

Plancherel formula for non-unimodular locally compact groups

(To the memory for the late Professor A. Kohari)

By

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§0. Introduction

In the papers of F. I. Mautner [9] [10] and I. E. Segal [13], a generalization of Plancherel formula for separable unimodular locally compact groups H , is established. This theory asserts the existence of so-called Plancherel measure μ over the reduced quasi-dual \mathcal{Q} of H , which satisfies for any function f in $L^1(H) \cap L^2(H)$,

$$(1) \quad \int_H |f(h)|^2 dh = \int_{\mathcal{Q}} \tau_{\omega}((U_f(\omega))^* U_f(\omega)) d\mu(\omega).$$

Here dh is the Haar measure over H , and $\tau_{\omega}((U_f(\omega))^* U_f(\omega))$ are traces over the positive parts $(\mathfrak{A}(\omega))^+$ of the von Neumann algebras $\mathfrak{A}(\omega)$ generated by the operators $U_f(\omega) \equiv \int_H f(h) U_h(\omega) dh$, ($f \in L^1(H)$), which correspond to the factor representations $\omega \equiv \{\mathfrak{D}(\omega), U_h(\omega)\}$ in \mathcal{Q} .

However, it is easily shown that for a non-unimodular group G the formula (1) is not true.

In 1961, A. Kohari [5] obtained an analogous formula for the motion group over the straight line. His theory gives a formula

$$(2) \quad \int_G |f(g)|^2 d_r g = \int_{\mathcal{G}} \tau_{\mathfrak{D}}((T(\mathfrak{D}) U_f(\mathfrak{D}))^* T(\mathfrak{D}) U_f(\mathfrak{D})) d\bar{\mu}(\mathfrak{D}),$$

instead of (1). Here $d_r g$ is a right Haar measure on this group G ,

and f shows any continuous function on G with a compact support. $T(\mathfrak{D})$ is an unbounded self-adjoint operator on the space $\mathfrak{H}(\mathfrak{D})$ of representation \mathfrak{D} . These operators are defined by means of the modular function Δ_G on G . In this case, $\tau_{\mathfrak{D}}$ are just given by the ordinary traces of operators.

The purposes of this paper are to construct the reduced quasi-dual of more general separable non-unimodular locally compact groups G except measure zero subset (§5. Theorem 5.1.), and to determine the operators $T(\mathfrak{D})$ by means of Δ_G . The elements of the reduced quasi-dual of G are given as representations induced from factor representations of some subgroup H of G , and are constructed on G -orbits in the reduced quasi-dual of H (§4, §5). Thus the reduced quasi-dual of G is considered as the G -orbits space X in the reduced quasi-dual of H except measure zero subset.

After these arguments, the regular representation of G is decomposed on X as the central decomposition. According to this decomposition the extended Plancherel formula (2) is proved for G , under adequate definitions of linear functional $\tau_{\mathfrak{D}}$ which are equal to the ordinary traces of operator for some good case (§6. Theorem 6.1). All these discussions are done on the base of the theories of F. I. Mautner and I. E. Segal.

The above mentioned central decomposition of regular representation of G raises also a decomposition of the regular double representation $\mathfrak{D} = \{L^2(G), R_{g_1}, L_{g_2}, J\}$ of G and a decomposition of the quasi-Hilbert algebra constructed on the convolution ring $C_0(G)$ of continuous functions with compact supports on G . We treat these problems in §7.

In the previous paper [14], we proved an invariance of the Plancherel measure under the Kronecker product operations, for unimodular groups of type I. We shall extend this to the invariance of the given "Plancherel measure" $\tilde{\mu}$ for separable locally compact groups G .

Thus we know that the Plancherel measure has a property like to the Haar measure on a locally compact group. Therefore, being suggested from the theory of Haar measure, naturally, the question arises:

whether the uniqueness up to constant is valid for such invariant measure or not. At last, we give an affirmative answer to this problem (§8).

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§1. Preliminary

In this section, for the sake of later uses, we shall explain the Plancherel formula for unimodular groups, which was established by F. I. Mautner [9], [10] and I. E. Segal [13].

Let H be a separable unimodular locally compact group, and $L^2(H)$ be the space of all square summable functions on H with respect to a Haar measure dh . On this space, a double representation $\mathfrak{D}_H \equiv \{L^2(H), R_h^H, L_k^H, J_H\}$ is constructed by the right translations R_h^H , the left translations L_k^H and an involution J_H on H , defined by

$$(1.1) \quad (J_H f)(h) = \overline{f(h^{-1})}.$$

As is shown by R. Godement [3], the centre \mathfrak{C} of the von Neumann algebra \mathfrak{A}_R generated by $\{R_h^H\}_{h \in H}$, is equal to the one of the von Neumann algebra generated by $\{L_h^H\}_{h \in H}$. So the central decomposition of the right regular representation $\mathfrak{R}_H \equiv \{L^2(H), R_h^H\}$ has not only the same decomposition of $L^2(H)$ as the central decomposition of the left regular representation $\mathfrak{L}_H \equiv \{L^2(H), L_h^H\}$, but also as the irreducible decomposition of \mathfrak{D}_H as a representation of $H \times H$. Take this decomposition over the dual \mathfrak{Q} of \mathfrak{C} , by a measure μ , as

$$(1.2) \quad \begin{aligned} \mathfrak{D}_H &= \{L^2(H), R_h^H, L_k^H, J_H\} \\ &\simeq \int_{\mathfrak{Q}} \{\mathcal{A}(\omega), W_h(\omega), V_k(\omega), J_H(\omega)\} d\mu(\omega). \end{aligned}$$

Here the Borel structure on \mathfrak{Q} is the one defined in a paper of G. W. Mackey [7]. From the separability of H and commutativity of \mathfrak{C} , μ

is a standard measure on Ω in the sense of G. W. Mackey (cf [7]. Th. 8.7. Cor.).

Here we shall define an operator on $L^2(H)$ as

$$(1.3) \quad (S_H f)(h) = f(h^{-1}).$$

It is easy to see that this operator S_H is decomposed by the decomposition (1.2).

$$(1.4) \quad S_H \sim \int_{\Omega} S_H(\omega) d\mu(\omega).$$

By virtue of (1.2), Ω may be considered as a set of factor representations $\omega \equiv \{\mathcal{A}(\omega), W_h(\omega)\}$ of H , and is called the reduced quasi-dual of H . According to (1.2), a vector f in $L^2(H)$ is represented as a vector field $\{v_f(\omega)\}$ on Ω , taking its value in $\mathfrak{F}(\omega)$ at ω , and

$$(1.5) \quad \|f\|^2 \equiv \int_H |f(h)|^2 dh = \int_{\Omega} \|v_f(\omega)\|^2 d\mu(\omega).$$

On the other hand, the bounded operator on $L^2(H)$,

$$(1.6) \quad R_f^H \equiv \int_H f(h) R_h^H dh, \quad (f \in L_1(H))$$

is decomposed as an operator field $\int_{\Omega} W_f(\omega) d\mu(\omega)$ on Ω .

Here

$$(1.7) \quad W_f(\omega) \equiv \int_H f(h) W_h(\omega) dh$$

are bounded operators on $\mathcal{A}(\omega)$'s.

Thus for any function f in $L^1(H) \cap L^2(H)$, we have decompositions of two kinds. One of which is as a vector field $\int_{\Omega} v_f(\omega) d\mu(\omega)$ on Ω , and the other is as an operator field $\int_{\Omega} W_f(\omega) d\mu(\omega)$ on Ω . Through the functions f in $L^1(H) \cap L^2(H)$, a correspondence between the opera-

tor fields $\int_{\mathcal{Q}} \mathcal{W}_f(\omega) d\mu(\omega)$ and the vector fields $\int_{\mathcal{Q}} v_f(\omega) d\mu(\omega)$ on \mathcal{Q} is obtained. And in the above cited papers [9], [10] and [13], the following facts are proved.

1) For μ -almost all ω , the correspondence of the component an operator $\mathcal{W}_f(\omega)$ and a vector $v_f(\omega)$ at ω , is independent of the selection of f . This correspondence gives one-to-one linear map from a dense space of the von Neumann algebra $\mathfrak{A}(\omega)$ generated by $\{\mathcal{W}_h(\omega)\}_{h \in H}$ to a dense space on $\mathcal{H}(\omega)$.

And $\mathfrak{A}(\omega)$ are factors.

2) For μ -almost all ω , the map

$$(1.8) \quad (\mathcal{W}_f(\omega))^* \mathcal{W}_f(\omega) \rightarrow \|v_f(\omega)\|^2$$

gives a faithful normal semi-finite trace on the positive part $\mathfrak{A}^+(\omega)$ of $\mathfrak{A}(\omega)$. We shall denote this trace by τ_ω , that is,

$$(1.9) \quad \|v_f(\omega)\|^2 = \tau_\omega((\mathcal{W}_f(\omega))^* \mathcal{W}_f(\omega)).$$

Combining (1.9) with (1.5), we obtain

$$(1.10) \quad \|f\|^2 = \int_{\mathcal{Q}} \tau_\omega((\mathcal{W}_f(\omega))^* \mathcal{W}_f(\omega)) d\mu(\omega).$$

This may be considered as a generalization of the Plancherel formula for unimodular group.

In the decomposition (1.2), the Plancherel measure μ can be determined only up to absolute continuity. Hence the traces τ_ω in (1.9) depend on the selection of μ . But because of the uniqueness of trace on factor, τ_ω are determined up to constant. Under such considerations, we give some normalization of μ and τ_ω as follows.

In general, it is possible that $\mathfrak{A}(\omega_1)$ and $\mathfrak{A}(\omega_2)$ are mutually spatially isomorphic, even if ω_1 and ω_2 are two different points in \mathcal{Q} . That is, there exists an isometric operator U from $\mathfrak{H}(\omega_1)$ onto $\mathfrak{H}(\omega_2)$, and the map $A \rightarrow UAU^{-1}$ gives an isomorphism from the von Neumann algebra $\mathfrak{A}(\omega_1)$ generated by $\{\mathcal{W}_h(\omega_1)\}_{h \in H}$ onto the von Neumann al-

gebra $\mathfrak{A}(\omega_2)$ generated by $\{W_h(\omega_2)\}_{h \in H}$. In such a case, for any trace τ_{ω_2} on $\mathfrak{A}^+(\omega_2)$, $\tau_1(A) \equiv \tau_{\omega_2}(UAU^{-1})$ gives a trace on $\mathfrak{A}^+(\omega_1)$ too.

From the definition of trace, it is easy to see that τ_1 does not depend on the selection of the isometric operator U . In such a way, if a trace is determined on $\mathfrak{A}^+(\omega_0)$ for some ω_0 , then the unique trace is given on any $\mathfrak{A}^+(\omega)$, for which $\mathfrak{A}(\omega)$ is spatially isomorphic to $\mathfrak{A}(\omega_0)$. And in (1.9), if such a normalization of τ_ω 's is done, then corresponding Plancherel measure μ is uniquely determined on such spatially isomorphic classes.

Moreover, when the factor $\mathfrak{A}(\omega)$ is of type I, a trace is given, using a minimal central projection P in $\mathfrak{A}(\omega)$ and the ordinary trace T_r of operators, as follows,

$$(1.11) \quad \tau_\omega(A) = T_r(PA).$$

obviously these traces satisfy the normalization stated above.

Hereafter, in the equation (1.10), we shall understand that for the set of τ_ω 's and μ , the above normalization is done already. That is,

$$(1.12) \quad \tau_{\omega_1}(A) = \tau_{\omega_2}(UAU^{-1}),$$

for any mutually spatially isomorphic pair (ω_1, ω_2) under an isometric operator U , and any operator A in $\mathfrak{A}(\omega_1)$.

Definition 1.1. *We call a separable unimodular group H has the reduced dual of type I, when μ -almost all ω in Ω is type I.*

For instance, if H is a group of type I, obviously H has the reduced dual of type I.

When H has the reduced dual of type I, we shall use the normalization given by (1.11). Thus the following equation is obtained in such a case, instead of (1.10).

$$(1.13) \quad \|f\|^2 = \int_{\Omega} T_r((W_f(\omega^0))^* W_f(\omega^0)) d\mu(\omega)$$

$$= \int_{\mathfrak{g}} \|W_f(\omega^0)\|^2 d\mu(\omega).$$

Here $W_f(\omega^0) \equiv \int_H f(h) W_h(\omega^0) dh$, the operator of the irreducible representation ω^0 as a minimal component of type I factor ω .

§2. Hilbert-Schmidt norms of operators in induced representations.

In this §, let G be a general locally compact group and H be an arbitrarily fixed closed subgroup of G . Since the unimodularities of the groups G and H are not assumed, there may be two Haar measures up to constant factors. Hereafter we use only the right Haar measures $d_r g$ and $d_r h$ on G and H respectively.

Notations. We denote, by $C_0(G)$ the space of all continuous functions with compact support on G , and by $L^p(G) (1 \leq p < \infty)$ the space of all measurable functions f such that

$$(2.1) \quad \int_G |f(g)|^p d_r g < +\infty.$$

Put $\omega = \{\mathcal{H}, W_h\}$ a given unitary representation of H . For this representation ω , we consider a trace τ_ω or the Hilbert-Schmidt norm of operators

$$(2.2) \quad W_k \equiv \int_H k(h) W_h d_r h,$$

corresponding to any continuous function k on H with compact support.

On the other hand, consider the representation $\mathfrak{D} \equiv \{\mathfrak{D}, U_g\} \equiv \text{Ind}_{H \uparrow G} \omega$ of G , induced from ω , and operators

$$(2.3) \quad U_f \equiv \int_G f(g) U_g d_r g,$$

corresponding to any continuous function f on G with compact support.

The aim of this § is to obtain the formula of a “trace” or the Hilbert-Schmidt norm of U_f written in terms of the traces or the Hilbert-Schmidt norms of the operators W_k . It is easy to see that these results are extendable to an adequate class of functions f .

Before entering discussions, we shall state the well-known lemmas about quasi-invariant measures on the factor space $H\backslash G$. (For the proof, see N. Bourbaki [1]).

Lemma 2.1. *There exists a continuous function $\phi(g)$ on G , such that*

$$(2.4) \quad 1) \quad \phi(g) > 0, \quad \text{for any } g \text{ in } G,$$

$$(2.5) \quad 2) \quad \phi(hg) = (\Delta_G(h)/\Delta_H(h))\phi(g),$$

for any h in H and any g in G .

Here $\Delta_G(g)$ and $\Delta_H(h)$ show the modular functions of right and left Haar measures on G and H defined by

$$(2.6) \quad \Delta_G(g_1) \equiv d_r(g_1g)/d_rg, \text{ and } \Delta_H(h_1) \equiv d_r(h_1h)/d_rh,$$

respectively.

Lemma 2.2. *Let ϕ be a function given in Lemma 2.1., then there exists a quasi-invariant measure ν on $H\backslash G$, such that*

$$(2.7) \quad \int_G f(g) d_rg = \int_{H\backslash G} d\nu(\tilde{g}) \int_H f(hg)\phi(hg) d_rh,$$

$$(2.8) \quad d\nu(gg_1)/d\nu(g) = \phi(g)/\phi(gg_1), \text{ for any } g, g_1 \text{ in } G.$$

Here f is any function in $L^1(G)$ and \tilde{g} shows the H -coset containing g in G .

Lemma 2.3. *All quasi-equivalent measures on the factor space $H\backslash G$ are mutually absolutely equivalent.*

Definition 2.1. Hereafter, put

$$(2.9) \quad w(g, g_1) \equiv \sqrt{\psi(g)/\psi(g g_1)} = \sqrt{d\nu(g g_1)/d\nu(g)}.$$

Next, we shall state simple sketches about the theory of induced representations given by G. W. Mackey [6].

