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Some remarks on a certain transformation of Macaulay rings

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Throughout this note a ring means a commutative ring with identity element. It is known that in a Macaulay local ring the number of the irreducible components of an ideal generated by a system of parameters is an invariant of the ring. A Macaulay local ring is called a Macaulay local ring of type n if the invariant is equal to n. (In Bass [1] it is called a *MC n* ring.) In this note, with a Macaulay ring R we associate a number called the global type given by the definition: the global type of R is the supremum of the types of local rings $R_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ in R. The purpose of this note is to show that the property "a Macaulay ring of global type at most n" is conserved under the transformation $R\left[\frac{x_1}{x}, \cdots, \frac{x_m}{x}\right]$ of R by an R-sequence $\{x, x_1, \cdots, x_m\}$ (Thoerem 1).

1. We recall some basic facts for the irreducible ideals of a ring. Let R be a ring and $\mathfrak{a}$ an ideal in R. We say that $\mathfrak{a}$ is irreducible in R if $\mathfrak{a}$ is not an intersection of two properly larger ideals in R. We also say that the representation $\mathfrak{a}=\mathfrak{b}_1\cap\cdots\cap\mathfrak{b}_n$ is a longest irredundant representation if every ideal $\mathfrak{b}_i$ is irreducible and if any ideal $\mathfrak{b}_i$ does not contain $\mathfrak{b}_1\cap\cdots\mathfrak{b}_{i-1}\cap\mathfrak{b}_{i+1}\cdots\cap\mathfrak{b}_n$. If $\mathfrak{a}=\mathfrak{b}_1\cap\cdots\cap\mathfrak{b}_n$ is a longest irredundant representation, the number n is said to be the length of the representation. In [5] E. Noether showed that any two longest irredundant representations of an ideal have the same length. Hence the length of a longest irredundant representation of an ideal is an invariant of the ideal and is called the index of reducibility according to Northcott [6].

Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$ and let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal. It is known that the index of reducibility of $\mathfrak{q}$ is equal to the dimension of the $R/\mathfrak{m}$-vector space $(\mathfrak{q}:\mathfrak{m})/\mathfrak{q}$ (cf. Satz 3, [3]). Hence $\mathfrak{q}$ is irreducible if and only if $\dim_{R/\mathfrak{m}}(\mathfrak{q}:\mathfrak{m})/\mathfrak{q}=1$ (A criterion of irreducibility of a primary ideal).

In order to see a link with homological algebra we need the following

Rees' Theorem (cf. [8]). *Let R be a ring, let N be an R-module and let $x_1, \cdots, x_d$ be elements of R such that the sequence $\{x_1, \cdots, x_d\}$ is an R-sequence and is also an N-sequence. Let $\mathfrak{a}$ be the ideal generated by $x_1, \cdots, x_d$ and let M be an R-module such that $\mathfrak{a}$ is contained in the annihilator of M. Then:*

$$\mathrm{Ext}_R^i(M, N)=0 \qquad \text{if } i<d$$

and

$$\mathrm{Ext}_R^i(M, N)\simeq \mathrm{Ext}_{R/\mathfrak{a}}^{i-d}(M, N/\mathfrak{a}N) \qquad \text{if } i\geq d.$$

This theorem gives a characterization of a Macaulay local ring as follows:

Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$ and of Krull dimension d. For R to be a Macaulay local ring it is necessary and sufficient that

$$\mathrm{Ext}_R^i(R/\mathfrak{m}, R)=0 \qquad \text{if } i<d$$

and

$$\mathrm{Ext}_R^d(R/\mathfrak{m}, R)\neq 0.$$

Moreover, if R is a Macaulay local ring, then the length of a maximal R-sequence is equal to d and every maximal R-sequence generates an $\mathfrak{m}$-primary ideal. Hence, if $\mathfrak{q}$ is such an $\mathfrak{m}$-primary ideal, by Rees' theorem we have

$$\mathrm{Ext}_R^d(R/\mathfrak{m}, R)\simeq (\mathfrak{q}:\mathfrak{m})/\mathfrak{q}.$$

This shows that in a Macaulay local ring R the index of reducibility of an $\mathfrak{m}$-primary ideal which is generated by a maximal R-sequence is an invariant of R and it is equal to the dimension of the $R/\mathfrak{m}$-vector space $\mathrm{Ext}_R^d(R/\mathfrak{m}, R)$. This invariant is called the *type* of R, and we say that R is a Macaulay local ring of type n if $n=\dim_{R/\mathfrak{m}}\mathrm{Ext}_R^d(R/\mathfrak{m},R)$.

