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Modules of high order differentials of topological rings

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In the present paper, we study the structure of modules of high order differentials of topological rings, especially we show the formal projectivity of the module of q-th order differentials $\Omega_{A/k}^{(q)}$ of a formally smooth k-algebra A under the assumption that A is preadmissible and the $q+1$-th powers of all open ideals are open in A (see, Theorem 2).

In many cases, we follow A. Grothendieck's terminology in [2].

As preparation, we study the relationship between the formal projectivity of modules and the formal smoothness of their symmetric algebras in Proposition 1, Proposition 2 and Proposition 3. We see how the formal projectivity and the fact that topologies are induced ones, are preserved by formally bimorphic homomorphisms in Lemma 5. In proposition 4, we show that the above mentioned condition "the $q+1$-th powers of all open ideals are open", furnishes $\Omega_{A/k}^{(q)}$ with the induced topology. This is an extension of Grothendieck's Lemma (20.4.4) in [2].

In Theorem 1, we find the relationship between the module of q-th order differentials of a ring and that of its completion. The proof is natural.

The formal projectivity of $\Omega_{A/k}^{(1)}$ was shown by Grothendieck in [2], without setting any assumption on the topology of A. Also, it is not very difficult to show the formal projectivity of $\Omega_{A/k}^{(q)}$ in restricted forms as extensions of the author's results, Theorem 3 and Theorem 4 in [5], that is, simply speaking, to discuss our assertion in case of unramified regular local rings, if we use the similar technique as used in the proofs of cited theorems. However, the author met a difficulty to prove the formal projectivity of $\Omega_{A/k}^{(q)}$ without setting any topological assumption on A, and so far we have proved it in the form of Theorem 2. To do that, we use Grothendieck's Theorem (19.5.3) in [2] (see, Proposition 5), which has compelled us to set an assumption that A is preadmissible. Also, it is the reason why we discussed on the symmetric algebra at the beginning of this paper. And we will find that, in the technique used in our proof, the assumption on $\Omega_{A/k}^{(q)}$ having the induced topology is essential.

1. All rings are assumed to be commutative rings with identity, all topological rings and modules are assumed to have linear topology, that is, there exist fundamental systems of neighbourhoods of (0) formed of ideals or submodules, respectively.

Definition 1. (Grothendieck [2]). Let A be a topological ring and let M be a topological A-module. M is called a formally projective A-module, if the following condition is satisfied. For every open ideal I of A, every couple of discrete A/I-module P, Q, every surjective A-homomorphism $u: P \to Q$ and every continuous A-homomorphism $v: M \to Q$, there exists a continuous A-homomorphism $w: M \to P$ such that $v = u \circ w$.

Definition 2. (Grothendieck [2]). Let A be a topological ring and let B be a topological A-algebra. B is called a formally smooth A-algebra, if the following condition is satisfied. For every discrete A-algebra C and every nilpotent ideal I of C, every continuous A-homomorphism $u: B \to C/I$ is decomposed as $B \xrightarrow{v} C \xrightarrow{\varphi} C/I$, where v is a continuous A-homomorphism and φ is the canonical homomorphism.

In definition 2, it is enough to assume that $I^2 = (0)$.

Let C be a topological ring and let V be a topological C-module. Let $S_C^{\cdot}(V) = \bigoplus_{n \geq 0} S_C^n(V)$ be the symmetric algebra. The topology in $S_C^{\cdot}(V)$ having the universality, is defined by a set of ideals of the form $\mathfrak{A} S_C^{\cdot}(V) + U S_C^{\cdot}(V)$, where $\mathfrak{A}$ (resp. U) runs over a fundamental system of open ideals (resp. open submodules) of C (resp. V). If the given topology of V is less fine than that induced by the topology in C, it is enough to take the couple $(\mathfrak{A}, U)$ such that $\mathfrak{A}V \subset U$ and we have $\mathfrak{A}S_C^{\cdot}(V) + US_C^{\cdot}(V) = \mathfrak{A}1_C + US_C^{\cdot}(V)$. The following proposition may be known but is not stated explicitly in the literature.