As is stated above, let $\mathfrak{D} \equiv \{\mathfrak{D}, U_g\}$ be the representation of G induced from the representation $\omega = \{\mathcal{H}, W_h\}$. The space \mathfrak{D} of representation \mathfrak{D} is defined as the totality of strongly measurable \mathcal{H} -valued functions $\mathbf{v}(g)$ on G satisfying

$$(2.10) \quad 1) \quad \mathbf{v}(hg) = W_h(\mathbf{v}(g)),$$

for any h in H and almost all g in G ,

$$(2.11) \quad 2) \quad \|\mathbf{v}\|_{\mathfrak{D}}^2 \equiv \int_{H \backslash G} \|\mathbf{v}(g)\|_{\mathcal{H}}^2 d\nu(\tilde{g}) < +\infty.$$

Here $\|\mathbf{v}\|_{\mathfrak{D}}$ and $\|\mathbf{v}(g)\|_{\mathcal{H}}$ show the norms in the spaces \mathfrak{D} and \mathcal{H} respectively. And on this space \mathfrak{D} , the operators of the representation operate as

$$(2.12) \quad (U_{g_1}\mathbf{v})(g) = w(g, g_1)\mathbf{v}(g g_1).$$

We refer to readers the Mackey's paper [6] for that $\{\mathfrak{D}, U_g\}$ gives a unitary representation of G , and we denote it by $\text{Ind}_{H \uparrow G} \omega$.

The following lemmas for which we do not mention the proofs, are given by G. W. Mackey [6].

Lemma 2.4. *The (right) regular representation \mathfrak{R} of G is equivalent to the representation induced by the (right) regular representation \mathfrak{R}_H of H .*

The correspondence of $L^2(G)$ to the space of $\text{Ind}_{H \uparrow G} \mathfrak{R}_H$, giving the unitary equivalence in Lemma 2.4, is defined by

$$(2.13) \quad L^2(G) \ni f(\cdot) \rightarrow f(\cdot g)\sqrt{\psi(\cdot g)}.$$

Here $f(\cdot, g)\sqrt{\psi(\cdot, g)}$ are considered as $L^2(H)$ -valued functions on $H \setminus G$.

Lemma 2.5. *If a representation ω_0 of H is represented as a direct integral over some measure space $\{\Omega_0, \mu\}$ as*

$$(2.14) \quad \omega_0 = \int_{\Omega_0} \omega_\alpha d\mu(\alpha),$$

then the direct integral $\int_{\Omega_0} \text{Ind}_{H \uparrow G} \omega_\alpha d\mu(\alpha)$ exists and is equivalent to $\text{Ind}_{H \uparrow G} \omega_0$.

Now take any positive character δ on G , trivial on H . That is, δ is a continuous function on G such that,

$$(2.15) \quad 1) \quad \delta(g) > 0, \quad \text{for any } g \text{ in } G,$$

$$(2.16) \quad 2) \quad \delta(g_1)\delta(g_2) = \delta(g_1g_2), \quad \text{for any } g_1, g_2 \text{ in } G,$$

$$(2.17) \quad 3) \quad \delta(h) = 1, \quad \text{for any } h \text{ in } H.$$

We consider a map on \mathfrak{H} defined by

$$(2.18) \quad (T_\delta \mathbf{v})(g) \equiv \delta(g)\mathbf{v}(g).$$

Lemma 2.6. *T_δ is a self-adjoint positive definite linear operator on \mathfrak{H} . This operator is bounded if and only if,*

$$(2.19) \quad \delta \equiv 1.$$

Proof. Indeed, T_δ gives a linear operator on \mathfrak{H} with the domain

$$(2.20) \quad \mathfrak{D}(T_\delta) = \left\{ \mathbf{v} \in \mathfrak{H}; \int_{H \setminus G} (\delta(g))^2 \|\mathbf{v}(g)\|^2 d\nu(g) < +\infty \right\}.$$

The assertions are deduced from this directly.

q.e.d.

Lemma 2.7.

$$(2.21) \quad T_\delta U_g = (\delta(g))^{-1} U_g T_\delta, \quad \text{for any } g \text{ in } G.$$

$$(2.22) \quad T_\delta U_f = U_{f \cdot \delta^{-1}} T_\delta, \quad \text{for any } f \text{ in } L^1(G) \text{ such that}$$

$$(2.23) \quad (f \cdot \delta^{-1})(g) \equiv f(g) \delta^{-1}(g)$$

is in $L^1(G)$.

Proof. At first, for any g_0 in G ,

$$\begin{aligned} (2.24) \quad \int_{H \setminus G} (\delta(g))^2 \|U_{g_0} \mathbf{v}(g)\|^2 d\nu(g) &= \int_{H \setminus G} (\delta(g))^2 \|w(g, g_0) \mathbf{v}(gg_0)\|^2 d\nu(g) \\ &= \int_{H \setminus G} (\delta(g))^2 \|\mathbf{v}(gg_0)\|^2 d\nu(gg_0) \\ &= (\delta(g_0))^{-2} \int_{H \setminus G} (\delta(g))^2 \|\mathbf{v}(g)\|^2 d\nu(g) < +\infty. \end{aligned}$$

This shows

$$(2.25) \quad \mathfrak{D}(T_\delta) = \mathfrak{D}(T_\delta U_g).$$

Next,

$$\begin{aligned} (2.26) \quad (T_\delta U_{g_0} \mathbf{v})(g) &= \delta(g) (U_{g_0} \mathbf{v})(g) = \delta(g) w(g, g_0) \mathbf{v}(gg_0) \\ &= (\delta(g_0))^{-1} w(g, g_0) \delta(gg_0) \mathbf{v}(gg_0) = (\delta(g_0))^{-1} w(g, g_0) (T_\delta \mathbf{v})(gg_0) \\ &= (\delta(g_0))^{-1} (U_{g_0} T_\delta \mathbf{v})(g). \end{aligned}$$

$$\begin{aligned} (2.27) \quad T_\delta U_f \mathbf{v} &= T_\delta \left(\int_G f(g) (U_g \mathbf{v}) d_r g \right) = \int_G f(g) (T_\delta U_g \mathbf{v}) d_r g \\ &= \int_G f(g) (\delta(g))^{-1} (U_g T_\delta \mathbf{v}) d_r g = U_{f \cdot \delta^{-1}} T_\delta \mathbf{v}. \quad \text{q.e.d.} \end{aligned}$$

Notations. We use the following notations for $t > 0$,

$$(2.28) \quad F_\delta(t) \equiv \{g: t^{-1} \leq \delta(g) \leq t\},$$

and

$$(2.29) \quad \mathfrak{D}_{\delta,t} \equiv \{\mathbf{v} = (\mathbf{v}(g)) \in \mathfrak{D}: [\text{support of } \mathbf{v}(g)] \subset F_\delta(t)\}.$$

Obviously, $\mathfrak{D}_{\delta,t}$ is a closed subspace of \mathfrak{D} and,

Lemma 2.8. T_δ leaves invariant $\mathfrak{D}_{\delta,t}$, and the restriction $T_{\delta,t}$ on $\mathfrak{D}_{\delta,t}$ of T_δ satisfies

$$(2.30) \quad \|T_{\delta,t}\| = \|(T_{\delta,t})^{-1}\| = t.$$

Proof. This is trivial from the definitions of T_δ and $\mathfrak{D}_{\delta,t}$.

q. e. d.

In general, $U_f T_\delta$ is unbounded. But for a suitable ω , a suitable δ and a suitable function f , $U_f T_\delta$ becomes not only bounded but also of Hilbert-Schmidt type.

Proposition 2.1. Let $\mathfrak{D} = \{\mathfrak{D}, U_g\}$ be the unitary representation of a locally compact group G induced from a unitary representation $\omega = \{\mathcal{H}, W_h\}$ of a closed subgroup H . And let T_δ be the operator on \mathfrak{D} defined by (2.18) for a given real character δ satisfying (2.15) ~ (2.17). Then, for any f in $L^1(G)$,

$$(2.31) \quad \begin{aligned} \|T_\delta U_f\|_{\mathfrak{D}}^2 &= \int_{H \setminus G} \int_{H \setminus G} \|W_f(g_1, g_2)\|_{\mathcal{H}}^2 (A_G(g_1))^{-2} \\ &\quad \times (\delta(g_1))^2 d\nu(\tilde{g}_1) d\nu(\tilde{g}_2). \end{aligned}$$

Here,

$$(2.32) \quad W_f(g_1, g_2) = \int_H f(g_1^{-1} h g_2) \sqrt{\psi(g_1) \psi(h g_2)} W_h d_r h.$$

And (2.31) means that if the integral of the right hand side converges, then the operator $T_\delta U_f$ is of Hilbert-Schmidt type and its Hilbert-Schmidt norm is given by (2.31).

Proof. At first, for a vector \mathbf{v} in \mathfrak{D} ,

$$\begin{aligned}
 (2.33) \quad (T_\delta U_f \mathbf{v})(g) &= \delta(g) \int_G f(g_1) w(g, g_1) \mathbf{v}(gg_1) d_r g_1 \\
 &= \delta(g) \int_G f(g^{-1}g_1) w(g, g^{-1}g_1) \mathbf{v}(g_1) (\Delta_G(g))^{-1} d_r g_1 \\
 &= \delta(g) \Delta_G(g^{-1}) \int_{H \setminus G} \left\{ \int_H f(g^{-1}hg_1) \sqrt{\psi(g)/\psi(hg_1)} \right. \\
 &\quad \left. \times \mathbf{v}(hg_1) \psi(hg_1) d_r h \right\} d\nu(\tilde{g}_1) \\
 &= \int_{H \setminus G} \left\{ \delta(g) \Delta_G(g^{-1}) \int_H f(g^{-1}hg_1) \sqrt{\psi(g)\psi(hg_1)} \right. \\
 &\quad \left. \times \mathbf{v}(g_1) d\nu(\tilde{g}_1) \right\} \\
 &= \int_{H \setminus G} \delta(g) \Delta_G(g^{-1}) W_f(g, g_1) \mathbf{v}(g_1) d\nu(\tilde{g}_1).
 \end{aligned}$$

Therefore, for fixed complete orthonormal systems $\{\mathbf{v}_\alpha\}_{\alpha \in A}$ in \mathfrak{D} and $\{\mathbf{u}_\beta\}_{\beta \in B}$ in \mathcal{H} , the followings are valid for almost all g .

$$\begin{aligned}
 (2.34) \quad \langle (T_\delta U_f \mathbf{v}_\alpha)(g), \mathbf{u}_\beta \rangle_{\mathcal{H}} &= \int_{H \setminus G} \langle \delta(g) \Delta_G(g^{-1}) W_f(g, g_1) \mathbf{v}_\alpha(g_1), \mathbf{u}_\beta \rangle_{\mathcal{H}} d\nu(\tilde{g}_1), \\
 &= \int_{H \setminus G} \langle \mathbf{v}_\alpha(g_1), \delta(g) \Delta_G(g^{-1}) (W_f(g, g_1))^* \mathbf{u}_\beta \rangle_{\mathcal{H}} d\nu(\tilde{g}_1) \\
 &= \int_{H \setminus G} \langle \mathbf{v}_\alpha(g_1), \mathbf{f}_g^\beta(g_1) \rangle_{\mathcal{H}} d\nu(\tilde{g}_1) = \langle \mathbf{v}_\alpha, \mathbf{f}_g^\beta \rangle_{\mathfrak{D}}.
 \end{aligned}$$

Here we put

$$(2.35) \quad \mathbf{f}_g^\beta(g_1) = \delta(g) \Delta_G(g^{-1}) (W_f(g, g_1))^* \mathbf{u}_\beta.$$

\mathbf{f}_g^β belongs to \mathfrak{D} for suitable functions f . And the last term in (2.34) means the scalar product in \mathfrak{D} . So,

$$\begin{aligned}
(2.36) \quad \sum_{\alpha} |\langle \mathbf{v}_{\alpha}, \mathbf{f}_{\mathbf{g}}^{\beta} \rangle_{\mathfrak{H}}|^2 &= \|\mathbf{f}_{\mathbf{g}}^{\beta}\|_{\mathfrak{H}}^2 = \int_{H \setminus G} \|\mathbf{f}_{\mathbf{g}}^{\beta}(g_1)\|_{\mathfrak{X}}^2 d\nu(\tilde{g}_1) \\
&= \int_{H \setminus G} (\delta(g))^2 (\Delta_G(g))^{-2} \|\mathcal{W}_f(g, g_1) * \mathbf{u}_{\beta}\|_{\mathfrak{X}}^2 d\nu(\tilde{g}_1).
\end{aligned}$$

And

$$\begin{aligned}
(2.37) \quad \|\| T_{\delta} U_f \|\|_{\mathfrak{H}}^2 &= \sum_{\alpha \in A} \|T_{\delta} U_f \mathbf{v}_{\alpha}\|_{\mathfrak{H}}^2 = \sum_{\alpha \in A} \int_{H \setminus G} \|T_{\delta} U_f \mathbf{v}_{\alpha}(g_1)\|_{\mathfrak{X}}^2 d\nu(\tilde{g}_1) \\
&= \int_{H \setminus G} \sum_{\alpha \in A} \sum_{\beta \in B} |\langle (T_{\delta} U_f \mathbf{v}_{\alpha})(g_1), \mathbf{u}_{\beta} \rangle_{\mathfrak{X}}|^2 d\nu(\tilde{g}_1) \\
&= \int_{H \setminus G} \sum_{\beta \in B} \sum_{\alpha \in A} |\langle \mathbf{v}_{\alpha}, \mathbf{f}_{\mathbf{g}_1}^{\beta} \rangle_{\mathfrak{H}}|^2 d\nu(\tilde{g}_1) \\
&= \int_{H \setminus G} \sum_{\beta \in B} \int_{H \setminus G} \|\mathcal{W}_f(g_1, g_2) * \mathbf{u}_{\beta}\|_{\mathfrak{X}}^2 (\Delta_G(g_1))^{-2} (\delta(g_1))^2 d\nu(\tilde{g}_2) d\nu(\tilde{g}_1) \\
&= \int_{H \setminus G} \int_{H \setminus G} \|\mathcal{W}_f(g_1, g_2) * \mathbf{u}_{\beta}\|_{\mathfrak{X}}^2 (\Delta_G(g_1))^{-2} (\delta(g_1))^2 d\nu(\tilde{g}_1) d\nu(\tilde{g}_2) \\
&= \int_{H \setminus G} \int_{H \setminus G} \|\mathcal{W}_f(g_1, g_2)\|_{\mathfrak{X}}^2 (\Delta_G(g_1))^{-2} (\delta(g_1))^2 d\nu(\tilde{g}_1) d\nu(\tilde{g}_2).
\end{aligned}$$

This completes the proof.

Corollary. *When H is a normal subgroup of G , and when we take a right Haar measure $d_{\tau} \tilde{g}$ on $H \setminus G$ as the quasi-invariant measure ν ,*

$$(2.38) \quad \mathcal{W}_f(g_1, g_2) = \int_H f(g_1^{-1} h g_2) \mathcal{W}_h d_{\tau} h.$$

Proof. In this case, the function ψ , which determines ν in Lemma 2.2., can be taken as the constant function 1. (cf. Lemma 3.3. and its remark). For such a ψ , the corresponding quasi-invariant measure ν is equal to $d_{\tau} \tilde{g}$. And the result follows from (2.32) directly.

q.e.d.

Hereafter we shall consider only such a measure ν (i.e. $d_r\tilde{g}$) for a normal subgroup H of G , especially for the fully unimodular subgroup defined in §3.

When the subgroup H is not of type I, the operators $W_f(g_1, g_2)$ in (2.32) are not of Hilbert-Schmidt type in general, and the integral on the right-hand side of (2.31) diverges.

