.}}{\sqrt{\operatorname{Re}\tau'}}\left(\int_{-\infty}^{\infty}|\tilde{g}(c\xi)|^2d\xi\right)^{1/2}$$

similary we have

(3.33) $$|\text{the 4-th term of (3.31)}| \leqq \frac{\text{const.}}{\sqrt{\operatorname{Re}\tau'}}\left(\int_{-\infty}^{\infty}|\tilde{f}(c\xi)|^2 d\xi\right)^{1/2}$$

(3.34) $$|\text{the 2-nd term of (3.31)}| \leqq \frac{\text{const.}}{\sqrt{\operatorname{Re}\tau'}}\left(\int_{-\infty}^{\infty}|\tilde{f}(c\xi)|^2 d\xi\right)^{1/2}$$

concerning the 1-st term of (3.31), let us choose a compensating function $\tilde{g}(c\xi)$ such that

(3.35) $$\sum_{j=1}^{q}\int_{-\infty}^{\infty}\left\{\frac{c_j^+(\tau',\eta')\mathcal{P}_0(\xi_j^+)}{\xi-\xi_j^+}\tilde{g}(c\xi)+\frac{c_j^-(\tau',\eta')\mathcal{P}_0(\xi_j^+)}{\xi-\xi_j^-}\tilde{f}(c\xi)\right\}d\xi=0$$

holds and that the following properties:

1°) $$\int_{-\infty}^{\infty}|\tilde{g}(c\xi)|^2 d\xi \leqq \text{const.}\int_{-\infty}^{\infty}|\tilde{f}(c\xi)|^2 d\xi$$

2°) the support of $\hat{g}(\tau, x, \eta)$ is contained in R_-^1

hold. The condition (3.35) is called "reflection condition".

We search $\tilde{g}(c\xi)$ under the form

$$G\cdot{}^t\left(\frac{1}{\xi-\bar{\xi}_1^+}, \frac{1}{\xi-\bar{\xi}_2^+}, \cdots, \frac{1}{\xi-\bar{\xi}_q^+}\right)$$

where G is an unknown $N\times q$-matrix having (j, k)-element g_{jk} $(j=1, 2, \cdots, N;\ k=1, 2, \cdots, q)$ and ^{t} denotes transposed, and let $I_{i,k}$ be

(3.36) $$I_{i,k}=\int_{-\infty}^{\infty}\frac{1}{\xi-\xi_i^+}\cdot\frac{1}{\xi-\bar{\xi}_k^+}d\xi \qquad (i, k=1, 2, \cdots, q).$$

Then the reflection condition (3.35) is written in the form

(3.37) $$\sum_{i=1}^{q}\mathcal{P}_0(\xi_i^+)\left\{c_i^+(\tau',\eta')G\cdot{}^t(I_{i1}, \cdots, I_{iq})\right.$$
$$\left.+c_i^-(\tau',\eta')\int_{-\infty}^{\infty}\frac{1}{\xi-\xi_i^-}\tilde{f}(c\xi)d\xi\right\}=0.$$

Now let us put

(3.38) $$c_i^+(\tau',\eta')G\cdot{}^t(I_{i1}, \cdots, I_{iq})+c_i^-(\tau',\eta')\int_{-\infty}^{\infty}\frac{1}{\xi-\xi_i^-}\tilde{f}(c\xi)d\xi=0$$
$$(i=1, 2, \cdots, q)$$

and from this formula (3.38), we can determine G.

In fact the k-th components of (3.38) are written by

$$(3.39)\quad \begin{cases} I_{11}g_{k1}+I_{12}g_{k2}+\cdots+I_{1q}g_{kq}=-\dfrac{c_1^-(\tau',\eta')}{c_1^+(\tau',\eta')}\displaystyle\int_{-\infty}^{\infty}\frac{1}{\xi-\xi_1^-}\tilde{f}_k(c\xi)d\xi \\ I_{21}g_{k1}+I_{12}g_{k2}+\cdots+I_{2q}g_{kq}=-\dfrac{c_2^-(\tau',\eta')}{c_2^+(\tau',\eta')}\displaystyle\int_{-\infty}^{\infty}\frac{1}{\xi-\xi_2^-}\tilde{f}_k(c\xi)d\xi \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ I_{q1}g_{k1}+I_{q2}g_{k2}+\cdots+I_{qq}g_{kq}=-\dfrac{c_q^-(\tau',\eta')}{c_q^+(\tau',\eta')}\displaystyle\int_{-\infty}^{\infty}\frac{1}{\xi-\xi_q^-}\tilde{f}_k(c\xi)d\xi \end{cases}$$