Proposition 1. Let C be a topological ring and let V be a topological C-module. Then, V is a formally projective C-module, if and only if $S_C^{\cdot}(V)$ is a formally smooth C-algebra.

Proof. Assume that V is formally projective. Let E be a discrete C-algebra with structure homomorphism $s: C \to E$. Let I be an ideal of E with $I^2 = (0)$. Let $u: S_C^{\cdot}(V) \to E/I$ be a continuous C-homomorphism and let $\varphi: E \to E/I$ be the canonical homomorphism. u induces on V a continuous C-module homomorphism u_V, because the given topology in V is finer than that induced from the topology in $S_C^{\cdot}(V)$. The kernel $\mathfrak{A}$ of s is an open ideal and E is annihilated by $\mathfrak{A}$ and so is E/I. Therefore, by our assumption, u_V is decomposed as $u_V: V \xrightarrow{v_V} E \xrightarrow{\varphi} E/I$, where v_V is a continuous C-module homomorphism. v_V is extended to a continuous C-algebra homomorphism $v: S_C^{\cdot}(V) \to E$. Then, we have $u = \varphi \cdot v$, as was asserted. Conversely, let $\mathfrak{A}$ be an open ideal of C, and let P and Q be discrete $C/\mathfrak{A}$-modules. Let $f: P \to Q$ be a surjective C-homomorphism and let $w_V: V \to Q$ be a continuous C-homomorphism. Put $C' = C/\mathfrak{A}$. Then, the ringfications $D_{C'}(Q)$ and $D_{C'}(P)$ are regarded as C-algebras. f can be extended to a surjective homomorphism $\bar{f}: D_{C'}(P) \to D_{C'}(Q)$ in the canonical way and the kernel of $\bar{f}$ is contained in P which is a nilpotent ideal. w_V can be extended to a continuous C-algebra homomorphism $w: S_C^{\cdot}(V) \to D_{C'}(Q)$. Hence by our assumption w is decomposed as $w: S_C^{\cdot}(V) \xrightarrow{h} D_{C'}(P) \xrightarrow{\bar{f}} D_{C'}(Q)$. Let h_V be the restriction of h to V. Since $\bar{f} \circ h(V) = w(V) = w_V(V) \subset Q$, $h_V(V) \subset P$ and we have the decomposition $w_V = f \circ h_V$.

Lemma 1. Let C be a ring. Let V be a projective C-module. Then $S_C^n(V)$ are projective C-modules for $n = 0, 1, 2, \cdots$.

Proof. The assertion is true for $n=0, 1$. Hence, there exists a free C-module F and C-homomorphism φ: $F\rightarrow V$ and ψ: $V\rightarrow F$ with $\varphi\circ\psi=id_V$. These induce C-algebra homomorphisms, preserving grades: $S_C^{\cdot}(V)\xrightarrow{S_C^{\cdot}(\psi)}S_C^{\cdot}(F)\xrightarrow{S_C^{\cdot}(\varphi)}S_C^{\cdot}(V)$, where we have $S_C^{\cdot}(\varphi)\circ S_C^{\cdot}(\psi)=id_{S_C^{\cdot}(V)}$. Our assertion follows from this, because $S_C^{\cdot}(F)$ is a polynomial ring over C.

Proposition 2. Let C be a ring with discrete topology and let V be a C-module. Then, the following three statements are equivalent.

(1) $S_C^{\cdot}(V)$ is a formally smooth C-algebra.

(2) V is a projective C-module.

(3) $S_C^n(V)$ are projective C-modules for $n=0, 1, 2, \cdots$.

Proof. This follows from Proposition 1 and Lemma 1.