However, if the von Neumann algebra \mathfrak{A}_ω generated by the operators $\{W_h\}_{h \in H}$, is of semi-finite type, a trace τ_ω on the positive part of \mathfrak{A}_ω may be used instead of the ordinary trace of operators. So we consider the space \mathfrak{A}_0 of operators of the form

$$(2.39) \quad A \equiv T_\delta U_f, \quad \text{for } f \text{ in } L^1(G),$$

and define the sesqui-linear form on \mathfrak{A}_0 by

$$(2.40) \quad \begin{aligned} \tau_{\mathfrak{D}}(A^*A) &= \int_{H \setminus G} \int_{H \setminus G} \tau_\omega((W_f(g_1, g_2))^* W_f(g_1, g_2)) (\Delta_G(g_1))^{-2} \\ &\quad \times (\delta(g_1))^2 d\nu(\tilde{g}_1) d\nu(\tilde{g}_2). \\ &= \int_{H \setminus G} \tau_\omega(W_{(f \times \delta^2)^* f}(g, g)) (\Delta_G(g))^{-1} (\delta(g))^2 d\nu(\tilde{g}). \end{aligned}$$

It is desirable that $\tau_{\mathfrak{D}}$ gives a semi-finite trace on the positive part of some von Neumann algebra. But here, only the followings are proved.

Lemma 2.9. *For any A in \mathfrak{A}_0 ,*

$$(2.41) \quad \tau_{\mathfrak{D}}(A^*A) = \tau_{\mathfrak{D}}(AA^*).$$

Proof. Let A be the operator in (2.39), then

$$(2.42) \quad A^* = U_f^* T_\delta = U_{f^*} T_\delta = T_\delta U_{f^* \delta}.$$

Here,

$$(2.43) \quad f^*(g) \equiv \overline{f(g^{-1})} \Delta_G(g^{-1}).$$

$$\begin{aligned}
(2.44) \quad & \mathcal{W}_{f \times \delta}(g_1, g_2) \Delta_G(g_1^{-1}) \delta(g_1) \\
&= \Delta_G(g_1^{-1}) \delta(g_1) \int_H \overline{f(g_2^{-1} h^{-1} g_1)} \Delta_G(g_2^{-1} h^{-1} g_1) \delta(g_1^{-1} h g_2) \\
&\quad \times \sqrt{\psi(g_1) \psi(h g_2)} \mathcal{W}_h d_r h \\
&= \Delta_G(g_2^{-1}) \delta(g_2) \int_H \overline{f(g_2^{-1} h g_1)} \\
&\quad \times \sqrt{\frac{\psi(g_1) \Delta_G(h^{-1}) \psi(g_2)}{\Delta_H(h^{-1})}} \Delta_G(h) \mathcal{W}_{h^{-1}} d_r(h^{-1}) \\
&= (\mathcal{W}_f(g_2, g_1))^* \Delta_G(g_2^{-1}) \delta(g_2).
\end{aligned}$$

Because τ_ω is a trace, for any B in \mathfrak{A}_ω

$$(2.45) \quad \tau_\omega(BB^*) = \tau_\omega(B^*B).$$

Therefore,

$$\begin{aligned}
(2.46) \quad & \tau_{\mathfrak{D}}(AA^*) = \tau_{\mathfrak{D}}((A^*)^* A^*) = \tau_{\mathfrak{D}}((T_\delta U_{f \times \delta})^* T_\delta U_{f \times \delta}) \\
&= \int_{H \setminus G} \int_{H \setminus G} \tau_\omega((\mathcal{W}_{f \times \delta}(g_1, g_2))^* \mathcal{W}_{f \times \delta}(g_1, g_2)) (\Delta_G(g_1^{-1}))^2 \\
&\quad \times (\delta(g_1))^2 d\nu(\tilde{g}_1) d\nu(\tilde{g}_2) \\
&= \int_{H \setminus G} \int_{H \setminus G} \tau_\omega((\mathcal{W}_f(g_2, g_1))^* (\mathcal{W}_f(g_2, g_1))^*) (\Delta_G(g_2^{-1}))^2 \\
&\quad \times (\delta(g_2))^2 d\nu(\tilde{g}_1) d\nu(\tilde{g}_2) \\
&= \int_{H \setminus G} \int_{H \setminus G} \tau_\omega((\mathcal{W}_f(g_2, g_1))^* \mathcal{W}_f(g_2, g_1)) (\Delta_G(g_2))^{-2} (\delta(g_2))^2 \\
&\quad \times d\nu(\tilde{g}_2) d\nu(\tilde{g}_1) \\
&= \tau_{\mathfrak{D}}(A^*A).
\end{aligned}$$

This completes the proof.

Lemma 2.10. For any k, f_1, f_2 in $C_0(G)$,

$$(2.47) \quad \tau_{\mathfrak{D}}((T_{\delta}U_{f_2})^*T_{\delta}U_{k*f_1}) = \tau_{\mathfrak{D}}((T_{\delta}U_{(k*\delta^2\mathcal{A}^{-1/2})*f_2})^*T_{\delta}U_{f_1}).$$

Here we use an abbreviation of notation as

$$(2.48) \quad (f \cdot \mathcal{A}^{-1/2})(g) \equiv f(g)(\mathcal{A}_G(g))^{-1/2}.$$

Proof. Denote the left hand side of (2.47) by I .

$$(2.49) \quad \begin{aligned} I &= \tau_{\mathfrak{D}}((U_{f_2})^*(T_{\delta})^2U_{k*}U_{f_1}) = \tau_{\mathfrak{D}}((U_{f_2})^*U_{k\delta^{-2}}(T_{\delta})^2U_{f_1}) \\ &= \tau_{\mathfrak{D}}((T_{\delta}(U_{k\delta^{-2}})^*U_{f_2})^*T_{\delta}U_{f_1}). \end{aligned}$$

On the other hand,

$$(2.50) \quad \begin{aligned} (U_{k\delta^{-2}})^* &= \int_G \overline{k(g)} \delta(g)^{-2} (U_{g^{-1}}) d_r g \\ &= \int_G \overline{k(g^{-1})} (\delta(g))^2 (\mathcal{A}_G(g))^{-1} U_g d_r g \\ &= \int_G k^*(g) (\delta(g))^2 (\mathcal{A}_G(g))^{-1/2} U_g d_r g = U_{k*\delta^2\mathcal{A}^{-1/2}}. \end{aligned}$$

Therefore,

$$(2.51) \quad I = \tau_{\mathfrak{D}}((T_{\delta}U_{(k*\delta^2\mathcal{A}^{-1/2})*f_2})^*T_{\delta}U_{f_1}). \quad \text{q.e.d.}$$

Lemma 2.11. For any f_1, f_2, f_3 in $C_0(G)$

$$(2.52) \quad \tau_{\mathfrak{D}}((T_{\delta}U_{(f_1\delta*f_2\delta)})^*T_{\delta}U_{f_3}) = \tau_{\mathfrak{D}}((T_{\delta}U_{f_1*f_2})^*T_{\delta}U_{f_3\delta}).$$

Proof. At first,

$$(2.53) \quad \begin{aligned} (f_1\delta)*(f_2\delta)(g) &= \int_G f_1(gg_1^{-1})\delta(gg_1^{-1})f_2(g_1)\delta(g_1)d_r g_1 \\ &= ((f_1*f_2)\delta)(g). \end{aligned}$$

$$\begin{aligned}
(2.54) \quad W_{f\delta}(g_1, g_2) &= \int_G f(g_1^{-1}h g_2) \delta(g_1^{-1}h g_2) \sqrt{\psi(g_1)\psi(h g_2)} W_h d_r h \\
&= \delta(g_1^{-1}g_2) W_f(g_1, g_2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(2.55) \quad & \tau_{\mathfrak{D}}((T_{\delta}U_{f_1\delta*f_2\delta})^* T_{\delta}U_{f_3}) \\
&= \int_{H \setminus G} \int_{H \setminus G} \tau_{\omega}((W_{(f_1*f_2)\delta}(g_1, g_2))^* W_{f_3}(g_1, g_2)) (\Delta_G(g_1))^{-2} \\
&\quad \times (\delta(g_1))^2 d\nu(\tilde{g}_1) d\nu(\tilde{g}_2) \\
&= \int_{H \setminus G} \int_{H \setminus G} \tau_{\omega}((\delta(g_1^{-1}g_2) W_{f_1*f_2}(g_1, g_2))^* W_{f_3}(g_1, g_2)) \\
&\quad \times (\Delta_G(g_1))^{-2} (\delta(g_1))^2 d\nu(\tilde{g}_1) d\nu(\tilde{g}_2) \\
&= \int_{H \setminus G} \int_{H \setminus G} \tau_{\omega}((W_{f_1*f_2}(g_1, g_2))^* W_{f_3\delta}(g_1, g_2)) (\Delta_G(g_1))^{-2} \\
&\quad \times (\delta(g_1))^2 d\nu(\tilde{g}_1) d\nu(\tilde{g}_2) \\
&= \tau_{\mathfrak{D}}((T_{\delta}U_{f_1*f_2})^* T_{\delta}U_{f_3\delta}). \qquad \text{q. e. d.}
\end{aligned}$$

We consider the case that H is a normal subgroup of G .

Lemma 2.12. *In this case, for fixed k in $C_0(G)$,*

$$\begin{aligned}
(2.56) \quad & |\tau_{\mathfrak{D}}((T_{\delta}U_{f_1})^* T_{\delta}U_k \cdot U_{f_2})| \\
& \leq c_k \{ \tau_{\mathfrak{D}}((T_{\delta}U_{f_1})^* T_{\delta}U_{f_1}) \cdot \tau_{\mathfrak{D}}((T_{\delta}U_{f_2})^* T_{\delta}U_{f_2}) \}^{1/2},
\end{aligned}$$

for any $f_1, f_2 \in C_0(G)$.

Here c_k is a positive constant depending only on the function k .

Proof. At first, from the non-negative definiteness of τ_{ω} ,

$$(2.57) \quad |\tau_\omega((W_{f_1}(g_1, g_2))^* W_{f_2}(g_1, g_2))| \\ \leq \tau_\omega((W_{f_1}(g_1, g_2))^* W_{f_1}(g_1, g_2)) \tau_\omega((W_{f_2}(g_1, g_2))^* W_{f_2}(g_1, g_2)).$$

Since

$$(2.58) \quad U_k \cdot U_{f_2} = U_{k*f_2},$$

$$(2.59) \quad I \equiv |\tau_\mathfrak{D}((T_\delta U_{f_1})^* T_\delta U_k \cdot U_{f_2})| = |\tau_\mathfrak{D}((T_\delta U_{f_1})^* T_\delta U_{k*f_2})| \\ = \left| \int_{H \setminus G} \int_{H \setminus G} \tau_\omega((W_{f_1}(g_1, g_2))^* W_{k*f_2}(g_1, g_2)) (\Delta_G(g_1))^{-2} \right. \\ \left. \times (\delta(g_1))^2 d_r \tilde{g}_1 d_r \tilde{g}_2 \right|.$$

Here,

$$(2.60) \quad W_{f_1}(g_1, g_2) = \int_H f_1(g_1^{-1} h g_2) W_h d_r h,$$

$$(2.61) \quad W_{k*f_2}(g_1, g_2) = \int_H (k*f)(g_1^{-1} h g_2) W_h d_r h \\ = \int_H \int_G k(g^{-1}) f_2(g g_1^{-1} h g_2) W_h d_r g d_r h \\ = \int_G k(g^{-1}) W_{f_2}(g_1 g^{-1}, g_2) d_r g.$$

Thus

$$(2.62) \quad I = \left| \int_{H \setminus G} \int_{H \setminus G} \int_G k(g^{-1}) \tau_\omega((W_{f_1}(g_1, g_2))^* W_{f_2}(g_1 g^{-1}, g_2)) \right. \\ \left. \times (\Delta_G(g_1))^{-2} (\delta(g_1))^2 d_r g d_r \tilde{g}_1 d_r \tilde{g}_2 \right| \\ \leq \int_G |k(g^{-1})| \left\{ \int_{H \setminus G} \int_{H \setminus G} |\tau_\omega((W_{f_1}(g_1, g_2))^* W_{f_2}(g_1 g^{-1}, g_2))| \right. \\ \left. \times (\Delta_G(g_1))^{-2} (\delta(g_1))^2 d_r \tilde{g}_1 d_r \tilde{g}_2 \right\} d_r g$$

$$\begin{aligned}
& \leq \int_G |k(g^{-1})| \left\{ \int_{H \setminus G} \int_{H \setminus G} (\tau_\omega((W_{f_1}(g_1, g_2))^* W_{f_1}(g_1, g_2))) \right. \\
& \quad \times \tau_\omega((W_{f_2}(g_1 g^{-1}, g_2))^* W_{f_2}(g_1 g^{-1}, g_2)))^{1/2} (\Delta_G(g_1))^{-2} \\
& \quad \left. \times (\delta(g_1))^2 d_r \tilde{g}_1 d_r \tilde{g}_2 \right\} d_r g \\
& \leq \int_G |k(g^{-1})| \left\{ \int_{H \setminus G} \int_{H \setminus G} \tau_\omega((W_{f_1}(g_1, g_2))^* W_{f_1}(g_1, g_2)) \right. \\
& \quad \times (\Delta_G(g_1))^{-2} (\delta(g_1))^2 d_r \tilde{g}_1 d_r \tilde{g}_2 \left. \right\}^{1/2} \\
& \quad \times \left\{ \int_{H \setminus G} \int_{H \setminus G} \tau_\omega((W_{f_2}(g_1 g^{-1}, g_2))^* W_{f_2}(g_1 g^{-1}, g_2)) (\Delta_G(g_1))^{-2} \right. \\
& \quad \left. \times (\delta(g_1))^2 d_r \tilde{g}_1 d_r \tilde{g}_2 \right\}^{1/2} d_r g \\
& = \int_G |k(g^{-1})| \left\{ \int_{H \setminus G} \int_{H \setminus G} \tau_\omega((W_{f_2}(g_1, g_2))^* W_{f_2}(g_1, g_2)) \right. \\
& \quad \times (\Delta_G(g_1))^{-2} (\delta(g_1))^2 (\Delta_G(g))^{-2} (\delta(g))^2 d_r \tilde{g}_1 d_r \tilde{g}_2 \left. \right\}^{1/2} d_r g \\
& \quad \times \{ \tau_{\mathfrak{D}}((T_{\delta} U_{f_1})^* T_{\delta} U_{f_1}) \}^{1/2} \\
& = \left\{ \int_G |k(g^{-1})| (\Delta_G(g))^{-2} (\delta(g))^2 d_r g \right\} \\
& \quad \times \{ \tau_{\mathfrak{D}}((T_{\delta} U_{f_1})^* T_{\delta} U_{f_1}) \tau_{\mathfrak{D}}((T_{\delta} U_{f_2})^* T_{\delta} U_{f_2}) \}^{1/2}.
\end{aligned}$$

Putting in (2.62),

$$(2.63) \quad c_k = \int_G |k(g^{-1})| (\Delta_G(g))^{-2} (\delta(g))^2 d_r g.$$

We obtain the result.

q. e. d.

When H is a unimodular group, the central decomposition (1.2) of the regular representation \mathfrak{R}_H of H gives semi-finite factor representations ω of H , except a subset of μ -measure zero. For such a factor re-

presentation ω , a canonical trace τ_ω is determined up to constant, as is stated in §1. Therefore, in this case, we obtain a canonical sesquilinear form $\tau_{\mathfrak{D}}$ by (2.40), using τ_ω for such a ω .

Lemma 2.13. *For μ -almost all ω ,
 $A=0$, if and only if,*

$$(2.64) \quad \tau_{\mathfrak{D}}(A^*A)=0.$$

Proof. From the theory of F. I. Mautner [9] [10] and I. E. Segal [13] (cf. §1), for μ -almost all ω , τ_ω is a faithful trace. Since ν is positive, (2.64) is equivalent to

$$(2.65) \quad W_f(g_1, g_2)=0, \quad \text{for almost all } (g_1, g_2).$$

But (2.33) shows that this is equivalent to

$$(2.66) \quad (A\mathbf{v})(g)=(T_\delta U_f \mathbf{v})(g)=\int_{H \setminus G} \delta(g) \Delta_G(g^{-1}) W_f(g, g_1) \mathbf{v}(g_1) d\nu(\tilde{g}_1) \\ =0, \quad \text{for any } \mathbf{v} \text{ and almost all } g.$$

This completes the proof.

§3. Automorphisms over unimodular groups.