We say that a noetherian ring R is a Macaulay ring if, for every prime ideal $\mathfrak{p}$ in R, the local ring $R_\mathfrak{p}$ is a Macaulay local ring. The *global type* of a Macaulay ring R is defined by the supremum of the types of local rings $R_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ in R.

Obviously the condition that R is a Gorenstein ring is equivalent to that R is a Macaulay ring of global type one.

A simple consequence of the definition is the following:

Let R be a ring, let $\{x_1, \cdots, x_d\}$ be an R-sequence and set $S=R/\mathfrak{a}$ where $\mathfrak{a}$ is the ideal generated by $x_1, \cdots, x_d$. If R is a Macaulay ring of global type at most n, then so is S. In particular, if R is a Macaulay local ring of type n, then so is S.

We shall use later this remark freely.

2. In this section we shall prove the following:

Proposition 1. *If R is a Macaulay ring of global type at most n, then so is the polynomial ring $R[X_1, \cdots, X_m]$. In particular, if R is a Gorenstein ring, then $R[X_1, \cdots, X_m]$ is also a Gorenstein ring.*

The Gorenstein ring case of this proposition was given in [9].

Before proving the proposition we need following considerations:

Let R be a ring and let $R[X]$ be the polynomial ring in an indeterminate X over R. Let U be the set of polynomials whose coefficients generate the unit ideal in R. Since U is a multiplicatively closed subset of $R[X]$, we can consider the ring $R[X]_U$, the quotient ring of $R[X]$ with respect to U, and we denote it by $R(X)$. This ring was firstly introduced by M. Nagata and the basic properties of the ring and the relationship between the ideals in R and the ideals in

$R(X)$ are mentioned in his book [4]. We recall here some of them:

(i) The ring $R(X)$ contains R.

(ii) If $\mathfrak{p}$ is a prime ideal in R, then $\mathfrak{p}R(X)$ is also a prime ideal in $R(X)$ and $\mathfrak{p}R(X)\cap R=\mathfrak{p}$. If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal in R, then $\mathfrak{q}R(X)$ is also a $\mathfrak{p}R(X)$-primary ideal in $R(X)$ and $\mathfrak{q}R(X)\cap R=\mathfrak{q}$. Hence, for $\mathfrak{p}$-primary ideals $\mathfrak{q}$ and $\mathfrak{q}'$, if $\mathfrak{q}\subsetneqq\mathfrak{q}'$, then $\mathfrak{q}R(X)\subsetneqq\mathfrak{q}'R(X)$.

(iii) If $\mathfrak{a}_1, \cdots, \mathfrak{a}_k$ are ideals in R, then $(\mathfrak{a}_1\cap\cdots\cap\mathfrak{a}_k)R(X)=\mathfrak{a}_1R(X)\cap\cdots\cap\mathfrak{a}_kR(X)$.

(iv) An ideal $\mathfrak{M}$ in $R(X)$ is a maximal ideal in $R(X)$ if and only if there exists a maximal ideal $\mathfrak{m}$ in R such that $\mathfrak{M}=\mathfrak{m}R(X)$.

(v) $R(X)$ is a flat R-algebra and therefore, if $\{x_1, \cdots, x_d\}$ is an R-sequence, then it is also an $R(X)$-sequence.

(vi) If R is a noetherian local ring with maximal ideal $\mathfrak{m}$, then $R(X)$ is the noetherian local ring $R[X]_{\mathfrak{m}R[X]}$. In this case $\mathfrak{m}R(X)$ is the maximal ideal of $R(X)$, and R and $R(X)$ have the same Krull dimension.

For the irreducibility of primary ideals we have the following:

Lemma 1. *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$ and let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal. For $\mathfrak{q}$ to be irreducible in R, it is necessary and sufficient that $\mathfrak{q}R(X)$ is irreducible in $R(X)$. More generally, the index of reducibility of $\mathfrak{q}R(X)$ is equal to that of $\mathfrak{q}$.*

Proof. The sufficiency follows immediately from (ii) and (iii). Assume that $\mathfrak{q}$ is irreducible in R. By a criterion of irreducibility of a primary ideal we have $(\mathfrak{q}:\mathfrak{m})/\mathfrak{q}\simeq R/\mathfrak{m}$. Hence we have $(\mathfrak{q}:\mathfrak{m})R(X)/\mathfrak{q}R(X)\simeq R(X)/\mathfrak{m}R(X)$. On the other hand, since $R(X)$ is R-flat, we have $(\mathfrak{q}:\mathfrak{m})R(X)=\mathfrak{q}R(X):\mathfrak{m}R(X)$. Therefore by a criterion of irreducibility of a primary ideal $\mathfrak{q}R(X)$ is irreducible in $R(X)$.