$(k=1, 2, \cdots, N)$ where $\tilde{f}_k(c\xi)$ denotes the k-th component of $\tilde{f}(c\xi)$. Let D be the determinant $|I_{ij}|_{1\leq i,j\leq q}$ formed by the coefficients of (3.39), then we get by the Cramer formula

$$(3.40)\quad g_{ki}=\frac{1}{D}\begin{vmatrix} I_{11},\cdots,I_{1\,i-1}, -\dfrac{c_1^-}{c_1^+}\displaystyle\int_{-\infty}^{\infty}\frac{1}{\xi-\xi_1^-}\tilde{f}_k(c\xi)d\xi, I_{1\,i+1},\cdots,I_{1q} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ I_{q1},\cdots,I_{q\,i-1}, -\dfrac{c_q^+}{c_q^-}\displaystyle\int_{-\infty}^{\infty}\frac{1}{\xi-\xi_q^-}\tilde{f}_k(c\xi)d\xi, I_{q\,i+1},\cdots,I_{qq}\end{vmatrix}$$

$$(k=1, 2, \cdots, N;\ i=1, 2, \cdots, q).$$

In order two prove the property 1°), we must estimate g_{ki}. If V' is small, then the inequality $|D|\geqslant\frac{1}{2}I_{11}\cdot I_{22}\cdots I_{qq}$ holds because I_{ii} becomes large as we want and I_{ik} $(i\neq k)$ is bounded by the definition of I_{ik} and lemma 4. Using the Laplace expansion, the Schwarz inequality and lemma 3 ii), we have

$$(3.41)\quad |g_{ki}|^2\leqq\text{const.}\left\{\frac{(\text{Im}[\xi_1^+])^2(\text{Im}[\xi_i^+])^2}{|\text{Im}[\xi_i^-]|}+\frac{(\text{Im}[\xi_2^+])^2(\text{Im}[\xi_i^+])^2}{|\text{Im}[\xi_i^-]|}\right.$$
$$\left.+\cdots+\frac{(\text{Im}[\xi_i^+])^2}{|\text{Im}[\xi_i^-]|}+\cdots+\frac{(\text{Im}[\xi_q^+])^2(\text{Im}[\xi_i^+])^2}{|\text{Im}[\xi_q^-]|}\right\}\int_{-\infty}^{\infty}|\tilde{f}_k(c\xi)|^2d\xi$$
$$(k=1, 2, \cdots, N, i=1, 2, \cdots, q).$$

On the other hand by the definition

$$(3.42)\qquad \tilde{g}_k(c\xi)=\sum_{i=1}^{q}\frac{g_{ki}}{\xi-\bar{\xi}_i^+}\qquad (k=1, 2, \cdots, N),$$

and we have by (3.41) and lemma 6

$$(3.43)\qquad \left(\int_{-\infty}^{\infty}|\tilde{g}_k(c\xi)|^2d\xi\right)^{1/2}\leqq\text{const.}\left(\int_{-\infty}^{\infty}|\tilde{f}_k(c\xi)|^2d\xi\right)^{1/2}$$
$$(k=1, 2, \cdots, N)$$

This proves the property 1°) and the construction of $\widetilde{g}(c\xi)$ shows the property 2°).

Thus we have the following

Proposition 2. *If we suppose the Condition I and II, then the inequality*

$$(3.44) \qquad |P\hat{u}_1(\tau;\ 0, \eta)| \leqq \frac{\text{const.}}{\sqrt{\text{Re}\,\tau}} \left(\int_{-\infty}^{\infty} |\widetilde{f}(\xi)|^2 d\xi \right)^{1/2}$$

holds for $(\tau, \eta) \in V \cap \{\text{Re}\,\tau > 0\}$ *where the constant does not depend on* τ *and* η.