Let σ be an automorphism^{*)} on a unimodular locally compact group H . For any unitary representation $\omega=\{\mathcal{H}, W_h(\omega)\}$ of H , $\{W_{\sigma^{-1}(h)}(\omega)\}$ gives a unitary representation of H on \mathcal{H} too. Put this representation

$$(3.1) \quad \sigma(\omega) \equiv \{\mathcal{H}, W_h(\sigma(\omega))\} = \{\mathcal{H}, W_{\sigma^{-1}(h)}(\omega)\}.$$

Obviously, σ preserves irreducibility and the property being a factor representation, so σ induces a conjugate transformation over the dual or quasi-dual of H . Moreover σ maps the regular representation \mathfrak{R}_H of H

^{*)} In the definition of automorphisms, we assume continuity. It is easily shown that only the measurability of an automorphism deduces its continuity and the continuity of its inverse.

to itself. So this conjugate transformation leaves invariant the reduced dual or reduced quasi-dual of H .

While, since the transformed measure $d\sigma(h)$ of the Haar measure dh is also a Haar measure, the modulus $\Delta_\sigma \equiv d\sigma(h)/dh$ is a constant on H .

In the same way, the restriction of σ on the reduced quasi-dual \mathcal{Q} of H transforms the Plancherel measure μ to $\sigma(\mu)$ defined by

$$(3.2) \quad \sigma(\mu)(E) = \mu(\sigma^{-1}(E)), \quad \text{for any measurable set } E \text{ in } \mathcal{Q}.$$

Now we shall show the followings.

Lemma 3.1. *The modulus $d\sigma(\mu)/d\mu$ is constant, and*

$$(3.3) \quad d\sigma(\mu)/d\mu = \Delta_\sigma.$$

Before stating the proof of this lemma, we introduce a notation

$$(3.4) \quad \sigma(k)(h) = k(\sigma^{-1}(h)),$$

for any function k on H , and show the following auxiliary lemma.

Lemma 3.2. *For any function k in $L^1(H)$,*

$$(3.5) \quad W_k(\sigma(\omega)) = \Delta_\sigma W_{\sigma^{-1}(k)}(\omega).$$

Here

$$(3.6) \quad W_k(\omega) = \int_H k(h) W_h(\omega) dh.$$

Proof.

$$\begin{aligned} (3.7) \quad W_k(\sigma(\omega)) &= \int_H k(h) W_h(\sigma(\omega)) dh = \int_H k(h) W_{\sigma^{-1}(h)}(\omega) dh \\ &= \int_H k(\sigma(h)) W_h(\omega) d\sigma(h) = \int_H \sigma^{-1}(k)(h) W_h(\omega) \Delta_\sigma dh \\ &= \Delta_\sigma W_{\sigma^{-1}(k)}(\omega). \end{aligned} \quad \text{q. e. d.}$$

Proof of Lemma 3.1. At first, it must be remarked that for two representations ω and $\sigma(\omega)$, the spaces of representations are common, moreover the von Neumann algebras generated by their operators are the same. Therefore, from the assumption of normalizations of traces τ_ω 's in §1, we can consider the traces τ_ω and $\tau_{\sigma(\omega)}$ are just the same.

Now, by the Plancherel formula given in §1, for any k in $L^1(H) \cap L^2(H)$,

$$\begin{aligned}
 (3.8) \quad & \int_H |k(\sigma(h))|^2 dh = \int_H |\sigma^{-1}(k)(h)|^2 dh \\
 & = \int_{\mathfrak{g}} \tau_\omega((W_{\sigma^{-1}(k)}(\omega))^* W_{\sigma^{-1}(k)}(\omega)) d\mu(\omega) \\
 & = \int_{\mathfrak{g}} \tau_{\sigma(\omega)}((W_k(\sigma(\omega)))^* W_k(\sigma(\omega))) (\Delta_\sigma)^{-2} d\mu(\omega) \\
 & = \int_{\mathfrak{g}} \tau_\omega((W_k(\omega))^* W_k(\omega)) (\Delta_\sigma)^{-2} d\mu(\sigma^{-1}(\omega)) \\
 & = (\Delta_\sigma)^{-2} \int_{\mathfrak{g}} \tau_\omega((W_k(\omega))^* W_k(\omega)) d\sigma(\mu)(\omega).
 \end{aligned}$$

On the other hand, the left hand side is equal to

$$\begin{aligned}
 (3.9) \quad & \int_H |k(h)|^2 d\sigma^{-1}(h) = \int_H |k(h)|^2 (\Delta_\sigma)^{-1} dh \\
 & = (\Delta_\sigma)^{-1} \int_{\mathfrak{g}} \tau_\omega((W_k(\omega))^* W_k(\omega)) d\mu(\omega).
 \end{aligned}$$

The arbitrariness of the function k leads us to

$$(3.10) \quad (\Delta_\sigma)^{-2} d\sigma(\mu)(\omega) = (\Delta_\sigma)^{-1} d\mu(\omega).$$

This completes the proof.

Now we shall restrict ourselves to the case that H is a normal

subgroup^{*)} of a locally compact group G . In this case, an inner automorphism on G induce an automorphism on H as

$$(3.11) \quad g(h) \equiv ghg^{-1}.$$

Use the notation Δ_g for Δ_σ as above.

But in this case, the factor space $H \backslash G$ becomes a group \tilde{G} . Denote the H -coset containing g by \tilde{g} . By the reason of Lemma 2.3, the right Haar measure $d_r \tilde{g}$ over \tilde{G} can be taken as a quasi-invariant measure over $H \backslash G$ and we can choose ψ in the Lemma 2.2, such that, for any f in $L^1(G)$,

$$(3.12) \quad \int_G f(g) d_r g = \int_{\tilde{G}} d_r \tilde{g} \int_H f(hg) \psi(hg) d_r h.$$

Put

$$(3.13) \quad \Delta_{\tilde{G}}(\tilde{g}_1) = d_r(\widetilde{g_1 g}) / d_r \tilde{g}.$$

Lemma 3.3. For ψ chosen as above,

$$(3.14) \quad \psi(g) = \text{constant}.$$

Proof. From (3.12),

$$(3.15) \quad \begin{aligned} \int_G f(g) \psi^{-1}(gg_1) d_r g &= \int_G f(gg_1^{-1}) \psi^{-1}(g) d_r g \\ &= \int_{H \backslash G} d_r \tilde{g} \left\{ \int_H f(hg g_1^{-1}) d_r h \right\} \\ &= \int_{H \backslash G} d_r(\widetilde{g g_1}) \left\{ \int_H f(hg) d_r h \right\} = \int_{H \backslash G} d_r \tilde{g} \left\{ \int_H f(hg) d_r h \right\} \\ &= \int_G f(g) \psi^{-1}(g) d_r g. \end{aligned}$$

^{*)} Here H does not need to be unimodular, so we consider the right Haar measure $d_r h$ on H .

This means that for continuous function ψ .

$$(3.16) \quad \psi^{-1}(gg_1) = \psi^{-1}(g), \quad \text{for any } g_1 \text{ in } G. \quad \text{q. e. d.}$$

By suitable normalization of the measure $d_r \tilde{g}$, we can consider this constant is equal to one, that is

$$(3.17) \quad \psi(g) \equiv 1,$$

Lemma 3.4.

$$(3.18) \quad \Delta_g = (\Delta_G(g) / \Delta_{\tilde{G}}(\tilde{g})).$$

Proof. For any f in $L^1(G)$,

$$\begin{aligned} (3.19) \quad \int_{H \setminus G} d_r \tilde{g} \left\{ \int_H f(hg) d_r h \right\} &= \int_G f(g) d_r g = \int_G f(g) (\Delta_G(g_1))^{-1} d_r (g_1 g) \\ &= (\Delta_G(g_1))^{-1} \int_G f(g_1^{-1} g) d_r g = (\Delta_G(g_1))^{-1} \int_{H \setminus G} d_r \tilde{g} \left\{ \int_H f(g_1^{-1} h g) d_r h \right\} \\ &= (\Delta_G(g_1))^{-1} \int_{H \setminus G} d_r \tilde{g} \left\{ \int_H f(g_1^{-1} h g_1 g_1^{-1} g) d_r h \right\} \\ &= (\Delta_G(g_1))^{-1} \int_{H \setminus G} d_r (\widetilde{g_1 g}) \left\{ \int_H f(hg) d_r (g_1 h g_1^{-1}) \right\} \\ &= (\Delta_G(g_1))^{-1} \int_{H \setminus G} \Delta_{\tilde{G}}(\tilde{g}_1) d_r(\tilde{g}) \left\{ \int_H f(hg) d_r(g_1(h)) \right\} \\ &= (\Delta_{\tilde{G}}(\tilde{g}_1) / \Delta_G(g_1)) \int_{H \setminus G} d_r(\tilde{g}) \left\{ \int_H f(hg) \Delta_{g_1} d_r h \right\}. \end{aligned}$$

This shows

$$(3.20) \quad (\Delta_G(g_1) / \Delta_{\tilde{G}}(\tilde{g}_1))^{-1} \Delta_{g_1} = 1.$$

This completes the proof.

Next, consider the more restricted case, That is, G is a non-unimodular locally compact group and H is the normal subgroup of G , which is equal to just the kernel of the modular function $\Delta_G(g)$.

$$(3.21) \quad H = \{g \in G; \Delta_G(g) = 1\}.$$

Lemma 3.5. *H is a unimodular group.*

Proof. In (3.18), put $g=h$ in H , then

$$(3.22) \quad \Delta_h = 1, \quad \text{for any } h \text{ in } H.$$

But since $\Delta_h = \Delta_H(h)$, the assertion follows.

Definition 3.1. *We shall call this subgroup H , the fully unimodular subgroup of G .*

For the fully unimodular subgroup H , the factor group $\tilde{G} = H \backslash G$ is (algebraically*) isomorphic to a subgroup D of the multiplicative group \mathbf{R}_+^* of all positive numbers by the mapping.

$$(3.23) \quad g \rightarrow \Delta_G(g).$$

Evidently D is abelian, then the Haar measure $d_r \tilde{g}$ on \tilde{G} is two-sided invariant and especially,

$$(3.24) \quad \Delta_{\tilde{G}}(\tilde{g}) = 1, \quad \text{for any } \tilde{g} \text{ in } \tilde{G}.$$

Therefore, by Lemma 3.4,

Lemma 3.6. *If H is the fully unimodular subgroup of G ,*

$$(3.25) \quad \Delta_g = \Delta_G(g).$$

Combining to Lemma 3.1, we obtain

*) In general this isomorphism is continuous but not topological.

Proposition 3.1. *For the fully unimodular subgroup H of a non-unimodular locally compact group G ,*

$$(3.26) \quad dg(\mu)/d\mu = \Delta_G(g).$$

§4. Orbits space on the quasi-dual of the fully unimodular subgroup.

Hereafter, we restrict ourselves only to the case that G is a separable non-unimodular locally compact group, and H is the fully unimodular subgroup of G .

From the arguments in §3, the restrictions of the inner automorphisms of G to H induce transformations over the quasi-reduced dual Ω of H . Thus G is considered as a group of transformations over Ω .

Definition 4.1. *For any ω in Ω , the subset*

$$(4.1) \quad x_\omega \equiv \{g(\omega) : g \in G\},$$

of Ω is called G -orbit passing through ω . And the set of all G -orbits is denoted by X . Moreover, define the map φ from Ω onto X by

$$(4.2) \quad \varphi(\omega) \equiv x_\omega.$$

We introduce a Borel structure in X which is generated by the subsets E of X such that $\varphi^{-1}(E)$ are measurable in Ω .

Lemma 4.1. *There exists a subset N of Ω of μ -measure zero, such that, any two different elements ω_1, ω_2 in $\Omega - N$, the double representations $\mathfrak{D}_{\omega_1} = \{\mathcal{H}(\omega_1), W_h(\omega_1), V_k(\omega_1)\}$ and $\mathfrak{D}_{\omega_2} = \{\mathcal{H}(\omega_2), W_h(\omega_2), V_k(\omega_2)\}$ are not equivalent.*

Proof. As is shown in §1, $\mathfrak{D}_H = \{L^2(H), R_h^H, L_k^H\}$ is a multiplicity free representation of $H \times H$ in the sense of G. W. Mackey[7], and (1.2) gives the central decomposition of \mathfrak{D}_H . So we can apply the results of A. Guichardet [3] and M. A. Naimark [11], and obtain the

assertion.

Corollary 1. *For any ω in $\Omega - N$, the isotropy subgroup*

$$(4.3) \quad G_\omega \equiv \{g; g^{-1}(\omega) \cong \omega\}$$

contains H .

Proof. Obviously, for any h_0 in H the operator $W_{h_0}(\omega)V_{h_0}(\omega)$ gives a unitary equivalence of double representations \mathfrak{D}_ω and $\mathfrak{D}_{h_0(\omega)}$. Therefore, the result is deduced from Lemma 4.1, directly.

Corollary 2. *For any ω in $\Omega - N$ and any g in G ,*

$$(4.4) \quad G_\omega = G_{g(\omega)}.$$

Proof. By the reason of the Corollary 1,

$$(4.5) \quad G_\omega \supset H.$$

But $H \setminus G$ is abelian. So for any g_0 in G_ω and any g in G , there exists an element h in H , such that

$$(4.6) \quad g_0 g = g g_0 h.$$

That is,

$$(4.7) \quad g_0(g(\omega)) = g_0 g(\omega) = g g_0 h(\omega) = g g_0(h(\omega)) = g g_0(\omega) = g(\omega).$$

This means $g_0 \in G_{g(\omega)}$, so

$$(4.8) \quad G_\omega \subseteq G_{g(\omega)}.$$

Changing the roles of ω and $g(\omega)$, we get the result.

Lemma 4.2. *The map*

$$(4.9) \quad (g, \omega) \rightarrow g^{-1}(\omega)$$

is a Borel measurable map of $G \times \Omega$ onto Ω .

Proof. The proof is analogous to that of the Theorem 7.3, in G. W. Mackey's paper [8].

To continue the discussions, we must put the following assumption with respect to a regularity of the G -orbits space X .

Assumption. *The G -orbits space X is countably separated in the sense of G. W. Mackey. (cf. [7]).*

That is, there are countable G -invariant measurable subsets of $\Omega - N$, and each G -orbit is the intersection of such sets containing it.

Lemma 4.3. (V. A. Rohlin [12]). *Under the above assumption, there are a measure $\tilde{\mu}$ on X and measures μ_x on Ω whose supports is in the corresponding G -orbit x , and*

$$(4.10) \quad \int_{\Omega} f(\omega) d\mu(\omega) = \int_X d\tilde{\mu}(x) \left\{ \int_x f(\omega) d\mu_x(\omega) \right\},$$

for any μ -summable function f on Ω .

Because of the assumption, each G -orbit is measurable in Ω . Therefore μ_x is considered as a measure on the space x .

According to the decomposition (1.2) of \mathfrak{D}_H , $L^2(H)$ is shown as a direct integral $\int_{\mathcal{Q}} \mathcal{H}(\omega) d\mu(\omega)$. And any element f in $L^2(H)$ corresponds to a vector-valued function $\mathbf{v}_f(\omega)$ on Ω . Moreover by the decomposition (4.10) of the measure μ , we obtain a weaker decomposition of $L^2(H)$.

$$(4.11) \quad L^2(H) \sim \int_X \mathfrak{F}^x d\tilde{\mu}(x).$$

Here,

$$(4.12) \quad \mathfrak{H}^x \sim \int_x \mathcal{H}(\omega) d\mu_x(\omega).$$

That is, \mathfrak{H}^x is considered as the space of vector-valued functions $\mathbf{v}_f(\omega)$ on x , such that

$$(4.13) \quad \|\mathbf{v}_f\|_x^2 \equiv \int_x \|\mathbf{v}_f(\omega)\|^2 d\mu_x(\omega) < +\infty.$$

While we must remark the followings. As is assumed in §1, for any ω in Ω and g in G , the spaces $\mathcal{H}(\omega)$ and $\mathcal{H}(g(\omega))$ must be identified in some canonical way. So that we fix a Hilbert space \mathcal{H}^x for each G -orbit x and $\mathbf{v}_f(\omega)$ is considered as \mathcal{H}^x -vector valued function, for which x is the G -orbit passing through ω .