To see the second part it is enough to show that if $\mathfrak{q}=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_n$ is a longest irredundant representation, then $\mathfrak{q}R(X)=\mathfrak{q}_1R(X)\cap\cdots\cap\mathfrak{q}_nR(X)$ is also a longest irredundant representation. This follows easily from the first part of the lemma and from (ii) and (iii). q.e.d.

From Lemma 1 we have the following:

Lemma 2. *If R is a Macaulay local ring of type n, then so is $R(X)$.*

Proof. From (v) and (vi) it follows easily that $R(X)$ is a Macaulay local ring. By Lemma 1 R and $R(X)$ have the same type.

q.e.d.

Proof of Proposition 1. It is sufficient to consider the case when $m=1$. Let $\mathfrak{P}$ be a prime ideal in $R[X]$ and $\mathfrak{p}=\mathfrak{P}\cap R$. We may assume that R is a local ring and $\mathfrak{p}$ is its maximal ideal. In case when $\mathfrak{P}=\mathfrak{p}R[X]$, since $R[X]_{\mathfrak{P}}$ is the local ring $R(X)$, Lemma 2 gives our assertion. We must therefore prove the proposition in case when $\mathfrak{P}\supsetneqq\mathfrak{p}R[X]$. In this case $\mathfrak{P}$ is a maximal ideal of $R[X]$. Let $\{x_1, \cdots, x_d\}$ be a system of parameters of R. Since $\{x_1, \cdots, x_d\}$ is an $R[X]_{\mathfrak{P}}$-sequence, we may further assume that the Krull dimension of R is zero. Hence to prove the proposition it is enough to show the following:

Let R be a Macaulay local ring with maximal ideal $\mathfrak{m}$ and of Krull dimension zero. If $\mathfrak{M}$ is a maximal ideal in $R[X]$ such that $\mathfrak{m}=\mathfrak{M}\cap R$, then $R[X]_{\mathfrak{M}}$ is a Macaulay local ring and has the same type as that of R,

The method of the proof is substantially the same as that of Proposition 1 (Part II) in [9]. We give here the proof for the convenience of the reader.

Since $\mathfrak{M}/\mathfrak{m}R[X]$ is a non zero prime ideal in $R[X]/\mathfrak{m}R[X]$ ($\simeq(R/\mathfrak{m})[X]$), $\mathfrak{M}=f(X)R[X]+\mathfrak{m}R[X]$ where $f(X)$ is a monic polynomial in $R[X]$ such that all the coefficients of $f(X)$ are units in R. Obviously $f(X)$ is a non zero-divisor in $R[X]_{\mathfrak{M}}$. Since the Krull dimension of $R[X]_{\mathfrak{M}}$ is one, $R[X]_{\mathfrak{M}}$ is a Macaulay local ring and $f(X)$ generates an $\mathfrak{M}R[X]_{\mathfrak{M}}$-primary ideal. Hence in order to show that $R[X]_{\mathfrak{M}}$ has the same type as that of R, it is sufficient to see that the dimension of the $R[X]/\mathfrak{M}$-vector space $(f(X)R[X]_{\mathfrak{M}}\colon \mathfrak{M}R[X]_{\mathfrak{M}})$

$/f(X)R[X]_{\mathfrak{M}}$ is equal to the dimension of the $R/\mathfrak{m}$-vector space $(0:\mathfrak{m})$. Let $\{u_1, \cdots, u_n\}$ be a system of minimal generators of the ideal $(0:\mathfrak{m})$. Assume that $u_1 g_1(X)+\cdots+u_n g_n(X)$ is in the ideal $f(X)R[X]$ where $g_i(X)\in R[X]$. Let $g_i(X)=h_i(X)f(X)+r_i(X)$ where $h_i(X)$ and $r_i(X)$ are in $R[X]$ and the degree of $r_i(X)<$ the degree of $f(X)$. Then $u_1r_1(X)+\cdots+u_nr_n(X)$ is in $f(X)R[X]$. This shows that $u_1r_1(X)+\cdots+u_nr_n(X)=0$. Hence, if a_{ij} is the coefficient of the term of degree j in $r_i(X)$, then $u_1a_{1j}+\cdots+u_na_{nj}=0$. Since $u_1, \cdots, u_n$ are linearly independent over $R/\mathfrak{m}$, we have $a_{ij}\in\mathfrak{m}$. Therefore $r_i(X)\in\mathfrak{m}R[X]$ and hence $g_i(X)\in\mathfrak{M}$. This shows that the residue classes of $u_1, \cdots, u_n$ modulo $f(X)R[X]$ are linearly independent over $R[X]/\mathfrak{M}$. On the other hand it is known that $(f(X)R[X]:\mathfrak{M})=f(X)R[X]+(0:\mathfrak{m})R[X]$ (see the proof of Proposition 1, Part II, [9]). Therefore the $R[X]/\mathfrak{M}$-vector space $(f(X)R[X]:\mathfrak{M})/f(X)R[X]$ has dimension n. Since $(f(X)R[X]:\mathfrak{M})R[X]_{\mathfrak{M}}=(f(X)R[X]_{\mathfrak{M}}:\mathfrak{M}R[X]_{\mathfrak{M}})$, the dimension of the $R[X]/\mathfrak{M}$-vector space $(f(X)R[X]_{\mathfrak{M}}:\mathfrak{M}R(X]_{\mathfrak{M}})/f(X)R[X]_{\mathfrak{M}}$ is equal to n. Thus our assertion is proved.