§4. Lopatinski determinant

In this section we consider the boundary value problems of the ordinary differential equations depending on parameters τ and η ($\text{Re}\,\tau > 0$, $\eta \in R^n$). As shown in §2, $\hat{v}(\tau;\ x, \eta) = \hat{u}(\tau;\ x, \eta) - \hat{u}_1(\tau;\ x, \eta)$ satisfies

$$(4.1) \qquad \begin{cases} \dfrac{d}{dx}\hat{v}(\tau;\ x, \eta) + M(\tau, \eta)\hat{v}(\tau;\ x, \eta) = 0 \\ P\hat{v}(\tau;\ 0, \eta) = -P\hat{u}_1(\tau;\ 0, \eta) \end{cases}$$

where $M(\tau, \eta) = A^{-1}(\tau I + i\eta B)$. Let $E^+(\tau, \eta)$ and $E^-(\tau, \eta)$ be the root spaces corresponding to the roots in ξ of $\det(i\xi I + M(\tau, \eta)) = 0$ with positive and negative imaginary parts respectively. $E^+(\tau, \eta)$ and $E^-(\tau, \eta)$ are named by the positive and negative root spaces respectively. In order to search for the solution of the problem (4.1) in $L^2(R^1_+)$, we construct a base of $E^+(\tau, \eta)$. Let (τ, η) be in $V \cap \{\text{Re}\,\tau > 0\}$ and $\xi_j^+(\tau, \eta)$ be the root mentioned above ($j = 1, 2 \cdots, q+s$). Then we can choose eigen vectors $h_1^+(\tau, \eta), \cdots, h_p^+(\tau, \eta)$ of $-M(\tau, \eta)$ corresponding to $\xi_1^+(\tau, \eta), \cdots, \xi_{q+s}^+(\tau, \eta)$, where p denotes the number of the roots including the multiplicity which approach the real axis when (τ, η) tends to $(i\gamma^0, \eta^0)$. In fact in the identity

$$(4.2) \qquad (\tau I + i\xi A + i\eta B) \cdot {}^t\text{cof}(\tau I + i\xi A + i\eta B) = \det(\tau I + i\xi A + i\eta B) \cdot I$$

where cof A denotes the matrix which consists of the cofactors of a matrix A, we substitute ξ by a root $\xi_j^+(\tau,\eta)$. If a root $\xi_j^+(\tau,\eta)$ is a p_1-tuple root, there exist p_1 eigen-vectors of $-M(\tau,\eta)$ corresponding to a root $\xi_j^+(\tau,\eta)$ by lemma 7 shown in the appendix. These vectors are continuous in $(\tau,\eta)\in V\cap\{\mathrm{Re}\,\tau\geqslant 0\}$ and homogeneous in τ and η. Normalizing these eigen vectors and setting them by $h_1^+(\tau,\eta),\cdots,h_p^+(\tau,\eta)$, $\{h_1^+(\tau,\eta),\cdots,h_p^+(\tau,\eta)\}$ are linearly independent, defined on $V'\cap\{\mathrm{Re}\,\tau\geqslant 0\}$, continuous and homogeneous of degree 0 in τ and η.

On the other hand the other root vectors of $-M(\tau,\eta)$ corresponding to the roots which lie a set away from the real axis can be constructed as following: let $(\tau,\eta)=(\tau_1,\eta_1)$ be fixed $(\mathrm{Re}\,\tau_1\geqslant 0)$, we choose a base $(h_{p+1}^+(\tau_1,\eta_1),\cdots,h_m^+(\tau_1,\eta_1))$ of the subspace generated by all root vectors of $-M(\tau_1,\eta_1)$ corresponding the above roots. Define

$$h_j^+(\tau,\eta)=\frac{1}{2\pi}\oint_c(i\xi I+M(\tau,\eta))^{-1}h_j^+(\tau_1,\eta_1)d\xi \qquad (4.3)$$