As a canonical method to do this, we take a representative $\omega(x)$ for each G -orbit x , and a Borel section $E(x)$ of $G_{\omega(x)}$ -left-coset space in G . And put

$$(4.14) \quad \mathcal{H}^x = \mathcal{H}(\omega(x)).$$

Then for any ω in Ω , there exist unique representative $\omega(x)$ of G -orbit x passing through ω , and unique g_1 in $E(x)$ such that,

$$(4.15) \quad \omega = g_1^{-1}(\omega(x)).$$

Thus, we realize $\{\mathcal{H}(\omega), W_h(\omega), V_k(\omega)\}$ by $\{\mathcal{H}^x, W_h(g_1^{-1}(\omega(x))), V_k(g_1^{-1}(\omega(x)))\}$.

While as is shown in §3, we can define a unitary operator

$$(4.16) \quad (U_\sigma f)(h) \equiv f(\sigma^{-1}(h)) A_\sigma^{-1/2},$$

on $L^2(H)$, for any automorphism σ on H . Evidently, the map $\sigma \rightarrow U_\sigma$ is an algebraic homomorphism of the group $A(H)$ of automorphisms on H into the group of unitary operators on $L^2(H)$. And if the topology of uniform convergence on any compact set is introduced into

$A(H)$, then $\{L^2(H), U_\sigma\}$ gives a unitary representation of $A(H)$. Especially, in the case that H is a normal subgroup of locally compact group G , the restriction to H of the inner automorphism induced by an element g of G deduces an unitary representation $\mathfrak{D} \equiv \{L^2(H), U_g\}$ of G in this way. Thus

$$(4.17) \quad (U_g f)(h) = f(g^{-1}(h))\Delta_g^{-1/2} = f(g^{-1}hg)(\Delta_G(g))^{-1/2}.$$

Lemma 4.4. *For any g in G , the operator U_g is decomposable with respect to the decomposition (4.11).*

Proof. The decomposition (4.11) is done by the abelian von Neumann algebra of all G -invariant operators in the centre of the von Neumann algebra generated by $\{R_h^H\}_{h \in H}$. But, since “ G -invariant” is equivalent to “commuting with any U_g ”, the assertion is trivial.

q. e. d.

Basing upon Lemma 4.4, Let \mathfrak{D} be decomposed as follows.

$$(4.18) \quad \mathfrak{D} \equiv \{L^2(H), U_g\} \simeq \int_X \{\mathfrak{F}^x, U_g(x)\} d\bar{\mu}(x).$$

Next we shall determine the forms of operators $U_g(x)$ on each \mathfrak{F}^x .

Lemma 4.5.

$$(4.19) \quad U_g R_h^H = R_{ghg^{-1}}^H U_g = R_{g(h)}^H U_g,$$

$$(4.20) \quad U_g L_h^H = L_{ghg^{-1}}^H U_g = L_{g(h)}^H U_g.$$

Proof.

$$\begin{aligned} (U_g R_h^H f)(h_1) &= (R_h^H f)(g^{-1}h_1g)(\Delta_G(g))^{-1/2} = f(g^{-1}h_1gh)(\Delta_G(g))^{-1/2} \\ &= f(g^{-1}h_1(ghg^{-1})g)(\Delta_G(g))^{-1/2} = (U_g f)(h_1ghg^{-1}) = (R_{ghg^{-1}}^H U_g f)(h_1). \end{aligned}$$

$$(U_g L_h^H f)(h_1) = f(h^{-1}g^{-1}h_1g)(\Delta_g(g))^{-1/2} = (L_{ghg^{-1}}^H U_g f)(h_1). \quad \text{q. e. d.}$$

We denote the decomposition of \mathfrak{D}_H according to (4.11) as

$$(4.21) \quad \{L^2(H), R_h^H, L_k^H\} \simeq \int_X \{\mathfrak{D}^x, W_h^x, V_k^x\} d\tilde{\mu}(x).$$

Then the components of this decomposition are shown by

$$(4.22) \quad \{\mathfrak{D}^x, W_h^x, V_k^x\} \simeq \int_x \{\mathcal{H}(\omega), W_h(\omega), V_k(\omega)\} d\mu_x(\omega).$$

Lemma 4.6. For $\tilde{\mu}$ -almost all x , $U_g(x)$ is given by

$$(4.23) \quad (U_g(x)\mathbf{v})(\omega) = U(\omega, g)\mathbf{v}(g^{-1}(\omega))\Delta_G(g)^{1/2}.$$

Here $U(\omega, g)$ are unitary operators on \mathcal{H}^x .

Proof. (4.19) and (4.20) lead us to

$$(4.24) \quad U_g(x)W_h^x = W_{g(h)}^x U_g(x),$$

$$(4.25) \quad U_g(x)V_k^x = V_{g(k)}^x U_g(x),$$

for $\tilde{\mu}$ -almost all x .

But these relations assert that $\{U_g(x)\}$ and the family of all decomposable projections on $\mathfrak{D}^x \sim \int_x \mathcal{H}(\omega) d\mu_x(\omega)$ give a transitive system of imprimitivity on the base x in the sense of G. W. Mackey (cf. [8], Th. 5.6). Therefore, there exists a family of unitary operators $\{U(\omega, g)\}$ on \mathcal{H}^x for which (4.23) is valid. q. e. d.

Proposition 4.1. For μ -almost all ω .

$$(4.26) \quad G_\omega = H.$$

Proof. It is sufficient to see that for any ω_0 in $\mathcal{Q} - N$, for which (4.23) is valid, G_{ω_0} is contained in H .

If it is not, then there is an element g_0 in G , which does not contained in H , and

$$(4.27) \quad g_0^{-1}(\omega_0) \cong \omega_0.$$

Because of Corollary 2 of Lemma 4.1, for any ω in the orbit x passing through ω_0 ,

$$(4.28) \quad g_0^{-1}(\omega) \cong \omega.$$

Hence, using (4.23), for any vector \mathbf{v} in \mathfrak{F}^x ,

$$(4.29) \quad (U_{g_0}(x)\mathbf{v})(\omega) = U(\omega, g_0)\mathbf{v}(\omega)(\Delta_G(g_0))^{1/2}$$

Since g_0 does not contained in H , $(\Delta_G(g_0))^{1/2}$ is not equal to one. This contradicts to that $U_{g_0}(x)$ is unitary. q.e.d.

Corollary. 1. *For μ -almost all ω , the map $\tilde{g} \rightarrow g(\omega)$ gives a Borel isomorphism of $H \backslash G$ onto x which passes through ω .*

Proof. Since G is separable, the separable locally compact space $H \backslash G$ is a standerd Borel space. While, since H is separable, $\mathcal{Q} - N'$ ($\mu(N')=0$) is a standerd Borel space (cf. §1), especially countable generated. Theorem 3.2 of G. W. Mackey's paper [7] results that the one-to-one map $\tilde{g} \rightarrow g(\omega)$ is Borel isomorphic. q.e.d.

Using Corollary 1 of Proposition 4.1, for μ -almost all ω we can introduce a measure μ_x^0 on $H \backslash G$, which is transfered from the measure μ_x on x by the map $\tilde{g} \rightarrow g(\omega)$. Thus

Corollary 2. *For $\bar{\mu}$ -almost all x ,*

$$(4.30) \quad d\mu_x^0(\tilde{g}) = c \Delta_G(g)^{-1} d\tilde{g}.$$

Here $d\tilde{g}$ shows a Haar measure on $\tilde{G} = H \backslash G$, and c is a positive constant.

Proof. (3.26) in Proposition 3.1 asserts that μ satisfies

$$(4.31) \quad dg(\mu)/d\mu = \Delta_G(g), \quad \text{for any } g.$$

Decompose μ to the integral (4.10) of measures μ_x , we get that for almost all x , μ_x satisfy analogous equations. This means that μ_x^0 is a relatively invariant measure over \tilde{G} , satisfying,

$$(4.32) \quad d\mu_x^0(\tilde{g}_1\tilde{g})/d\mu_x^0(\tilde{g}) = \Delta_G(g_1)^{-1}, \quad \text{for any } g_1.$$

That is, $\Delta_G(g)d\mu_x^0(\tilde{g})$ is a Haar measure on \tilde{G} . The uniqueness of Haar measure on \tilde{G} deduces the result. q. e. d.

For the sake of later uses, we shall determine the form of operators $U(\omega, g)$ in (4.23).

By the reason of Proposition 4.1, we can assume that $G_\omega = H$. So we take a Borel section E of H -cosets G in independently on x , and give the above mentioned canonical form of $\{\mathcal{A}(\omega), W_h(\omega), V_k(\omega)\}$ by this Borel section. Moreover we can take E in such a way that

$$(4.33) \quad E \cap H = \{e\}.$$

Lemma 4.7. *By adequate normalization of $\mathcal{A}(\omega)$, multiplying a number of absolute value 1, for all g and μ -almost all ω ,*

$$(4.34) \quad U(\omega, g) = W_{h_0}(\omega(x))V_{h_0}(\omega(x)).$$

Here

$$(4.35) \quad \omega = g_0^{-1}(\omega(x)) \quad (g_0 \in E),$$

and

$$(4.36) \quad h_0^{-1}g_0g \in E.$$

Proof. At first, since

$$(4.37) \quad (U_{h_1}f)(h) = f(h_1^{-1}hh_1) = (R_{h_1}^H L_{h_1}^H f)(h), \quad \text{for } h_1 \text{ in } H,$$

$$(4.38) \quad U(\omega, h_1) = W_{h_1}(\omega)V_{h_1}(\omega).$$

The commutation relations (4.24), (4.25) lead us to

$$(4.39) \quad U(\omega, \mathbf{g})W_h(\mathbf{g}^{-1}(\omega)) = W_{\mathbf{g}(h)}(\omega)U(\omega, \mathbf{g}),$$

$$(4.40) \quad U(\omega, \mathbf{g})V_h(\mathbf{g}^{-1}(\omega)) = V_{\mathbf{g}(h)}(\omega)U(\omega, \mathbf{g}).$$

And by the definition of the canonical forms of $W_{\mathbf{g}}(\omega)$ and $V_{\mathbf{g}}(\omega)$,

$$(4.41) \quad W_{\mathbf{g}(h)}(\omega) = W_{\mathbf{g}(h)}(\mathbf{g}_0^{-1}(\omega(x))) = W_{\mathbf{g}_0\mathbf{g}(h)}(\omega(x)),$$

$$(4.42) \quad \begin{aligned} W_h(\mathbf{g}^{-1}(\omega)) &= W_h(\mathbf{g}^{-1}\mathbf{g}_0^{-1}(\omega(x))) = W_{\mathbf{g}'(h)}(\omega(x)) \\ &= W_{(h_0^{-1}\mathbf{g}_0\mathbf{g})(h)}(\omega(x)) \\ &= W_{h_0^{-1}(\mathbf{g}_0\mathbf{g}(h))h_0}(\omega(x)) \\ &= W_{h_0^{-1}}(\omega(x))W_{\mathbf{g}_0\mathbf{g}(h)}(\omega(x))W_{h_0}(\omega(x)). \end{aligned}$$

Here

$$(4.43) \quad \mathbf{g}' = h_0^{-1}\mathbf{g}_0\mathbf{g}, \quad \text{is an element in } E.$$

And analogously

$$(4.44) \quad V_{\mathbf{g}(h)}(\omega) = V_{\mathbf{g}_0\mathbf{g}(h)}(\omega(x)),$$

$$(4.45) \quad V_h(\mathbf{g}^{-1}(\omega)) = V_{h_0^{-1}}(\omega(x))V_{\mathbf{g}_0\mathbf{g}(h)}(\omega(x))V_{h_0}(\omega(x)).$$

Combining (4.41)~(4.45) with (4.39), (4.40),

$$(4.46) \quad \begin{aligned} (U(\omega, \mathbf{g})W_{h_0^{-1}}(\omega(x))V_{h_0^{-1}}(\omega(x)))W_{\mathbf{g}_0\mathbf{g}(h)}(\omega(x)) \\ = W_{\mathbf{g}_0\mathbf{g}(h)}(\omega(x))(U(\omega, \mathbf{g})W_{h_0^{-1}}(\omega(x))V_{h_0^{-1}}(\omega(x))), \end{aligned}$$

$$(4.47) \quad \begin{aligned} (U(\omega, \mathbf{g})W_{h_0^{-1}}(\omega(x))V_{h_0^{-1}}(\omega(x)))V_{\mathbf{g}_0\mathbf{g}(h)}(\omega(x)) \\ = V_{\mathbf{g}_0\mathbf{g}(h)}(\omega(x))(U(\omega, \mathbf{g})W_{h_0^{-1}}(\omega(x))V_{h_0^{-1}}(\omega(x))). \end{aligned}$$

The irreducibility of the representation $\{\mathcal{A}^x, W_{\mathbf{g}_0\mathbf{g}(h)}(\omega(x)), V_{\mathbf{g}_0\mathbf{g}(h)}(\omega(x))\}$ of the group $H \times H$ results that the operator $U(\omega, \mathbf{g})W_{h_0^{-1}}(\omega(x))V_{h_0^{-1}}(\omega(x))$ must be a scalar operator $c(\omega, \mathbf{g})$. That is

$$(4.48) \quad U(\omega, g) = c(\omega, g) W_{h_0}(\omega(x)) V_{h_0}(\omega(x)).$$

$$(4.49) \quad |c(\omega, g)| = 1, \quad \text{for all } g \text{ and almost all } \omega.$$

Put $g \equiv h_1 \in H$, then $h_0 = g_0 h_1 g_0^{-1} = g_0(h_1)$ and

$$(4.50) \quad W_{h_1}(\omega) = W_{h_1}(g_0^{-1}(\omega(x))) = W_{g_0(h_1)}(\omega(x)) = W_{h_0}(\omega(x)).$$

Similarly

$$(4.51) \quad V_{h_1}(\omega) = V_{h_0}(\omega(x)).$$

Therefore, comparing (4.38) with (4.48), we obtain

$$(4.52) \quad c(\omega, h) = 1, \quad \text{for any } h \text{ in } H.$$

On the other hand, because $\{U(\omega, g)\}$ gives a representation $\{\mathfrak{D}^x, U_g\}$ of G , it must satisfy the following relation for all g_1, g_2 and almost all ω .

$$(4.53) \quad U(\omega, g_1) U(g_1^{-1}(\omega), g_2) = U(\omega, g_1 g_2).$$

If we denote by h_1, h_2 and h_3 the corresponding elements of H in (4.43) to $(\omega, g_1), (g_1^{-1}(\omega), g_2)$ and $(\omega, g_1 g_2)$ respectively, then it is easy to see,

$$(4.54) \quad h_1 h_2 = h_3.$$

Since U is given by the form (4.48) and since by (4.54) the corresponding parts of (4.53) of $W_h(\omega(x)), V_h(\omega(x))$ satisfy the analogous equation as (4.53), we obtain

$$(4.55) \quad c(\omega, g_1) c(g_1^{-1}(\omega), g_2) = c(\omega, g_1 g_2).$$

This shows that if we determine the value of $c(\omega(x), g)$ for any g , then $c(\omega, g)$ is calculated by

$$(4.56)^*) \quad c(g_1^{-1}(\omega), g_2) = (c(\omega, g))^{-1} c(\omega, g_1 g_2).$$

With (4.52), if we take vectors $c(\omega(x), g)v(g(\omega(x)))$ ($g \in E$) in $\mathcal{H}^x = \mathcal{H}(g(\omega(x)))$ instead of vectors $v(g(\omega(x)))$, then we obtain

$$(4.57) \quad c(\omega, g) \equiv 1.$$

This completes the proof.

§ 5. Reduced quasi-dual of non-unimodular group.