Lemma 2. Let C be a ring, let V be a C-module and let $\mathfrak{A}$ be an ideal of C. Put $C'=C/\mathfrak{A}$ and $V'=V/\mathfrak{A}V$. Then $S_{C'}^{\cdot}(V')\cong S_C^{\cdot}(V)/\mathfrak{A}S_C^{\cdot}(V)$.

Proof. Since we have a natural C-homomorphism of V' into $S_C^{\cdot}(V)/\mathfrak{A}S_C^{\cdot}(V)$, we have a canonical homomorphism: $S_{C'}^{\cdot}(V')\rightarrow S_C^{\cdot}(V)/\mathfrak{A}S_C^{\cdot}(V)$. Clearly, we have an inverse homomorphism of it.

Proposition 3. Let C be a topological ring. Let V be a C-module with the induced topology. Then the following four statements are equivalent.

(1) $S_C^{\cdot}(V)$ is a formally smooth C-algebra.

(2) V is a formally projective C-module.

(3) $S_C^n(V)$ are formally projective C-modules for $n=0, 1, 2, \cdots$.

(4) For every open ideal $\mathfrak{A}$ of C, $S_{C'}^{\cdot}(V')$ is a formally smooth C'-module, where $C'=C/\mathfrak{A}$ and $V'=V/\mathfrak{A}V$.

Proof. By Lemma 2, we have $S_{C'}(V')\cong S_C(V)/\mathfrak{A}S_C(V)$. Hence, our proposition follows from Proposition 2.

Let A be a topological ring. Let M and N be topological A-modules. A continuous A-homomorphism u: $M\rightarrow N$ is called formally epimorphic, if $u(M)$ is dense in N. u is called formally monomorphic, if the topology in M is the reciprocal image of the topology in N. u is called formally bimorphic, if it is formally epimorphic and formally monomorphic. (Grothendieck [2]). These notions can be applied to ring homomorphisms, because if $\varphi: A \to B$ is a continuous ring homomorphism, B can be regarded as an A-module through φ.

Let B be a topological ring and let I be an ideal of B. As the topology in $gr_I^n(B) = I^n/I^{n+1}$, we take the topology induced from B, that is, first we consider the topology in I^n as a subspace of B and then take the quotient topology modulo I^{n+1} or equivalently, we first take the quotient topology in B/I^{n+1} and then take the induced topology in I^n/I^{n+1}. The following Lemma 3 will not be used in this paper.

Lemma 3. Let B and B' be topological rings. Let I and I' be ideals in B and B', respectively. Let φ be a continuous homomorphism: $B \to B'$ such that $\varphi(I) \subset I'$. Assume that φ induces formal epimorphisms $gr^n(\varphi): gr_I^n(B) \to gr_{I'}^n(B')$ for $n = 0, 1, \cdots, q$. Then φ induces formal epimorphisms $\varphi_q: B/I^{q+1} \to B'/I'^{q+1}$ and $\varphi_q': I/I^{q+1} \to I'/I'^{q+1}$.

Proof. We use the induction on q. For $q=0$, our assertions are trivial. Assume that φ_{q-1}, φ_{q-1}' and $gr^q(\varphi)$ are formal epimorphisms. We have only to prove that for a given open ideal $\mathfrak{B}'$ of B'/I^{q+1}, it holds that $\varphi_q(B/I^{q+1}) + \mathfrak{B}' = B'/I^{q+1}$ and $\varphi_q(I/I^{q+1}) + \mathfrak{B}' \supset I'/I'^{q+1}$. By our assumption, we have $\varphi_q(B/I^{q+1}) + I^q/I^{q+1} + \mathfrak{B}' = B'/I'^{q+1}$ and $\varphi_q(I/I^{q+1}) + I^q/I^{q+1} + \mathfrak{B}' \supset I'/I'^{q+1}$. Since $gr^q(\varphi)$ is formally epimorphic, we have $\varphi_q(I^q/I^{q+1}) + \mathfrak{B}' \supset I'^q/I'^{q+1}$. Therefore, it holds that $\varphi_q(B/I^{q+1}) + \mathfrak{B}' = B'/I'^{q+1}$ and $\varphi_q(I/I^{q+1}) + \mathfrak{B}' \supset I'/I'^{q+1}$.