$$(j=p+1,\cdots,m)$$

where (τ,η) is near (τ_1,η_1) and c is a simple closed curve containing only the roots away from the real axis. As $M(\tau,\eta)$ is continuous in τ and η, these $h_j^+(\tau,\eta)$ $(j=p+1,\cdots,m)$ vary continuously in τ and η. By the construction (4.3), $h_j^+(\tau,\eta)$ is defined in $V\cap\{\mathrm{Re}\,\tau\geqslant 0\}$ and homogeneous of degree zero in τ and η. Thus we can obtain a base $\{h_1^+(\tau,\eta),\cdots,h_m^+(\tau,\eta)\}$ of $E^+(\tau,\eta)$ in $V\cap\{\mathrm{Re}\,\tau>0\}$ which is linearly independent, continuous and homogeous of degree zero with respect to τ and η in $V\cap\{\mathrm{Re}\,\tau\geqslant 0\}$. (see, S. Mizohata [6] and M. Matsumura [4])

We estimate the solution $\hat{v}(\tau;x,\eta)$ in $L^2(R_+^1)$ of the problem (4.1). As $\hat{v}(\tau;0,\eta)$ should be in $E^+(\tau,\eta)$, $\hat{v}(\tau;0,\eta)$ can be written in the form

$$\hat{v}(\tau;0,\eta)=c_1h_1^+(\tau,\eta)+c_2h_2^+(\tau,\eta)+\cdots+c_mh_m^+(\tau,\eta) \qquad (4.4)$$

and by the boundary condition

$$\begin{aligned} P\hat{v}(\tau;0,\eta)&=c_1Ph_1^+(\tau',\eta')+c_2Ph_2^+(\tau',\eta')+\cdots+c_mPh_m^+(\tau',\eta') \\ &=-P\hat{u}_1(\tau;0,\eta). \end{aligned} \qquad (4.5)$$

Using the Condition III and the Cramer formula

$$(4.6)\qquad |c_i(\tau', \eta')| \leqq \text{const.} |P\hat{u}_1(\tau; 0, \eta)|.$$

The solution $\hat{v}(\tau; x, \eta)$ in $L^2(R^1_+)$ of the problem (4.1) is

$$(4.7)\qquad \hat{v}(\tau; x, \eta) = \sum_{j=1}^{q+s} \frac{1}{2\pi} \oint_{c_j} e^{i\xi x}(i\xi I + M(\tau, \eta))^{-1} \hat{v}(\tau; 0, \eta) d\xi$$
$$+ \frac{1}{2\pi} \oint_{c_0} e^{i\xi x}(i\xi I + M(\tau, \eta))^{-1} \hat{v}(\tau; 0, \eta) d\xi$$

where c_j is a small circle with center ξ_j^+ $(j=1, 2, \cdots, q+s=p)$ and c_0 is a simple closed curve containing only ξ_j^+ $(j=p+1, \cdots, m)$. The L^2-norm of the first term of (4.7) is

$$(4.8)\qquad \int_0^\infty |e^{i\xi_j^+(\tau,\eta)x} \hat{v}(\tau, 0, \eta)|^2 dx$$
$$\leqq \text{const.} |P\hat{u}_1(\tau; 0, \eta)|^2 \int_0^\infty e^{-2\operatorname{Im}\xi_j^+(\tau,\eta)x} dx$$
$$\leqq \frac{\text{const.}}{\operatorname{Im}[\xi_j^+(\tau, \eta)]} |P\hat{u}_1(\tau; 0, \eta)|^2 \leqq \frac{\text{const.}}{\operatorname{Re}\tau} |P\hat{u}_1(\tau; 0, \eta)|^2$$

and that of the second term of (4.7) is

$$(4.9)\qquad \int_0^\infty \left| \oint_{c_0} e^{i\xi x}(i\xi I + M(\tau, \eta))^{-1} \hat{v}(\tau; 0, \eta) d\xi \right|^2 dx$$
$$= \int_0^\infty \left| \oint_{c_0'} e^{ic\xi x}(i\xi I + M(\tau', \eta'))^{-1} \hat{v}(c\tau'; 0, c\eta') d\xi \right|^2 dx$$
$$\leqq \frac{\text{const.}}{c \operatorname{Re}\tau'} |\hat{v}(c\tau'; 0, c\eta')|^2 \leqq \frac{\text{const.}}{\operatorname{Re}\tau} |P\hat{u}_1(\tau; 0, \eta)|^2$$

by lemma 5 and (3.6) where $c = \sqrt{|\tau|^2 + |\eta|^2}$. Consequently from proposition 2, we have

$$(4.10)\qquad \int_0^\infty |\hat{v}(\tau; x, \eta)|^2 dx \leqq \frac{\text{const.}}{(\operatorname{Re}\tau)^2} \int_{-\infty}^\infty |\tilde{f}(\xi)|^2 d\xi.$$