As in §4, G is a separable non-unimodular locally compact group, and H is the fully unimodular subgroup of G . We put the same assumption on the G -orbits space X , as in §4, too.

Consider the right regular representation $\mathfrak{R} = \{L^2(G), R_g\}$ of G , here R_g is the operator on $L^2(G)$ defined by

$$(5.1) \quad (R_{g_0} f)(g) \equiv f(g g_0).$$

By Lemma 2.4, \mathfrak{R} is unitary equivalent to the representation of G induced from the regular representation \mathfrak{R}_H of H . That is,

$$(5.2) \quad \mathfrak{R} \cong \text{Ind}_{H \uparrow G} \mathfrak{R}_H$$

On the other hand, by the central decomposition (1.2),

$$(5.3) \quad \mathfrak{R}_H \cong \int_{\mathfrak{g}} \omega d\mu(\omega).$$

So, using Lemma 2.5.,

$$(5.4) \quad \mathfrak{R} \cong \text{Ind}_{H \uparrow G} \left\{ \int_{\mathfrak{g}} \omega d\mu(\omega) \right\} \cong \int_{\mathfrak{g}} (\text{Ind } \omega) d\mu(\omega).$$

And since μ is described as an integral of measures μ_x on x with respect to the measure $\tilde{\mu}$ on the orbit space, we obtain,

*) Of course, since (4.56) is valid only except measure zero, we must discuss more carefully. But here we talk about only outline for brevity. The corrections of these arguments are routine.

$$(5.5) \quad \mathfrak{R} \cong \int_X d\tilde{\mu}(x) \left\{ \int_x (\text{Ind } \omega) d\mu_x(\omega) \right\}.$$

Lemma 5.1. For any element g_0 in G ,

$$(5.6) \quad \text{Ind}_{G \uparrow H} g_0(\omega) \cong \text{Ind}_{G \uparrow H} \omega.$$

Proof. In this case, the quasi-invariant measure ν on the abelian factor group $H \backslash G$, can be taken as the two-sided Haar measure on $H \backslash G$. So the map

$$(5.7) \quad \mathbf{v}(g) \rightarrow \mathbf{v}(g_0 g)$$

gives the unitary equivalence of (5.6). q. e. d.

Thus, the first integral on x in (5.5) is a direct integral of mutually equivalent representations. Therefore, it is equivalent to a discrete direct sum of the representation $\text{Ind}_{H \uparrow G} \omega$ with the multiplicity of the dimension of $L^2(x, \mu_x)$. And this dimension is countably infinite except $\tilde{\mu}$ -measure zero. Hence,

Lemma 5.2.

$$(5.8) \quad \mathfrak{R} \cong \int_X \left\{ \sum_{H \uparrow G}^{\infty} \text{Ind } \omega_x \right\} d\tilde{\mu}(x)$$

$$(5.9) \quad \cong \int_X \text{Ind} \left\{ \sum_{H \uparrow G}^{\infty} \omega_x \right\} d\tilde{\mu}(x).$$

Here ω_x are elements of Ω , passed through by x .

Notation. By the reason of Lemma 5.1., the component $\text{Ind}_{H \uparrow G} \left\{ \sum_{H \uparrow G}^{\infty} \omega_x \right\}$ of \mathfrak{R} in the decomposition (5.9) is determined up to unitary equivalence, depending only on the orbit x passing through ω . We denote this representation by $\mathfrak{D}_x = \{ \tilde{\mathfrak{D}}(x), \hat{\mathfrak{W}}_g(x) \}$. Thus,

$$(5.10) \quad \mathfrak{R} \cong \int_X \mathfrak{D}_x d\tilde{\mu}(x).$$

Consider a vector $\mathbf{v} \equiv (\mathbf{v}(g))$ in the space \mathfrak{S}° of the induced representation $\text{Ind}_{H \uparrow G} \omega$. As is remarked in §2, in our case we can put that the quasi-invariant measure $d\nu$ on $H \backslash G$ is equal to a Haar measure $d\tilde{g}$ on $H \backslash G$. Then $\phi \equiv 1$ in (2.7), and by (2.9) $w(g, g_1) \equiv 1$ in the operator (2.12) of the induced representation. Thus

$$(5.11) \quad (U_{g_1}^\circ \mathbf{v})(g) = \mathbf{v}(gg_1).$$

Evidently for any h in H , by arguments in §4,

$$(5.12) \quad \begin{aligned} (U_h^\circ \mathbf{v})(g) &= \mathbf{v}(gh) = W_{ghg^{-1}}(\omega)(\mathbf{v}(g)) = W_{g(h)}(\omega)(\mathbf{v}(g)) \\ &= W_h(g_1^{-1}(\omega))(\mathbf{v}(g)). \end{aligned}$$

Here,

$$(5.13) \quad g_1 \in E \text{ and } g_1 g^{-1} \in H.$$

From the equation

$$(5.14) \quad \|\mathbf{v}\|^2 = \int_{H \backslash G} \|\mathbf{v}(g)\|^2 d\tilde{g},$$

the restriction $\text{Ind}_{H \uparrow G} \omega|_H$ to the subgroup H of the above mentioned induced representation is decomposed to a direct integral of the representations $g_1^{-1}(\omega) = \{\mathcal{H}^x, W_h(g_1^{-1}(\omega))\}$ (cf. §4), on $H \backslash G$ with respect to the measure $d\tilde{g}$. And the map

$$(5.15) \quad \mathbf{v} \rightarrow \{\mathbf{v}(g_1)\}_{g_1 \in E}$$

gives the decomposition

$$(5.16) \quad \text{Ind}_{H \uparrow G} \omega|_H \equiv \{\mathfrak{S}^\circ, U_h^\circ\} \cong \int_{H \backslash G} g_1^{-1}(\omega) d\tilde{g}.$$

On the other hand, Corollary 2 of Proposition 4.1 asserts that the

integral on $H \backslash G$ with respect to $d\tilde{g}$ can be transferred to the integral on the G -orbit x in Ω passing through ω with respect to μ_x . Thus,

Lemma 5.3. *For μ -almost all ω ,*

$$(5.17) \quad \text{Ind}_{H \uparrow G} \omega |_H \cong \int_x \sigma d\mu_x(\sigma).$$

Here x is the G -orbit in Ω passing through ω .

Proof. It is evident from the above arguments and from the fact that each representation σ of H in x is given as the form $g_1^{-1}(\omega)$ by some g_1 in E . q. e. d.

Now, in the central decomposition (1.2),

$$(5.18) \quad (P_F \mathbf{v})(\omega) \equiv \chi_F(\omega) \mathbf{v}(\omega), \quad \left(\mathbf{v} \in \int_{\Omega} \mathcal{H}(\omega) d\mu(\omega) \right)$$

define central projections in the von Neumann algebra \mathfrak{A}_R generated by

$$(5.19) \quad \{R_h^F\}_{h \in H} \sim \left\{ \int_X W_h^x d\tilde{\mu}(x) \right\}_{h \in H} \quad \text{on } \mathfrak{S}^x.$$

Here F are measurable subsets of Ω , and χ_F is the characteristic function of F . Following the weaker decomposition (4.11), P_F is decomposed as

$$(5.20) \quad P_F \sim \int_X P_F(x) d\tilde{\mu}(x).$$

By the theory of von Neumann algebras (cf. J. Dixmier [2] Chap. II.), for $\tilde{\mu}$ -almost all x , $\{P_F(x)\}_F$ is contained in the centre of the von Neumann algebra generated by $\{W_h^x\}_{h \in H}$, and induces the decomposition

$$(5.21) \quad \{\mathfrak{S}^x, W_h^x, V_k^x\} \cong \int_x \{\mathcal{H}(\omega), W_h(\omega), V_k(\omega)\} d\mu_x(\omega).$$

Lemma 5.4. For $\tilde{\mu}$ -almost all x , a bounded operator B on \mathfrak{S}^x commuting with $\{W_h^x\}_{h \in H}$, is decomposable, i.e.

$$(5.22) \quad B \cong \int_x B(\omega) d\mu_x(\omega).$$

Proof. Since B commutes with $\left\{ \int_x W_h(\omega) d\mu_x(\omega) \right\}_{h \in H}$, B commutes with $\{P_F(x)\}_F$ too. This means that B is decomposable. (cf. J. Dixmier [2], p. 169.) q. e. d.

Lemma 5.5. For μ -almost all ω , the representations $\text{Ind}_{H \uparrow G} \omega$ are factor representations.

Moreover, when H has the reduced dual of type I, $\text{Ind}_{H \uparrow G} \omega^0$ are irreducible for μ -almost all ω . Here ω^0 show the minimal (therefore irreducible) components of ω .

Proof. Let $\text{Ind}_{H \uparrow G} \omega \equiv \{\mathfrak{S}^\omega, U_g^\omega\}$ and $\text{Ind}_{H \uparrow G} \omega^0 \equiv \{\mathfrak{S}^{\omega^0}, U_g^{\omega^0}\}$. For the first half, it is sufficient to show that, for μ -almost all ω , any bounded operator A in the centre of the von Neumann algebra generated by $\{U_g^\omega\}_{g \in G}$ is a scalar operator. And for the last half, it is enough to show that, for μ -almost all ω , any bounded operator A^0 commuting with any operator $U_g^{\omega^0} (g \in G)$ is a scalar operator.

Consider a bounded operator B on \mathfrak{S}^ω of the representation $\text{Ind}_{H \uparrow G} \omega$, which commutes with any $U_g^\omega (g \in G)$. Especially, B commutes with any $U_h^\omega (h \in H)$. By Lemma 5.3, $\{\mathfrak{S}^\omega, U_h^\omega\}$ is equivalent to $\int_X \{\mathcal{H}(\sigma), W_h(\sigma)\} d\mu_x(\sigma)$, hence is equivalent to $\{\mathfrak{S}^x, W_h^x\}$, for μ -almost all ω . So from Lemma 5.4, B is decomposable. That is, there are bounded operators $B(\sigma)$ on $\mathcal{H}(\sigma)$ and

$$(5.23) \quad B \sim \int_x B(\sigma) d\mu_x(\sigma) \sim \int_{H \setminus G} B(g^{-1}(\omega)) d\tilde{g}.$$

But B commutes with any operator U_g^ω of the form (5.11), so

$$(5.24) \quad \begin{aligned} B(g_1^{-1}g^{-1}(\omega))\mathbf{v}(gg_1) &= U_{g_1}^\omega(B\mathbf{v})(g) = (BU_{g_1}^\omega\mathbf{v})(g) \\ &= B(g^{-1}(\omega))\mathbf{v}(gg_1). \end{aligned}$$

By the reason of the arbitrariness of \mathbf{v} , we obtain,

$$(5.25) \quad B(g_1^{-1}g^{-1}(\omega)) = B(g^{-1}(\omega)), \quad \text{for any } g_1 \text{ and almost all } g.$$

This means that there exists a bounded operator B_0 on $\mathcal{H}(\omega)$ such that

$$(5.26) \quad (B\mathbf{v})(g) = B_0(\mathbf{v}(g)), \quad \text{for almost all } g.$$

Moreover the commutation relation of B with any $U_h^\omega (h \in H)$ leads us to that B_0 must commute with any $W_h(\omega) (h \in H)$.

Conversely, for any bounded operator B_0 on $\mathcal{H}(\omega)$ commuting with any $W_h(\omega) (h \in H)$, (5.26) defines an operator B on \mathfrak{H}^ω which commutes with any U_g^ω .

Next, consider an operator A in the centre of the von Neumann algebra generated by $\{U_g^\omega\}_{g \in G}$. Since A commutes with any $U_g^\omega (g \in G)$, from the above arguments, there exists an operator A_0 on $\mathcal{H}(\omega)$, which commutes with any $W_h(\omega) (h \in H)$, and A is the form of

$$(5.27) \quad (A\mathbf{v})(g) = A_0(\mathbf{v}(g)).$$

And A commutes with any B of the form (5.26), we obtain,

$$(5.28) \quad A_0B_0 = B_0A_0.$$

But B_0 is any operator commuting with any $W_h(\omega) (h \in H)$. Hence A_0 is in the centre of von Neumann algebra generated by $\{W_h(\omega)\}_{h \in H}$. And this algebra is a factor, so A_0 and A must be a scalar operator. This shows the first half of the proposition.

When H has the reduced dual of type I, by the arguments as above, there exists an operator A_0^0 on $\mathcal{H}(\omega^0)$, and A^0 is the form of

$$(5.29) \quad (A^0\mathbf{v})(g) = A_0^0(\mathbf{v}(g)).$$

A_0^0 commutes with any operator of ω^0 . Therefore the irreducibility of ω^0 leads us to that A_0^0 is a scalar operator, then A^0 is a scalar operator too.

This completes the proof.

Proposition 5.1. *$\bar{\mu}$ -almost all components of the decomposition (5.10) of \mathfrak{R} , are factor representations of G .*

Proof. This is direct result of Lemma 5.5. q. e. d.

Proposition 5.2. *The decomposition (5.10) gives the central decomposition of \mathfrak{R} .*

Proof. Basing upon Proposition 5.1., it is sufficient to show, that any diagonal operator in this decomposition is in the centre of the von Neumann algebra \mathfrak{A} generated by $\{R_g\}_{g \in G}$. But this is enough to prove that projections P_F defined by

$$(5.29) \quad (P_F w)(x) = \chi_F(x) w(x),$$

are in the centre of \mathfrak{A} . Here F is any measurable set in X and χ_F is the characteristic function of F . And P_F is a projection on the space of $\int_X \mathfrak{D}_x d\bar{\mu}(x)$.

But because of the form of the operator P_F , evidently P_F commutes with any operator $R_g \sim \int_X \widehat{W}_g(x) d\bar{\mu}(x)$ ($g \in G$). According to the central decomposition of $\mathfrak{R}|_H \cong \sum \mathfrak{R}_H \cong \sum_{\mathcal{Q}} \int_{\mathcal{Q}} \{\mathcal{A}(\omega), W_h(\omega)\} d\mu(\omega)$, obviously, P_F is decomposed as,

$$(5.30) \quad (P_F v)(\omega) = \chi_{\varphi^{-1}(F)}(\omega) v(\omega),$$

with the map φ defined in §4. Since $\varphi^{-1}(F)$ is measurable in \mathcal{Q} , this projection is in the centre of the von Neumann algebra generated by $\{R_h\}_{h \in H}$, hence of course, in the centre of \mathfrak{A} .

This completes the proof.

Summarizing the above arguments we obtain the following main theorem.

Theorem 5.1. *Let G be a separable non-unimodular locally compact group. Let G satisfy the "Assumption" in §4 on the space X of G -orbits in the reduced quasi-dual of the fully unimodular subgroup H of G .*

Then the reduced quasi-dual of G is constructed on X except measure zero, as the space of factor representations

$\mathfrak{D}_x \equiv \text{Ind}_{H \uparrow G} \{ \Sigma \oplus \omega \} (\omega \in x)$. *Here $\text{Ind}_{H \uparrow G} \Sigma \oplus \omega$ depends only on x up to unitary equivalence.*

Moreover, if H has the reduced dual of type I, then the reduced dual of G is constructed on X , as the space of irreducible representations $\mathfrak{D}_x^0 \equiv \text{Ind}_{H \uparrow G} \omega^0$, except measure zero. Here ω^0 is the minimal component of ω . Especially, G has the reduced dual of type I, too.

§6. Plancherel formula for non-unimodular groups.

Now we are on the step to prove an extension of Plancherel formula.

By the arguments in §4 and §5, the reduced quasi-dual of a separable non-unimodular locally compact group G satisfying the regularity assumption in §4, is constructed on the space X of G -orbits in the reduced quasi-dual \mathcal{Q} of its fully unimodular subgroup H , except measure zero set. So we identify the reduced quasi-dual of G with X , and factor representations $\mathfrak{D} \equiv \{ \tilde{\mathfrak{H}}(\mathfrak{D}), \hat{\mathfrak{W}}_g(\mathfrak{D}) \}$ with corresponding orbits x in X respectively.