Remark. (1) If R is a Macaulay local ring of type n, then so is the formal power series ring $R[[X_1, \cdots, X_m]]$.

For, by Rees' theorem we have

$$\mathrm{Ext}_S^i(S/\mathfrak{M}, S)=0 \qquad \text{if } i<m$$

and

$$\mathrm{Ext}_S^i(S/\mathfrak{M}, S)\simeq\mathrm{Ext}_R^{i-m}(R/\mathfrak{m}, R) \qquad \text{if } i\geq m$$

where $S=R[[X_1, \cdots, X_m]]$, $\mathfrak{M}$ is the maximal ideal of S and $\mathfrak{m}$ is the maximal ideal of R. From this our assertion follows immediately.

(2) Let R be a noetherian local ring and $\hat{R}$ its completion. For R to be a Macaulay local ring of type n, it is necessary and sufficient that $\hat{R}$ is a Macaulay local ring of type n.

This follows from that $\mathrm{Ext}_{\hat{R}}^i(K, \hat{R})\simeq\mathrm{Ext}_R^i(K, R)\otimes_R\hat{R}$ and from that $\mathrm{Ext}_{\hat{R}}^i(K, \hat{R})=0$ if and only if $\mathrm{Ext}_R^i(K, R)=0$ (because $\hat{R}$ is a faithfully flat R-algebra) where K is the residue field of R.

3. Let R be a ring and let $\{x, x_1, \cdots, x_m\}$ be a system of elements in R such that x is a non zero-divisor in R. Set $S=R\left[\frac{x_1}{x}, \cdots, \frac{x_m}{x}\right]$, a subring of the total quotient ring of R, and $P=R[X_1, \cdots, X_m]$, the polynomial ring in m indeterminates over R. Let $\mathfrak{J}$ be the kernel of the ring homomorphism $\varphi: P\to S$ defined by $\varphi(X_i)=\frac{x_i}{x}$.

The aim of this section is to prove the following theorem which is partly a more precise result than Theorem 2.4 in Ratliff [7].

Theorem 1. *If R is a Macaulay ring of global type at most n and if $\{x, x_1, \cdots, x_m\}$ is an R-sequence, then the ring $S=R\left[\frac{x_1}{x}, \cdots, \frac{x_m}{x}\right]$ is also a Macaulay ring of global type at most n. In particular, if R is a Gorenstein ring, then so is S.*

First we need the following:

Lemma (Davis [2]). *With the same notation as above, let R be an arbitrary ring. If $\{x, x_1, \cdots, x_m\}$ is an R-sequence, then the kernel $\mathfrak{J}$ of the map $\varphi: P\to S$ is generated by $xX_1-x_1, \cdots, xX_m-x_m$.*

By this lemma and by Proposition 1, in order to prove Theorem 1 it is enough to show the following:

Proposition 2. *With the same notation as above, let R be an arbitrary ring. If $\{x, x_1, \cdots, x_m\}$ is an R-sequence, then $\{xX_1-x_1, \cdots, xX_m-x_m\}$ is a P-sequence.**)

Proof. Set $Y_i=xX_i-x_i$. Let $\mathfrak{J}_k$ be the ideal generated by $Y_1, \cdots, Y_k$ and $\mathfrak{J}_0=(0)$. Obviously $\mathfrak{J}_m=\mathfrak{J}$ and $\mathfrak{J}\neq P$. Hence, for the proof it is sufficient to show that $(\mathfrak{J}_k: Y_{k+1})=\mathfrak{J}_k$ for $k=0, \cdots, m-1$. Let $f(X)\in(\mathfrak{J}_k: Y_{k+1})$ and write $f(X)=g_t(X)X_{k+1}^t+\cdots+g_0(X)$, $g_i(X)\in R[X_1, \cdots, X_k, X_{k+2}, \cdots, X_m]$. We shall first show that $g_t(X)\in\mathfrak{J}_k$.