We have obtained this estimate (4.10) locally in τ and η. However we can also obtain a global estimate by using the homogeneity and the continuity of a base $\{h_1^+(\tau, \eta), \cdots, h_m^+(\tau, \eta)\}$ and a partition of unity in $|\tau|^2 + |\eta|^2 = 1$ $(\operatorname{Re}\tau \geqq 0)$.

By proposition 1, (4.10) and $\hat{u}(\tau; x, \eta) = \hat{u}_1(\tau; x, \eta) + \hat{v}(\tau; x, \eta)$

we have the following;

Theorem *Let $\hat{u}(\tau;\ x, \eta)$ be a solution of* (3.1) *and the Conditions I, II and III be supposed, then the inequality*

$$(4.11)\qquad \|\hat{u}(\tau;\ x,\eta)\|_{L^2(R^1_+)}\leqq\frac{\text{const.}}{\text{Re}\,\tau}\|\hat{f}(\tau;\ x,\eta)\|_{L^2(R^1_+)}$$

holds where the constant does not depend on τ and η. Also we have

$$(4.12)\qquad \|\hat{u}(\tau;\ x,y)\|_{L^2(R^{n+1}_+)}\leqq\frac{\text{const.}}{\text{Re}\,\tau}\|\hat{f}(\tau;\ x,y)\|_{L^2(R^{n+1}_+)}\,.$$

Remark 4. In the case that the Lopatinski determinant $\det P\mathcal{H}(\tau',\eta')$ tends to infinity when (τ',η') tends to $(i\gamma^{0'},\eta^{0'})$, the solution $c=(c_1,\cdots,c_m)$ of (4.5) is the following;

$$c_i(\tau',\eta')=\sum_{j=1}^{n}\frac{\Delta_{ji}(\tau',\eta')}{\det P\mathcal{H}(\tau',\eta')}\{-P\hat{u}_1(\tau;\ 0,\eta)\}_j\quad(i=1,\cdots,m)$$

where u_j denotes the j-component of a vector $u={}^t(u_1,\cdots,u_m)$. Let us suppose that

$$\left|\frac{\Delta_{ji}(\tau',\eta')}{\det P\mathcal{H}(\tau',\eta')}\right|\leqq\text{const.}\qquad(j=1,\cdots,m;\ i=1,\cdots,p)$$

$$\left|\frac{\Delta_{ji}(\tau',\eta')}{\det P\mathcal{H}(\tau',\eta')}\right|\leqq\frac{\text{const.}}{(\text{Re}\,\tau')^{1/2}}\qquad(j=1,\cdots,m;\ i=p+1,\cdots,m)$$

hold in $(\tau',\eta')\in V'\cap\text{Re}\{\tau'>0\}$ where the constants do not depend on (τ',η'). Since $\xi_j^+(\tau',\eta')$ $(j=p+1,\cdots,m)$ do not approach the real axis by the definition, there exists a positive constant a such that $\min_{p<j\leqq m}\text{Re}[\xi_j^+(\tau',\eta')]>a$ in $(\tau',\eta')\in V'\cap\{\text{Re}\,\tau'\geqq 0\}$. Therefore (4.9) follows immediately. In fact

$$\int_0^\infty\Big|\oint_c e^{i\xi x}(i\xi I+M(\tau,\eta))^{-1}\{c_{p+1}(\tau',\eta')h^+_{p+1}(\tau',\eta')+\cdots$$
$$+c_m(\tau',\eta')h^+_m(\tau';\ \eta')\}\,d\xi\Big|^2dx$$
$$=\int_0^\infty\Big|\oint_{c_0'} e^{i c\xi x}(i\xi I+M(\tau',\eta'))^{-1}\{c_{p+1}(\tau',\eta')h^+_{p+1}(\tau',\eta')+\cdots$$
$$+c_m(\tau',\eta')h^+_m(\tau',\eta')\,d\xi\Big|^2dx$$

$$\leqq \frac{\text{const.}}{ca\,\mathrm{Re}\,\tau'}\,|P\hat{u}_1(\tau;\,0,\eta)|^2 \leqq \frac{\text{const.}}{\mathrm{Re}\,\tau}\,|P\hat{u}_1(\tau;\,0,\eta)|^2.$$

Hence under the above assumption, the same inequalities (4.11) and (4.12) in the theorem hold as well.