As is shown in §5, μ -almost all elements \mathfrak{D} in X are induced representations from some representation ω of H in \mathcal{Q} , and moreover, when H has the reduced dual of type I, minimal (therefore, irreducible) components $\mathfrak{D}^0 \equiv \{ \tilde{\mathfrak{H}}(\mathfrak{D}^0), \hat{\mathfrak{W}}_g(\mathfrak{D}^0) \}$ of μ -almost all elements \mathfrak{D} in X are induced from minimal components ω^0 of ω . Thus for such \mathfrak{D} or \mathfrak{D}^0 , we can define the operators $T_\delta(\mathfrak{D})$ or $T_\delta(\mathfrak{D}^0)$ defined in §2 for a positive character δ which is trivial on H . Especially, the modular func-

tion Δ_G and its real powers are positive character which are trivial on H . Hence, put

$$(6.1) \quad T_{\Delta_G^{1/2}}(\mathfrak{D}) \equiv T(\mathfrak{D}).$$

Moreover, we can define, for induced representations \mathfrak{D} in X , sesqui-linear forms $\tau_{\mathfrak{D}}$ on the vector spaces \mathfrak{A}_0 of operators $\{T(\mathfrak{D})\widehat{W}_f(\mathfrak{D}) : f \in C_0(G)\}$ by (2.40). And if H has the reduced dual of type I, these sesqui-linear forms $\tau_{\mathfrak{D}}$ are considered as the ordinary traces of operators of the minimal components \mathfrak{D}^0 of \mathfrak{D} .

Thus we obtain the following main theorem.

Theorem 6.1. *For any f in $C_0(G)$,*

$$(6.2) \quad \int_G |f(g)|^2 d_r g = \int_X \tau_{\mathfrak{D}}((T(\mathfrak{D})\widehat{W}_f(\mathfrak{D}))^* T(\mathfrak{D})\widehat{W}_f(\mathfrak{D})) d\bar{\mu}(\mathfrak{D})$$

$$(6.3) \quad = \int_X \tau_{\mathfrak{D}}(T(\mathfrak{D})(\widehat{W}_{f\Delta^{-1/2}}(\mathfrak{D}))^* \widehat{W}_{f\Delta^{-1/2}}(\mathfrak{D})T(\mathfrak{D})) d\bar{\mu}(\mathfrak{D})$$

$$(6.4) \quad = \int_X \tau_{\mathfrak{D}}((T(\mathfrak{D}))^2 \widehat{W}_{(f^* \Delta)_* f}(\mathfrak{D})) d\bar{\mu}(\mathfrak{D}).$$

Here

$$(6.5) \quad f^*(g) = f(\overline{g^{-1}})(\Delta_G(g))^{-1},$$

and

$$(6.6) \quad (f\Delta^{-1/2})(g) = f(g)(\Delta_G(g))^{-1/2},$$

$$(6.7) \quad (f\Delta)(g) = f(g)\Delta_G(g).$$

Especially, if H has the reduced dual of type I,

$$(6.8) \quad \int_G |f(g)|^2 d_r g = \int_X \|T(\mathfrak{D}^0)\widehat{W}_f(\mathfrak{D}^0)\|^2 d\bar{\mu}(\mathfrak{D})$$

$$(6.9) \quad = \int_X \|\widehat{W}_{f_d^{-1/2}}(\mathfrak{D}^0) T(\mathfrak{D}^0)\|^2 d\tilde{\mu}(\mathfrak{D})$$

$$(6.10) \quad = \int_X T_r((T(\mathfrak{D}^0))^2 \widehat{W}_{(f^*d)^*f}(\mathfrak{D}^0)) d\tilde{\mu}(\mathfrak{D}).$$

Here $\|\cdot\|$ shows the Hilbert-Schmidt norm of operators, and $T_r(\cdot)$ shows the ordinary trace of operators.

Proof. Let f be a function in $L^1(G) \cap L^2(G)$, then,

$$(6.11) \quad \|f\|^2 = \int_G |f(g)|^2 d_r g = \int_{H \setminus G} d\tilde{g} \int_H |f(hg)|^2 dh.$$

Since H is unimodular, the Plancherel formula (1.10) is available,

$$(6.12) \quad I(g) \equiv \int_H |f(hg)|^2 dh = \int_{\mathcal{Q}} \tau_{\omega}((W_{f_g}(\omega))^* W_{f_g}(\omega)) d\mu(\omega).$$

Here

$$(6.13) \quad f_g(h) \equiv f(hg)$$

are considered as functions in $L^1(H) \cap L^2(H)$ for almost all g . As in §4, \mathcal{Q} is divided as the union of G -orbits, and the measure μ over \mathcal{Q} is decomposed to the integration of measures μ_x on x with respect to the measure $\tilde{\mu}$ on X . Hence,

$$(6.14) \quad I(g) = \int_X d\tilde{\mu}(x) \int_x \tau_{\omega}((W_{f_g}(\omega))^* W_{f_g}(\omega)) d\mu_x(\omega).$$

But by Corollary 2 of Proposition 4.1, the measures μ_x is transferred to a quasi-invariant measure on the homogeneous group $\tilde{G} = H \setminus G$, as

$$(6.15) \quad d\mu_x(g(\omega(x))) = c_x \Delta_{\tilde{G}}^{-1}(g) d\tilde{g}.$$

Here $\omega(x)$ is a representative in x given in §4, and c_x is a constant. It is easy to see that by adequate selection of the measure $\tilde{\mu}$ on X , c_x can be taken one. Thus, substituting the notations \mathfrak{D} for x ,

$$(6.16) \quad I(g) = \int_X d\tilde{\mu}(\mathfrak{D}) \int_{H \backslash G} \tau_\omega((W_{f_o}(g_1(\omega(x))))^* W_{f_o}(g_1(\omega(x)))) \\ \times \Delta_G^{-1}(g_1) d\tilde{g}_1.$$

On the other hand, for any g_1 in the Borel section E of $H \backslash G$ given in §4,

$$(6.17) \quad W_{f_o}(g_1(\omega(x))) = \int_H f(hg) W_h(g_1(\omega(x))) dh \\ = \int_H f(hg) W_{g_1^{-1}hg_1}(\omega(x)) dh \\ = \int_H f(g_1h g_1^{-1}g) W_h(\omega(x)) d(g_1(h)) \\ = \int_H f(g_1h g_1^{-1}g) W_h(\omega(x)) \Delta_G(g_1) dh \\ = W_f^z(g_1^{-1}, g_1^{-1}g) \Delta_G(g_1).$$

Here $W_f^z(g_1, g_2)$ is the operator $W_f(g_1, g_2)$ given in Corollary of Proposition 2.1 for $\omega(x)$. Thus,

$$(6.18) \quad \|f\|^2 = \int_{H \backslash G} I(g_2) d\tilde{g}_2 = \int_{H \backslash G} d\tilde{g}_2 \left\{ \int_X d\tilde{\mu}(\mathfrak{D}) \right. \\ \left. \times \int_{H \backslash G} \tau_\omega((W_f^z(g_1^{-1}, g_1^{-1}g_2))^* W_f^z(g_1^{-1}, g_1^{-1}g_2)) \Delta_G(g_1) d\tilde{g}_1 \right\} \\ = \int_X d\tilde{\mu}(\mathfrak{D}) \left\{ \int_{H \backslash G} \int_{H \backslash G} \tau_\omega((W_f^z(g_1, g_1g_2))^* W_f^z(g_1, g_1g_2)) \right. \\ \left. \times (\Delta_G(g_1))^{-1} d\tilde{g}_1 d\tilde{g}_2 \right\} \\ = \int_X d\tilde{\mu}(\mathfrak{D}) \left\{ \int_{H \backslash G} \int_{H \backslash G} \tau_\omega((W_f^z(g_1, g_2))^* W_f^z(g_1, g_2)) \right. \\ \left. \times (\Delta_G(g_1))^{-1} d\tilde{g}_1 d\tilde{g}_2 \right\}.$$

Evidently the integrand dose not change by the substitution of g_1 with $hg_1 (h \in H)$. Thus we can take off the assumption that g_1 is in E .

If we put in Proposition 2.1,

$$(6.19) \quad \delta = \Delta_G^{1/2},$$

then

$$(6.20) \quad \|f\|^2 = \int_X d\tilde{\mu}(\mathfrak{D}) \left\{ \int_{H \setminus G} \int_{H \setminus G} \tau_\omega((W_f^z(g_1, g_2))^* W_f^z(g_1, g_2)) \right. \\ \left. \times (\Delta_G(g_1))^{-2} (\delta(g_1))^2 d\tilde{g}_1 d\tilde{g}_2 \right\}.$$

This formula (6.20) assures the convergence of the following integrals for $\tilde{\mu}$ -almost all \mathfrak{D} .

$$(6.21) \quad \tau_{\mathfrak{D}}((T(\mathfrak{D}) \widehat{W}_f(\mathfrak{D}))^* T(\mathfrak{D}) \widehat{W}_f(\mathfrak{D})) = \int_{H \setminus G} \int_{H \setminus G} \tau_\omega((W_f^z(g_1, g_2))^* \\ \times W_f^z(g_1, g_2)) (\Delta_G(g_1))^{-1} d\tilde{g}_1 d\tilde{g}_2.$$

And (6.2) is obtained by substituting (6.21) into (6.20).

The equality (6.4) is deduced by direct calculations. (cf. Lemma 2.7.)

$$(6.22) \quad (T(\mathfrak{D}) \widehat{W}_f(\mathfrak{D}))^* T(\mathfrak{D}) \widehat{W}_f(\mathfrak{D}) = (\widehat{W}_f(\mathfrak{D}))^* (T(\mathfrak{D}))^2 \widehat{W}_f(\mathfrak{D}) \\ = \widehat{W}_{f^*}(\mathfrak{D}) (T(\mathfrak{D}))^2 \widehat{W}_f(\mathfrak{D}) = (T(\mathfrak{D}))^2 \widehat{W}_{f^* \Delta}(\mathfrak{D}) \widehat{W}_f(\mathfrak{D}) \\ = (T(\mathfrak{D}))^2 \widehat{W}_{(f^* \Delta)^* f}.$$

When H has the reduced dual of type I, we can take the ordinary trace of operators of minimal components ω^0 of ω instead of τ_ω in (6.12) and for $\tilde{\mu}$ -almost all \mathfrak{D} , the minimal components \mathfrak{D}^0 of \mathfrak{D} are obtained as the induced representation from ω^0 . Thus, the last half of the proposition is proved just in the same way.

This completes the proof.

Corollary 1. For functions f which are linear combinations of the functions $f_1 * f_2$ ($f_1, f_2 \in C_0(G)$),

$$(6.23) \quad f(e) = \int_X \tau_{\mathfrak{D}}((T(\mathfrak{D}))^2 \widehat{W}_f(\mathfrak{D})) d\tilde{\mu}(\mathfrak{D}).$$

Especially, if H has the reduced dual of type I,

$$(6.24) \quad f(e) = \int_X T_r((T(\mathfrak{D}^0))^2 \widehat{W}_f(\mathfrak{D}^0)) d\tilde{\mu}(\mathfrak{D})$$

Proof. For the linearity of the both side, it is sufficient to prove for the function of the form $(f * \Delta) * f$. By (6.4),

$$(6.25) \quad (f * \Delta) * f(e) = \int_G \overline{(f(g))} (\Delta_G(g))^{-1} \Delta_G(g) f(g) d_r g$$

$$(6.25) \quad = \int_G |f(g)|^2 d_r g$$

$$(6.26) \quad = \int_X \tau_{\mathfrak{D}}((T(\mathfrak{D}))^2 \widehat{W}_{(f * \Delta) * f}(\mathfrak{D})) d\tilde{\mu}(\mathfrak{D}).$$

When H has the reduced dual of type I, this is equal to

$$(6.27) \quad \int_X T_r((T(\mathfrak{D}^0))^2 \widehat{W}_{(f * \Delta) * f}(\mathfrak{D}^0)) d\tilde{\mu}(\mathfrak{D}).$$

This completes the proof.

Corollary 2. Let f be a function as in Corollary 1. Then,

$$(6.28) \quad f(g) = \int_X \tau_{\mathfrak{D}}((T(\mathfrak{D}))^2 \widehat{W}_f(\mathfrak{D}) \widehat{W}_{g^{-1}}(\mathfrak{D})) d\tilde{\mu}(\mathfrak{D}).$$

And if H has the reduced dual of type I,

$$(6.29) \quad f(g) = \int_X T_r((T(\mathfrak{D}^0))^2 \widehat{W}_f(\mathfrak{D}^0) \widehat{W}_{g^{-1}}(\mathfrak{D}^0)) d\tilde{\mu}(\mathfrak{D}).$$

Proof. Since,

$$(6.30) \quad f(g) = f * \delta_{g^{-1}}(e),$$

and

$$(6.31) \quad \widehat{W}_{f * \delta_{g^{-1}}}(\mathfrak{D}) = \widehat{W}_f(\mathfrak{D}) \widehat{W}_{g^{-1}}(\mathfrak{D}),$$

the result follows from Corollary 1, soon.

q. e. d.

Obviously, for unimodular group, if we take the operator $T(\mathfrak{D})$ as the identity operator on the space of representation \mathfrak{D} in its reduced quasi-dual X , the formulae in Theorem 6.1 and its corollaries are valid in the same form. Therefore, we may consider that Theorem 6.1 and its corollaries give an extension of Plancherel formula for any separable locally compact group satisfying the assumption in §4.

§7. Decompositions of the regular double representation and the regular quasi-Hilbert algebra of G .

Here we shall discuss an analogue to the decomposition theory of F. I. Mautner [9] [10], I. E. Segal [13] and J. Dixmier [2], which is proved in the case of unimodular groups. We use the notations given in §§4~6.

Consider the regular double representation $\{L^2(G), R_{g_1}, L_{g_2}, J\}$ of G , defined by

$$(7.1) \quad (R_{g_1}f)(g) = f(gg_1),$$

$$(7.2) \quad (L_{g_2}f)(g) = (\Delta_G(g_2))^{-1/2} f(g_2^{-1}g),$$

$$(7.3) \quad (Jf)(g) = (\Delta_G(g))^{-1/2} \overline{f(g^{-1})}.$$

J is an involution operator on $L^2(G)$, such that

$$(7.4) \quad JR_gJ = L_g, \quad \text{for any } g \text{ in } G.$$

We define the following operators on $L^2(G)$, too.

$$(7.5) \quad (Sf)(g) = (\Delta_G(g))^{-1/2} f(g^{-1}),$$

$$(7.6) \quad (\boldsymbol{\delta}f)(g) = \delta(g)f(g).$$

Here $\delta(g)$ is a positive character on G , satisfying

$$(7.7) \quad \delta(h) = 1, \quad \text{for any } h \text{ in } H.$$

Obviously S is a unitary operator on $L^2(G)$, and $\boldsymbol{\delta}$ is a 1-to-1 self-adjoint positive definite operator with dense domain and dense range in $L^2(G)$. And the followings are trivial.

$$(7.8) \quad S^2 = I, \quad (\text{identity operator on } L^2(G)).$$

$$(7.9) \quad SR_gS = L_g,$$

$$(7.10) \quad \boldsymbol{\delta}R_{g_0} = (\delta(g_0))^{-1}R_{g_0}\boldsymbol{\delta},$$

$$(7.11) \quad \boldsymbol{\delta}L_{g_0} = \delta(g_0)L_{g_0}\boldsymbol{\delta}.$$

By the way, the space $C_0(G)$ of continuous functions with compact carrier on G , becomes a quasi-Hilbert algebra in the sense of J. Dixmier [2], by the ordinary structure of a vector space on \mathbf{C} , and the scalar product in $L^2(G)$, the product of convolution. The involution is defined by the same form as J in (7.3), that is,

$$(7.12) \quad f^*(g) = (Jf)(g) = \overline{f(g^{-1})}\Delta_G(g)^{1/2}$$

The linear map $f \rightarrow \hat{f}$ is given by

$$(7.13) \quad \hat{f}(g) = (\Delta_G^{1/2}f)(g) \equiv \Delta_G(g)^{1/2}f(g)$$

We call this quasi-Hilbert algebra *the regular quasi-Hilbert algebra* on G .