Lemma 4. Let A be a topological ring. Let M and N be topological A-modules. Let $\varphi: M \to N$ be a continuous homomorphism. Then, φ is formally monomorphic, if and only if (1) φ is an open map of M to the subspace $\varphi(M)$ of N and (2) $\text{Ker}(\varphi)$ is contained in every open neighbourhood of (0) in M.

Proof. Assume that φ is formally monomorphic. Let U be an open submodule of M, then there exists an open submodule V in N such that $U\supset\varphi^{-1}(V)$. Hence, (2) U contains $\mathrm{Ker}(\varphi)$ and $\varphi(U)\supset V\cap\varphi(M)$, hence (1) $\varphi(U)$ is an open submodule of $\varphi(M)$. Conversely, assume that (1) and (2) are true. Let U be an open submodule of M. Then, $\varphi(U)$ is an open submodule in $\varphi(M)$, hence there exists an open set V in N, such that $\varphi(U)=V\cap\varphi(M)$. Since $U\supset\mathrm{Ker}(\varphi)$, $U=U+\mathrm{Ker}(\varphi)$ $=\varphi^{-1}(\varphi(U))=\varphi^{-1}(V)$, which proves our assertion.

Lemma 5. Let A be a topological ring. Let φ: $M\to N$ be a formally bimorphic homomorphism of topological A-modules. Then we have:

(1) M is formally projective if and only if N is formally projective, and

(2) if φ is surjective and M has the induced topology, N has the induced topology.

Proof. (1) Let $\mathfrak{A}$ be an open ideal of A. Let f: $P\to Q$ be a surjective homomorphism of discrete $A/\mathfrak{A}$-modules. First, assume that M is formally projective and there exists a continuous A-homomorphism u: $N\to Q$. Then, there exists a continuous A-homomorphism v: $M\to P$ such that $f\circ v=u\circ\varphi$ and an open submodule M' on which v vanishes. Since φ is formally monomorphic, we may choose M' so small that there exists an open submodule N' of N such that $M'=\varphi^{-1}(N')$, moreover, we may assume that u vanishes on N'. From the definition of formal bimorphisms, it is easy to see that φ induces an isomorphism: $M/M'\simeq N/N'$. Hence, v induces a continuous A-homomorphism $\bar{u}$: $N\to P$ such that $u=f\circ\bar{u}$. Next, assume that N is formally projective and there exists a continuous A-homomorphism w: $M\to Q$. Then, as above, there exists an open submodule N'' such that w vanishes on $\varphi^{-1}(N'')$. And φ induces an isomorphism $M/\varphi^{-1}(N'')\simeq N/N''$. Hence w is decomposed as $M\xrightarrow{\varphi}N\xrightarrow{l}Q$, where l is a continuous A-homomorphism. Then there exists a continuous A-homomorphism m: $N\to P$ such that $l=f\circ m$. Hence, $w=l\circ\varphi=f\circ(m\circ\varphi)$. Hence, M is formally projective. (2) By Lemma 4, (1), $\varphi(\mathfrak{A}M)=\mathfrak{A}N$ is an open set in N. Let V be an open neighbourhood of (0) in N. Then, by our assumption there exists an open ideal $\mathfrak{B}$ of A, such that $\varphi^{-1}(V)\supset\mathfrak{B}M$. Hence $V\supset\varphi(\mathfrak{B}M)=\mathfrak{B}N$, which proves our assertion.