Appendix

Puiseux Expansion

Proof of lemma 6

We recall the Condition I:

(A.1) $\det(\tau I + i\xi A + i\eta B) = \prod_{j=1}^{s}(\tau - i\lambda_j(\xi,\eta))^{p_j}$ for any real (ξ,η)

where $\lambda_j(\xi,\eta)$ $(j=1, 2, \cdots, s)$ are real and distinct for any real $(\xi,\eta) \neq (0,0)$, p_j $(j=1, 2, \cdots, s)$ do not depend on ξ, η and $p_1+p_2+\cdots+p_s=N$. Let us $\tau=i\gamma^0$ and $\eta=\eta^0$ $(\gamma^0, \eta^0$: real) and suppose that $M(i\gamma^0, \eta^0)$ admits a pure imaginary characteristic root $i\xi^0$. Then we can suppose $j=1$ without loss of generality. Let us set

(A.2) $\mu + i\nu = \tau - i\gamma^0 = i\lambda_1(\xi,\eta) - i\lambda_1(\xi^0,\eta^0)$ $(\mu, \nu$: real; $\mu>0)$

and expand (A.2) in a small neighbourhood of (ξ^0, η^0):

(A.3) $$\nu - i\mu = \lambda_1(\xi^0,\eta) - \lambda_1(\xi^0,\eta^0) + \frac{\partial}{\partial\xi}\lambda_1(\xi^0,\eta)(\xi-\xi^0) + \frac{1}{2!}\frac{\partial^2}{\partial\xi^2}\lambda_1(\xi^0,\eta)(\xi-\xi^0)^2 + \cdots$$

and write

(A.4) $\nu - i\mu = b_0(\eta) + b_1(\eta)(\xi-\xi^0) + b_2(\eta)(\xi-\xi^0)^2 + \cdots$

where $b_0(\eta) = \lambda_1(\xi^0,\eta) - \lambda_1(\xi^0,\eta^0)$, $b_k(\eta) = \frac{1}{k!}\frac{\partial^k}{\partial\xi^k}\lambda_1(\xi^0,\eta)$ $(k=1, 2, \cdots)$, $b_k(\eta)$ $(k=0, 1, \cdots)$ are all real valued and $b_1(\eta^0) \neq 0$ or $b_2(\eta^0) \neq 0$ by the Condition II (see §2).

We consider the cases that $b_1(\eta)$ vanishes or not.

1) The case that $b_1(\eta^0) \neq 0$.

Let $|\nu - i\mu|$ and $|\eta - \eta^0|$ be sufficiently small. Then $b_1(\eta)$ never vanishes and the solution with respect to $\xi - \xi^0$ of (A.4) is uniquely determined in $|\xi-\xi^0|<\delta_1$ for some small $\delta_1>0$. This ξ is a simple root in our sense of $\det(\tau I+i\xi A+i\eta B)=0$ when (τ,η) varies in a small neighbourhood of $(i\gamma^0,\eta^0)$.

2) The case that $b_1(\eta^0)=0$.