*) Cf. Theorem 2.4 of Ratliff [7], where R is assumed to be a Macaulay ring.

Let $\varphi_k: P \to R\left[\frac{x_1}{x}, \cdots, \frac{x_k}{x}, X_{k+1}, \cdots, X_m\right]$ be the ring homomorphism defined by $\varphi_k(X_i) = \frac{x_i}{x}$ for $i=1, \cdots, k$ and $\varphi_k(X_i) = X_i$ for $i = k+1, \cdots, m$ and let $\varphi_0: P \to P$ be the identity map. Since $\{x, x_1, \cdots, x_k\}$ is an R-sequence, by the above lemma we see that $\mathfrak{J}_k$ is the kernel of φ_k. Thus $\varphi_k(f(X)Y_{k+1}) = 0$ and from this it follows that $x\varphi_k(g_t(X)) = 0$. Write $g_t(X) = \sum h_s(X_1, \cdots, X_k)M_s(X)$ where $M_s(X)$ are monomials in $X_{k+2}, \cdots, X_m$ and $h_s(X_1, \cdots, X_k) \in R[X_1, \cdots, X_k]$. Then we have $xh_s\left(\frac{x_1}{x}, \cdots, \frac{x_k}{x}\right) = 0$ and hence $h_s(X_1, \cdots, X_k) \in \mathfrak{J}_k$. This shows that $g_t(X) \in \mathfrak{J}_k$. Next, set $f_1(X) = f(X) - g_t(X)X_{k+1}^t$. Since $f_1(X)Y_{k+1} \in \mathfrak{J}_k$, by the same argument as above, we have $g_{t-1}(X) \in \mathfrak{J}_k$. Whence by induction we can show that $g_i(X) \in \mathfrak{J}_k$ for all i. Therefore we have $f(X) \in \mathfrak{J}_k$ and hence $(\mathfrak{J}_k: Y_{k+1}) \subseteq \mathfrak{J}_k$. Since the opposite inclusion is obvious, the proof is complete.

Let R be a noetherian ring and let $\mathfrak{a}$ be an ideal in R which is generated by $a_1, \cdots, a_m$. Let t be an indeterminate and set $u = t^{-1}$. The graded noetherian ring $R[ta_1, \cdots, ta_m, u]$ is called the Rees ring of R with respect to $\mathfrak{a}$. If $\{a_1, \cdots, a_m\}$ is an R-sequence, then $\{u, a_1, \cdots, a_m\}$ is an $R[u]$-sequence. Hence by Proposition 1 and by Theorem 1 we have the following:

Corollary. *If R is a Macaulay ring of global type at most n and if $\{a_1, \cdots, a_m\}$ is an R-sequence, then the Rees ring $R[ta_1, \cdots, ta_m, u]$ is also a Macaulay ring of global type at most n.*

We end this section with a few remarks for complete intersections. Let R be a ring. We say that R is a complete intersection if R is a residue ring of a regular ring A by an ideal which is generated by an A-sequence. Hence if R is a complete intersection, then for every prime ideal $\mathfrak{p}$ in R the local ring $R_{\mathfrak{p}}$ is a complete intersection in ordinary sense.

The following results follow directly from the definition:

Let R be a ring and $\mathfrak{a}$ an ideal which is generaed by an R-sequence. If R is a complete intersection, then so is $R/\mathfrak{a}$.

If R is a complete intersection, then so is the polynomial ring $R[X_1, \cdots, X_m]$.

In fact, let $R=A/\mathfrak{a}$ where A is a regular ring and $\mathfrak{a}$ is the ideal generated by an A-sequence $\{x_1, \cdots, x_d\}$. It is well known that the polynomial ring $A[X_1, \cdots, X_m]$ is a regular ring. Since $A[X_1, \cdots, X_m]$ is A-flat, $\{x_1, \cdots, x_d\}$ is also an $A[X_1, \cdots, X_m]$-sequence. Hence our assertion follows from the fact that $A[X_1, \cdots, X_m]/\mathfrak{a}A[X_1, \cdots, X_m] \simeq R[X_1, \cdots, X_m]$.

From these and from Proposition 2 we have the following similar result to Theorem 1:

Theorem 2. *If R is a complete intersection and if $\{x, x_1, \cdots, x_m\}$ is an R-sequence, then $R\left[\frac{x_1}{x}, \cdots, \frac{x_m}{x}\right]$ is also a complete intersection.*

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