2. Let K be a topological ring. Let A be a topological k-algebra. Let j_1 and j_2: $A\rightarrow A\otimes_k A$ be the two canonical k-homomorphisms such that $j_1(a)=a\otimes 1$ and $j_2(a)=1\otimes a$ for $a\in A$. Let p: $A\otimes_k A\rightarrow A$ be the canonical k-homomorphism such that $p(a\otimes b)=ab$. If we furnish the tensor product topology in $A\otimes_k A$, j_1, j_2 and p are continuous. Let $I_{A/k}$ be the kernel of P. Denote by $\Omega^{(q)}_{A/k}$ the module of high differentials of order q. Then, $\Omega^{(q)}_{A/k}=I_{A/k}/I^{q+1}_{A/k}$ and the canonical derivations $d^q_{A/k}$ is j_2-j_1 modulo $I^{q+1}_{A/k}$ (Osborn [4], Nakai [3]). We consider the topology in $\Omega^{(q)}_{A/k}$ as that induced from the topology in $A\otimes_k A$. We regard $\Omega^{(q)}_{A/k}$ as a topological A-module through j_1. A similar technique used in (20.4.5) of [2] can be used to prove the following.

Proposition 4. The topology of $\Omega^{(q)}_{A/k}$ is less fine than the topology induced from that of A. If in A the $q+1$-th powers of all open ideals are open, these two topologies coincide to each other.

Proof. The first assertion is trivial. To prove the second assertion, it is enough to show that if $\mathfrak{R}$ is an open ideal in A, we have $(\mathfrak{R}^{q+1}\otimes A+A\otimes\mathfrak{R}^{q+1})\cap I\subset\mathfrak{R}I+I^{q+1}$, where $I=I_{A/k}$. First, we prove, by induction on q, that if $x_i\in\mathfrak{R}$ ($i=1, 2, \cdots, q+1$) and $w\in A$, it holds that $x_1\cdots x_{q+1}\otimes w-w\otimes x_1\cdots x_{q+1}\in\mathfrak{R}I+I^{q+1}$. We have

$$\begin{aligned}
&x_1(x_2\cdots x_{q+1})\otimes w-w\otimes x_1(x_2\cdots x_{q+1})\\
&\quad=x_1\otimes 1\cdot((x_2\cdots x_{q+1})\otimes w-w\otimes(x_2\cdots x_{q+1}))\\
&\qquad+x_1w\otimes x_2\cdots x_{q+1}-w\otimes x_1(x_2\cdots x_{q+1})\\
&\quad=x_1\otimes 1\cdot(x_2\cdots x_{q+1}\otimes w-w\otimes x_2\cdots x_{q+1})\\
&\qquad+w\otimes 1\cdot(x_1\otimes x_2\cdots x_{q+1}-1\otimes x_1(x_2\cdots x_{q+1}))
\end{aligned}$$

$$=x_1\otimes 1\cdot(x_2\cdots x_{q+1}\otimes w-w\otimes x_2\cdots x_{q+1})$$

$$+w\otimes 1\cdot(x_1\otimes 1-1\otimes x_1)\cdot 1\otimes x_2\cdots x_{q+1}$$

$$=x_1\otimes 1\cdot(x_2\cdots x_{q+1}\otimes w-w\otimes x_2\cdots x_{q+1})$$

$$+x_2\cdots x_{q+1}w\otimes 1(x_1\otimes 1-1\otimes x_1)$$

$$-w\otimes 1\cdot(x_1\otimes 1-1\otimes x_1)\cdot(x_2\cdots x_{q+1}\otimes 1-1\otimes x_2\cdots x_{q+1}).$$

The first and the second terms belong to $\mathfrak{K}I$ and since the last component of the last term belongs to $\mathfrak{K}I+I^q$ by our induction assumption, the last term belongs to $\mathfrak{K}I+I^{q+1}$. Therefore, we prove our assertion.