We consider first the case:

i) $b_1(\eta)\equiv 0$ in a small neighbourhood of η^0. (A. 4) becomes to

(A. 5) $$\nu-b_0(\eta)-i\mu=b_2(\eta)(\xi-\xi^0)^2+b_3(\eta)(\xi-\xi^0)^3+\cdots.$$

By $b_2(\eta^0)\neq 0$ and the analyticity of $\lambda_1(\xi,\eta)$ in ξ and η there exists ρ_1 and $\rho_2>0$ such that when $|\eta-\eta^0|<\rho_1$ and $|\nu-b_0(\eta)-i\mu|<\rho_1$, we have the development

(A. 6) $$\xi-\xi^0=z^{1/2}+c_2(\eta)z^{2/2}+c_3(\eta)z^{3/2}+\cdots \quad \text{for } |\xi-\xi^0|<\rho_2$$

where $z=(\nu-b_0(\eta)-i\mu)/b_2(\eta)$ and $c_i(\eta)$ $(i=1,2,\cdots)$ may be taken all real values. Define ζ^+ and ζ^- by the branches of $\sqrt{z}$ with positive and negative imaginary part respectively, then

(A. 7) $$\zeta^+=\frac{1}{2}\left\{\mp\sqrt{|z|-\operatorname{Im}z}-\sqrt{|z|+\operatorname{Im}z}\right\}+\frac{i}{2}\left\{\mp\sqrt{|z|-\operatorname{Im}z}+\sqrt{|z|+\operatorname{Im}z}\right\} \quad \text{for } \operatorname{Re}z\gtrless 0,$$

(A. 8) $$\zeta^-=\frac{1}{2}\left\{\pm\sqrt{|z|-\operatorname{Im}z}+\sqrt{|z|+\operatorname{Im}z}\right\}+\frac{i}{2}\left\{\pm\sqrt{|z|-\operatorname{Im}z}-\sqrt{|z|+\operatorname{Im}z}\right\} \quad \text{for } \operatorname{Re}z\gtrless 0.$$

We have

(A. 9) $$\delta_1|z|^{1/2}\geqslant|\operatorname{Im}\zeta^\pm|\geqslant\delta_2|z|^{1/2} \qquad \text{for } \operatorname{Re}z\leqslant 0$$

and

(A. 10) $$\delta_1'\mu|z|^{-1/2}\geqq|\operatorname{Im}\zeta^\pm|\geqq\delta_2'\mu|z|^{-1/2} \qquad \text{for } \operatorname{Re}z>0$$

where δ_1, δ_2, δ_1' and δ_2', are some positive constants. Let us define

(A. 11) $$\xi^\pm-\xi^0=\zeta^\pm+c_2(\eta)(\zeta^\pm)^2+c_3(\eta)(\zeta^\pm)^3+\cdots$$

where $\xi^\pm$ denote the branches such that $\operatorname{Im}[\xi]>0$ and $\operatorname{Im}[\xi]<0$ respectively for each of $\zeta^\pm$. Using (A. 10) and (A. 11) we have also for small $|z|$

(A. 12) $K_1|z|^{1/2} \geqq |\mathrm{Im}[\xi^{\pm}]| > K_2|z|^{1/2}$ for $\mathrm{Re}\, z \leqq 0$

and

(A. 13) $K_1'\mu|z|^{1/2} > |\mathrm{Im}[\xi^{\pm}]| > K_2'\mu|z|^{-1/2}$ for $\mathrm{Re}\, z > 0$

where K_1, K_2, K_1' and K_2' are some positive constants.

ii) $b_1(\eta) \neq 0$.

Since $\xi=\xi^0$ is a double root, that is, $b_2(\eta^0)=\dfrac{1}{2!}\dfrac{\partial^2}{\partial\xi^2}\lambda_1(\xi^0,\eta^0)\neq 0$, there exists a real analytic function $\xi=\xi^0(\eta)$ satisfying $\dfrac{\partial}{\partial\xi}\lambda_1(\xi^0(\eta),\eta)=0$ in a neighbourhood of (ξ^0,η^0). Let $|\eta-\eta^0|$ be small and regard $\lambda_1(\xi,\eta)$ as a function of ξ and we expand $\lambda_1(\xi,\eta)$ in a neighbourhood of $\xi=\xi^0(\eta)$, there holds

$$\text{(A. 14)} \quad \lambda_1(\xi,\eta)-\lambda_1(\xi^0(\eta),\eta)=\frac{\partial}{\partial\xi}\lambda_1(\xi^0(\eta),\eta)(\xi-\xi^0) + \frac{1}{2!}\frac{\partial^2}{\partial\xi^2}\lambda_1(\xi^0(\eta),\eta)(\xi-\xi^0)^2+\cdots.$$