Now, elements in $(\mathfrak{K}^{q+1}\otimes A+A\otimes\mathfrak{K}^{q+1})\cap I$ are of the form

$$\sum a_i\otimes w_i+\sum w_j'\otimes a_j', \text{ where } a_i,\ a_j'\in\mathfrak{K}^{q+1} \text{ and } \sum a_iw_i+\sum w_j'a_j'=0.$$

Then,

$$\sum a_i\otimes w_i+\sum w_j'\otimes a_j'=\sum a_i\otimes w_i+\sum a_j'\otimes w_j'+\sum(w_j'\otimes a_j'-a_j'\otimes w_j')$$

$$=\sum a_i\otimes 1\cdot(1\otimes w_i-w_i\otimes 1)+\sum a_j'\otimes 1\cdot(1\otimes w_j'-w_j'\otimes 1)$$

$$+\sum(w_j'\otimes a_j'-a_j'\otimes w_j').$$

The first and the second terms belong to $\mathfrak{K}I$, and the last term belongs to $\mathfrak{K}I+I^{q+1}$ by the above argument. Hence, we prove our proposition.

Let A be a topological ring. A is called preadmissible, if there exists an open ideal J of A (which is called an ideal of definition) such that for every open neighbourhood V of (0) in A there exists an integer $n>0$ such that $J^n\subset V$. (Grothendieck [1]).

Theorem 1. Let k be a topological ring. Let A and A' be topological k-algebras. Let ψ: $A\to A'$ be a formally bimorphic k-homomorphism. Then, ψ induces a formally bimorphic A-module homomorphism λ: $\Omega^{(q)}_{A/k}\to\Omega^{(q)}_{A'/k}$ $(q=1, 2, \cdots)$.

Proof. Let $\mathfrak{A}'$ be an open ideal of A'. We remark that the ideals of the form $\mathfrak{A}'\otimes A'+A'\otimes\mathfrak{A}'$ form a fundamental system of neighbourhoods of (0) in $A'\otimes_k A'$ and by our assumption the ideals of the form $\psi^{-1}(\mathfrak{A}')\otimes A+A\otimes\psi^{-1}(\mathfrak{A}')$ form that in $A\otimes_k A$. Since $A/\psi^{-1}(\mathfrak{A}')\otimes_k A/\psi^{-1}(\mathfrak{A}')\simeq A'/\mathfrak{A}'\otimes_k A'/\mathfrak{A}'$, we have $\psi^{-1}(\mathfrak{A}')\otimes A+A\otimes\psi^{-1}(\mathfrak{A}')=(\psi\otimes\psi)^{-1}(\mathfrak{A}'\otimes A+A\otimes\mathfrak{A}')$.

We have the diagram:

$$\begin{array}{ccccc}
0\rightarrow I/(\psi^{-1}(\mathfrak{A}')\otimes A+A\otimes\psi^{-1}(\mathfrak{A}'))\cap I & \rightarrow & A\otimes A/(\psi^{-1}(\mathfrak{A}')\otimes A+A\otimes\psi^{-1}(\mathfrak{A}')) & \rightarrow & A/\psi^{-1}(\mathfrak{A}')\rightarrow 0\\
 & & \downarrow\wr & & \downarrow\wr\\
0\rightarrow I'/(\mathfrak{A}'\otimes A+A\otimes\mathfrak{A}')\cap I' & \rightarrow & A'\otimes A'/\mathfrak{A}'\otimes A+A\otimes\mathfrak{A}' & \longrightarrow & A'/\mathfrak{A}'\longrightarrow 0
\end{array}$$

It follows that

$$I/(\psi^{-1}(\mathfrak{A}')\otimes A+A\otimes\psi^{-1}(\mathfrak{A}'))\cap I\simeq I'/(\mathfrak{A}'\otimes A'+A'\otimes\mathfrak{A}')\cap I'.$$

Therefore,

$$\begin{aligned}
&I/I^{q+1}\Big/(\psi^{-1}(\mathfrak{A}')\otimes A+A\otimes\psi^{-1}(\mathfrak{A}'))\cap I+I^{q+1}/I^{q+1}\\
&\quad\simeq I/I^{q+1}\\
&\quad\Big/(\psi^{-1}(\mathfrak{A}')\otimes A+A\otimes\psi^{-1}(\mathfrak{A}'))\cap I/(\psi^{-1}(\mathfrak{A}')\otimes A+A\otimes\psi^{-1}(\mathfrak{A}'))\cap I^{q+1}\\
&\quad\simeq I/(\psi^{-1}(\mathfrak{A}')\otimes A+A\otimes\psi^{-1}(\mathfrak{A}'))\cap I\\
&\quad\Big/I^{q+1}/(\psi^{-1}(\mathfrak{A}')\otimes A+A\otimes\psi^{-1}(\mathfrak{A}'))\cap I^{q+1}\\
&\quad\simeq I/(\psi^{-1}(\mathfrak{A}')\otimes A+A\otimes\psi^{-1}(\mathfrak{A}'))\cap I\\
&\quad\Big/(I/\psi^{-1}(\mathfrak{A}')\otimes A+A\otimes\psi^{-1}(\mathfrak{A}')\cap I)^{q+1}\\
&\quad\simeq I'/\mathfrak{A}'\otimes A+A'\otimes\mathfrak{A}'\cap I'\Big/(I'/\mathfrak{A}'\otimes A'+A'\otimes\mathfrak{A}'\cap I')^{q+1}\\
&\quad\simeq I'/I'^{q+1}\Big/(\mathfrak{A}'\otimes A'+A'\otimes\mathfrak{A}')\cap I'+I'^{q+1}/I'^{q+1}
\end{aligned}$$

This shows that λ: $\Omega^{(q)}_{A/k}\simeq I/I^{q+1}\rightarrow\Omega^{(q)}_{A'/k}\simeq I'/I'^{q+1}$ is formally bimorphic.

Corollary. Let k be a topological ring. Let R be a topological k-algebra. Assume that there exists an ideal m of R with finite generators and m^n $(n=0, 1, 2, \cdots)$ form a fundamental system of neighbourhoods of (0) in R (that is, R is an m-adic ring in the classical terminology). Let R^* be a completion of R. Then $\Omega^{(q)}_{R/k}$ and $\Omega^{(q)}_{R^*/k}$ are formally bimorphic.

Proof. In our case, R and R^* be formally bimorphic. Hence our assertion follows from Theorem 1.

Grothendieck proves the following.

Proposition 5. ([2], Theorem (19.5.3)). Let A be a topological ring. Let B be a topological A-algebra. Let I be an ideal in B. Put $C=B/I$. Assume that C is a formally smooth A-algebra. Then we have,

(1) If B is a formally smooth A-algebra, I/I^2 is a formally projective C-module.

(2) Assume that B is a formally smooth A-algebra and is preadmissible.

We put the canonical surjections φ: $S^{\cdot}_C(I/I^2)\rightarrow gr^{\cdot}_I(B)$ and φ_n: $S^n_C(I/I^2)\rightarrow gr^n_I(B)$ $(n=0, 1, 2, \cdots)$. Then, the φ_n are formally bimorphic.

Lemma 6. Let k be a topological ring. Let A be a topological k-algebra. Assume that the topology in A is preadmissible. Then, the topology in $B=A\otimes_k A$ is preadmissible.

Proof. Let J be an ideal of definition of A. We put $J'=J\otimes A+A\otimes J$. We prove that J' is an ideal of definition of B. Let $\mathfrak{K}$ and $\mathfrak{B}$ be open ideals of A. Then, by our assumption there exists an integer n such that $J^n\subset\mathfrak{K}$ and $J^n\subset\mathfrak{B}$. Then, $J'^{2n}\subset J^n\otimes A+A\otimes J^n\subset\mathfrak{K}\otimes A+A\otimes\mathfrak{B}$. Since ideals of the form $\mathfrak{K}\otimes A+A\otimes\mathfrak{B}$ form a fundamental system of neighbourhoods of (0) in B, we prove our assertion.