Since $\dfrac{\partial}{\partial\xi}\lambda_1(\xi^0(\eta),\eta)=0$,

$$\text{(A. 14)}' \quad \frac{\tau}{i}-\lambda_1(\xi^0(\eta),\eta)=b_2(\eta)(\xi-\xi^0(\eta))^2+b_3(\eta)(\xi-\xi^0(\eta))^3+\cdots$$

holds, where $b_2(\eta^0)\neq 0$, $\xi^0(\eta^0)=\xi^0$ and $\lambda_1(\xi^0(\eta^0),\eta^0)=\lambda_1(\xi^0,\eta^0)=\gamma^0$. Put $\tau-i\gamma^0=\mu+i\nu$, (A. 14)′ becomes to

(A. 15) $b_0(\eta)=\lambda_1(\xi^0(\eta),\eta)-\gamma^0$ and $b_0(\eta^0)=0$.

This is the case treated before. Thus (A. 12) and (A. 13) show lemma 3.

Remark 5. The proof of this lemma shows that one of the two roots which approach a real double root when (τ,η) tends to $(i\gamma^0,\eta^0)$ has a positive imaginary part and the other has a negative imaginary part in $V\cap\{\mathrm{Re}\,\tau>0\}$.

Lemma 7. *If we suppose the Conditions I and II then the rank of* $\tau I+i\xi A+i\eta B$ *is* $N-p_1$ *in a small neighbourhood of* $(\tau,\xi,\eta)=(i\lambda_1(\xi^0,\eta^0),\xi^0,\eta^0)$ *when* (τ,ξ,η) *satisfies* $\det(\tau I+i\xi A+i\eta B)=0$.

Proof. From the proof of lemma 6,

(A. 16) $\xi^{\pm}=\xi^{0}(\eta)+z^{1/2}+c_{2}(\eta)z^{2/2}+c_{3}(\eta)z^{3/2}+\cdots$

where $z=\left(\frac{\tau}{i}-\lambda_{1}(\xi^{0}(\eta),\eta)+\lambda_{1}(\xi^{0},\eta^{0})\right)/b_{2}(\eta)$. Let $T(\tau;\ \xi,\eta)$ be a minor of the matrix $(\tau I+i\xi A+i\eta B)$ of order $N-p_{1}+1$, $T(\tau;\ \xi,\eta)$ is a homogeneous polynomial in τ, ξ and η and let η be fixed in a neighbourhood of η^{0} and substitute ξ in $T(\tau;\ \xi,\eta)$ by (A. 16), then we can regard $T(\tau;\ \xi,\eta)$ as a function of z, that is, $T(z)=T(\tau;\ \xi,\eta)$. $T(z)$ can be written by the puiseux expansion of z. In a segment where z is positive real, τ is pure imaginary and $\xi^{\pm}$ is also pure real. From the Condition I, $M(z)=0$ for some small non-negative real value z. Hence $M(z)=0$ for $\varepsilon_{0}>|z|\geqslant 0$ where ε_{0} is some positive constant. (q.e.d.)

Bibliography

[1] Agmon, S., Problêmes mixtes pour les équations hyperbolique d'ordre supérieur. Colloques internationaux du centre national de la recherche scientifique, No. 117, Paris, (1962) pp. 13-18.

[2] Hersh, R., Mixed problems in several variables. J. Math. and Mech. Vol. **12**, (1963) pp. 317-334.

[3] Lax, P. D., Differential equations, difference equations and matrix theory. Comm. Pure Appl. Math. Vol. **11**, (1958) pp. 175-194.

[4] Matsumura. M., Comportement des solutions de quelques problêmes mixtes pour certains systêmes hyperboliques symétriques à coefficients constants. Publ. RIMS, Kyoto Univ. Ser. A, **4**, (1968) pp. 309-359.

[5] Mizohata, S., On Agmon's results for hyperbolic equations. unpublished. (In Japanese).

[6] Mizohata, S., An Introduction to Integral Equations. (Book) Asakura pub. (1968).

[7] Sarason, L., On hyperbolic mixed problems. Arch. Rational Mech. Anal. Vol. **18**, (1965) pp. 311-334.

Kyoto University