Theorem 2. Let k be a topological ring and let A be a formally smooth k-algebra. Assume that A is a preadmissible ring and that the $q+1$-th powers of all open ideals are open in A. Then, $\Omega^{(q)}_{A/k}$ is a formally projective A-module.

Proof. Put $B=A\otimes_k A$. Then B is a formally smooth A-algebra by (19.3.5), (iii) in [2]. By Lemma 6, B is preadmissible. Therefore, Proposition 5 is true if we put $I=I_{A/k}$ and $C=A$. We note that by Proposition 4 I/I^{n+1} have the topology induced from that in A for $n=0, 1, \cdots, q$. Especially, I/I^2 has an induced topology and so does $S^{\cdot}_A(I/I^2)$. Therefore, by Proposition 5, (1) and Proposition 3 $S^n_A(I/I^2)$ are formally projective A-modules and by Proposition 5, (2) and Lemma 5. (1), $gr^n_I(B)$ are formally projective A-modules, for $n=0, 1, 2, \cdots$. By Lemma 5, (2), the $gr^n_I(B)$ have induced topology. We shall prove that I/I^{1+l} are formally projective A-modules for $l\leq q$, by induction on l. Our assertion is true for $l=1$. Assume that $l\leq q$ and I/I^l is formally projective. We have only to show that for every open ideal of $\mathfrak{A}$ of A, the $A/\mathfrak{A}$-module $I/I^{l+1}/\mathfrak{A}I/I^{l+1}\simeq I/I^{l+1}+\mathfrak{A}I$ is projective. Since $\mathfrak{A}I^l/I^{l+1}$ is an open set in I^l/I^{l+1} and I^l/I^{l+1} is a subspace of I/I^{l+1}, there exists an open ideal $\mathfrak{B}$ of A such that $\mathfrak{B}\subset\mathfrak{A}$ and $\mathfrak{A}I^l/I^{l+1}\supset\mathfrak{B}I/I^{l+1}\cap I^l/I^{l+1}$, that is, $\mathfrak{A}I^l+I^{l+1}\supset\mathfrak{B}I\cap I^l$. We have the exact sequence of $A/\mathfrak{B}$-modules:

$$0\rightarrow I^l+\mathfrak{B}I/I^{l+1}+\mathfrak{B}I\rightarrow I/I^{l+1}+\mathfrak{B}I\rightarrow I/I^l+\mathfrak{B}I\rightarrow 0.$$

Since $I/I^l+\mathfrak{B}I$ is a projective $A/\mathfrak{B}$-module, we have a decomposition:

$$I/I^{l+1}+\mathfrak{B}I\cong(I/I^l+\mathfrak{B}I)\oplus(I^l+\mathfrak{B}I/I^{l+1}+\mathfrak{B}I).$$

Taking residues modulo $\mathfrak{A}$, we have:

$$I/I^{l+1}+\mathfrak{A}I\cong(I/I^l+\mathfrak{A}I)\oplus(I^l+\mathfrak{B}I/I^{l+1}+\mathfrak{B}I+\mathfrak{A}I^l).$$

$$\cong(I/I^l+\mathfrak{A}I)\oplus(I^l/(I^{l+1}+\mathfrak{B}I+\mathfrak{A}I^l)\cap I^l)$$

$$\cong (I/I^l + \mathfrak{A}I) \oplus (I^l/I^{l+1} + \mathfrak{A}I^l + (\mathfrak{B}I \cap I^l))$$

$$\cong (I/I^l + \mathfrak{A}I) \oplus (I^l/I^{l+1} + \mathfrak{A}I^l).$$

The last two terms are projective $A/\mathfrak{A}$-modules by our assumption. Therefore $I/I^{l+1} + \mathfrak{A}I$ is a projective $A/\mathfrak{A}$-module.

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