Now, we put the central decomposition of the right regular representation $\mathfrak{R} = \{R_g\}_{g \in G}$, given by (5.10), as

$$(7.14) \quad \{L^2(G), R_g\} \sim \int_X \{\tilde{\mathfrak{F}}(\mathfrak{D}), \hat{W}_g(\mathfrak{D})\} d\tilde{\mu}(\mathfrak{D}).$$

Because the operators $\mathfrak{L} = \{L_g\}_{g \in G}$ commute with any R_g ($g \in G$), the operators L_g is decomposed by this decomposition, too.

$$(7.15) \quad L_g \sim \int_X \hat{V}_g(\mathfrak{D}) d\tilde{\mu}(\mathfrak{D}).$$

We shall determine the form of the operators $\hat{V}_g(\mathfrak{D})$, and show that the operators, S, J, \mathfrak{d} are decomposed also by (5.10). Lastly, we can obtain the decomposition of the regular quasi-Hilbert algebra $C_0(G)$ on G , according to this central decomposition.

At first, we must decide the concrete form of the equivalence map of the decomposition (5.10).

The unitary map in (5.2) is defined as a map of $L^2(G)$ onto the space of $L^2(H)$ -valued functions on G , which satisfy,

$$(7.16) \quad \mathbf{f}_H(hg) = R_h^H(\mathbf{f}_H(g)), \quad (g \in G, h \in H),$$

$$(7.17) \quad \|\mathbf{f}_H\|^2 \equiv \int_{H \setminus G} \|\mathbf{f}_H(g)\|^2 d\tilde{g} < +\infty.$$

And this map is given by

$$(7.18) \quad f \rightarrow \mathbf{f}_H(g) \equiv (f(hg)),$$

Here $f(hg)$ is an element of $L^2(H)$ as a function of h , and

$$(7.19) \quad \|\mathbf{f}_H(g)\|^2 = \int_H |f(hg)|^2 dh, \quad \text{for almost all } g \text{ in } G.$$

Next, each vector of $L^2(H)$ is decomposed by (5.3), to a vector-valued function on Ω , which take its value in $\mathcal{H}(\omega)$ at ω . We obtain by this step.

$$(7.20) \quad \mathbf{f}_H(g) \rightarrow \mathbf{f}(\omega, g) \quad (\omega \in \Omega, g \in G)$$

From (7.16), (7.17), this function must satisfy

$$(7.21) \quad \mathbf{f}(\omega, hg) = W_h(\omega)(\mathbf{f}(\omega, g)), \quad (g \in G, h \in H, \omega \in \Omega)$$

and, since

$$(7.22) \quad \|\mathbf{f}_H(g)\|^2 = \int_{\mathfrak{g}} \|\mathbf{f}(\omega, g)\|^2 d\mu(\omega),$$

$$(7.23) \quad \|\mathbf{f}_H\|^2 = \int_{H \setminus G} \left(\int_{\mathfrak{g}} \|\mathbf{f}(\omega, g)\|^2 d\mu(\omega) \right) d\tilde{g} < +\infty.$$

Thus the equivalence between \mathfrak{R} and $\int_{H \uparrow G} \text{Ind } \omega d\mu(\omega)$ is given by

$$(7.24) \quad f \rightarrow \mathbf{f}(\omega, g), \quad (\omega \in \Omega, g \in G),$$

as the map from $L^2(G)$ to $\int_{H \setminus G} \left(\int_{\mathfrak{g}} \mathcal{H}(\omega, g) d\mu(\omega) \right) d\tilde{g}$.

Here $\mathcal{H}(\omega, g) \sim \mathcal{H}(\omega)$.

By the reason of Lemma 5.1., the representations $\text{Ind}_{H \uparrow G} \omega$ are mutually equivalent when ω 's are passed through by the same orbit $x(=\mathfrak{D})^*$ in X . And the decomposition (5.10) is obtained by summing up these equivalent representations. That is, the space $\tilde{\mathfrak{F}}(\mathfrak{D})$ of $\mathfrak{D} \left(\cong \int_{\mathfrak{D} \uparrow H \uparrow G} \text{Ind } \omega \times d\mu_{\mathfrak{D}}(\omega) \right)$ is considered as the space of vector valued functions $\mathbf{f}(\omega, g)$ on $\mathfrak{D} \times G$ satisfying (7.21) and

$$(7.25) \quad \|\mathbf{f}\|_{\mathfrak{D}}^2 \equiv \int_{H \setminus G} \left(\int_{\mathfrak{D}} \|\mathbf{f}(\omega, g)\|^2 d\mu_{\mathfrak{D}}(\omega) \right) d\tilde{g} < +\infty.$$

Hence,

$$(7.26) \quad \begin{aligned} \|\mathbf{f}\|^2 &= \|\mathbf{f}_H\|^2 = \int_X \|\mathbf{f}\|_{\mathfrak{D}}^2 d\tilde{\mu}(\mathfrak{D}) \\ &= \int_X d\tilde{\mu}(\mathfrak{D}) \left\{ \iint_{\mathfrak{D} \times H \setminus G} \|\mathbf{f}(\omega, g)\|^2 d\mu_{\mathfrak{D}}(\omega) d\tilde{g} \right\}. \end{aligned}$$

Comparing with (6.2) and (6.8), and by the arbitrariness of f ,

^{*)} We identify a factor representation \mathfrak{D} to the corresponding G -orbit x in Ω , as in §6.

$$(7.27) \quad \|f\|_{\mathfrak{D}}^2 = \tau_{\mathfrak{D}}((T(\mathfrak{D})\widehat{W}_f(\mathfrak{D}))^* T(\mathfrak{D})\widehat{W}_f(\mathfrak{D})),$$

or if H has the reduced dual of type I,

$$(7.28) \quad \|f\|_{\mathfrak{D}} = \|T(\mathfrak{D}^0)\widehat{W}_f(\mathfrak{D}^0)\|,$$

for almost all \mathfrak{D} .

Here we must remark that under the normalization of the isomorphisms between $\mathcal{H}^{\mathfrak{D}}$ and $\mathcal{H}(\omega)$'s ($\omega \in \mathfrak{D}$) given in §4, $f(\omega, g)$ are considered as vectors in the same space $\mathcal{H}^{\mathfrak{D}}$ when ω are in the same orbit \mathfrak{D} . So $\mathfrak{H}(\mathfrak{D})$ can be considered as a space of $\mathcal{H}^{\mathfrak{D}}$ -valued functions on $\mathfrak{D} \times G$. Hereafter we shall use this normalization.

Lemma 7.1.

$$(7.29) \quad J_H U_g = U_g J_H, \quad \text{for any } g \text{ in } G,$$

$$(7.30) \quad S_H U_g = U_g S_H, \quad \text{for any } g \text{ in } G.$$

Here J_H, S_H, U_g are operators on $L^2(H)$ defined by

$$(7.31) \quad (J_H f)(h) = \overline{f(h^{-1})},$$

$$(7.32) \quad (S_H f)(h) = f(h^{-1}),$$

$$(7.33) \quad (U_g f)(h) = f(g^{-1}hg)(\Delta_G(g))^{-1/2},$$

(cf. §1, §4).

Proof. It is clear from the forms of operators. q. e. d.

The operators J_H and S_H are decomposed by the central decomposition (1.2) of \mathfrak{K}_H , as

$$(7.34) \quad J_H \sim \int_{\mathfrak{a}} J_H(\omega) d\mu(\omega),$$

$$(7.35) \quad S_H \sim \int_{\mathfrak{a}} S_H(\omega) d\mu(\omega).$$

Lemma 7.2. *The operators $J_H(\omega)$ and $S_H(\omega)$ are depend only on the orbit \mathfrak{D} passing through ω as operators on $\mathcal{H}^{\mathfrak{D}}$, respectively.*

Proof. From Lemmata 4.6 and 4.7,

$$(7.36) \quad (U_g(\mathfrak{D})\mathbf{v})(\omega) = W_{h_0}(\omega(\mathfrak{D}))V_{h_0}(\omega(\mathfrak{D}))\mathbf{v}(g^{-1}(\omega))\Delta_G(g)^{1/2}.$$

Here

$$(7.37) \quad \omega = g_0^{-1}(\omega(\mathfrak{D})) \quad (g_0 \in E) \text{ and } h_0^{-1}g_0g \in E.$$

So (7.29) shows

$$(7.38) \quad J_H(\omega)W_{h_0}(\omega(\mathfrak{D}))V_{h_0}(\omega(\mathfrak{D})) = W_{h_0}(\omega(\mathfrak{D}))V_{h_0}(\omega(\mathfrak{D}))J_H(g^{-1}(\omega)),$$

for any h_0 in H . In (7.38), if we substitute hg to g , $(hg)^{-1}(\omega) = g^{-1}(\omega)$ and h_0 is exchanged for $g_0hg_0^{-1}h_0 \equiv h_1h_0$. So

$$(7.39) \quad J_H(\omega)W_{h_1h_0}(\omega(\mathfrak{D}))V_{h_1h_0}(\omega(\mathfrak{D})) = W_{h_1h_0}(\omega(\mathfrak{D})) \times \\ V_{h_1h_0}(\omega(\mathfrak{D}))J_H(g^{-1}(\omega)).$$

Because of arbitrariness of h , we can put $h_1h_0 = e$ and we obtain

$$(7.40) \quad J_H(\omega) = J_H(g^{-1}(\omega)), \quad \text{for any } g \text{ in } G.$$

Analogously,

$$(7.41) \quad S_H(\omega) = S_H(g^{-1}(\omega)), \quad \text{for any } g \text{ in } G. \quad \text{q.e.d.}$$

Definition 7.1. *We denote by $J_H^{\mathfrak{D}}$, $S_H^{\mathfrak{D}}$, the operators $J_H(\omega)$, $S_H(\omega)$ on $\mathcal{H}^{\mathfrak{D}}$ respectively, which depend only on the orbit \mathfrak{D} passing through ω .*

Lemma 7.3. *According to the decomposition (7.24) of the space $L^2(G)$, the operators on $L^2(G)$ are represented as follows.*

$$(7.42) \quad \text{i) } R_{g_0}f \rightarrow \mathbf{f}(\omega, gg_0).$$

$$(7.43) \quad \text{ii) } L_{g_0}f \rightarrow \Delta_G(g_0)^{1/2} W_{h_0}(\omega(\mathfrak{D})) V_{h_0}(\omega(\mathfrak{D})) \mathbf{f}(g_0^{-1}(\omega), g_0^{-1}g),$$

$$\omega = g_1^{-1}(\omega(\mathfrak{D})) \quad (g_1 \in E), h_0^{-1}g_1g_0 \in E.$$

$$(7.44) \quad \text{iii) } Jf \rightarrow \Delta_G(g)^{1/2} W_{h_0}(\omega(\mathfrak{D})) V_{h_0}(\omega(\mathfrak{D})) J_H^{\mathfrak{D}} \mathbf{f}(g^{-1}(\omega), g^{-1}),$$

$$\omega = g_1^{-1}(\omega(\mathfrak{D})) \quad (g_1 \in E), h_0^{-1}g_1g \in E.$$

$$(7.45) \quad \text{iv) } Sf \rightarrow \Delta_G(g)^{1/2} W_{h_0}(\omega(\mathfrak{D})) V_{h_0}(\omega(\mathfrak{D})) S_H^{\mathfrak{D}} \mathbf{f}(g^{-1}(\omega), g^{-1}),$$

$$\omega = g_1^{-1}(\omega(\mathfrak{D})) \quad (g_1 \in E), h_0^{-1}g_1g \in E.$$

$$(7.46) \quad \text{v) } \mathfrak{d}f \rightarrow \delta(g)\mathbf{f}(\omega, g).$$

Proof. These correspondences are confirmed by tracing the steps given in the beginning of this §. That is,

$$(7.47) \quad \text{i) } (R_{g_0}f)(g) = f(gg_0) \rightarrow (f(hg)) = \mathbf{f}_H(gg_0) \rightarrow \mathbf{f}(\omega, gg_0).$$

$$(7.48) \quad \text{ii) } (L_{g_0}f)(g) = \Delta_G(g_0)^{-1/2} f(g_0^{-1}g) \rightarrow (\Delta_G(g_0)^{-1/2} f(g_0^{-1}hg))$$

$$= (\Delta_G(g_0)^{-1/2} f(g_0^{-1}hg_0 \cdot g_0^{-1}g)) = (U_{g_0} \mathbf{f}_H)(g_0^{-1}g)$$

$$\rightarrow \Delta_G(g_0)^{1/2} W_{h_0}(\omega(\mathfrak{D})) V_{h_0}(\omega(\mathfrak{D})) \mathbf{f}(g_0^{-1}(\omega), g_0^{-1}g).$$

Here

$$\omega = g_1^{-1}(\omega(\mathfrak{D})) (g_1 \in E), h_0g_1g_0 \in E.$$

$$(7.49) \quad \text{iii) } (Jf)(g) = (\Delta_G(g))^{-1/2} \overline{f(g^{-1})} \rightarrow (\Delta_G(hg))^{-1/2} \overline{f(g^{-1}h^{-1})}$$

$$= (\Delta_G(g))^{-1/2} \overline{f(g^{-1}h^{-1}g \cdot g^{-1})} = (U_g J_H \mathbf{f}_H)(g^{-1})$$

$$\rightarrow \Delta_G(g)^{1/2} W_{h_0}(\omega(\mathfrak{D})) V_{h_0}(\omega(\mathfrak{D})) J_H^{\mathfrak{D}} \mathbf{f}(g^{-1}(\omega), g^{-1})$$

Here

$$\omega = g_1^{-1}(\omega(\mathfrak{D})) \quad (g_1 \in E), h_0 g_1 g \in E.$$

vi) is proved analogously to iii).

$$(7.50) \quad \text{v) } (\mathfrak{d}f)(g) = \delta(g)f(g) \rightarrow (\delta(hg)f(hg)) = \delta(g)f_H(g) \\ \rightarrow \delta(g)f(\omega, g).$$

This completes the proof.

Lemma 7.4. *According to the central decomposition (5.10), the operators J and S are decomposed as follows.*

$$(7.51) \quad J \sim \int_X J_{\mathfrak{D}} d\tilde{\mu}(\mathfrak{D}),$$

$$(7.52) \quad S \sim \int_X S_{\mathfrak{D}} d\tilde{\mu}(\mathfrak{D}).$$

Here $J_{\mathfrak{D}}, S_{\mathfrak{D}}$ are operators on $\tilde{\mathfrak{H}}(\mathfrak{D})$ defined by

$$(7.53) \quad (J_{\mathfrak{D}}f)(\omega, g) = \Delta_G(g)^{1/2} W_{h_0}(\omega(\mathfrak{D})) V_{h_0}(\omega(\mathfrak{D})) J_H^{\mathfrak{D}} f(g^{-1}(\omega), g^{-1}),$$

$$(7.54) \quad (S_{\mathfrak{D}}f)(\omega, g) = \Delta_G(g)^{1/2} W_{h_0}(\omega(\mathfrak{D})) V_{h_0}(\omega(\mathfrak{D})) S_H^{\mathfrak{D}} f(g^{-1}(\omega), g^{-1}),$$

$$\omega = g_1^{-1}(\omega(\mathfrak{D})) \quad (g_1 \in E), h_0^{-1} g_1 g \in E.$$

Proof. Because of (7.44) and (7.45), it is sufficient to see that $J_{\mathfrak{D}}$ and $S_{\mathfrak{D}}$ define isometric operators of $\tilde{\mathfrak{H}}(\mathfrak{D})$ onto itself.

$$(7.55) \quad (J_{\mathfrak{D}}f)(\omega, hg) = \Delta_G(hg)^{1/2} W_{h_1}(\omega(\mathfrak{D})) V_{h_1}(\omega(\mathfrak{D})) J_H^{\mathfrak{D}} \\ \times f((g^{-1}h^{-1})(\omega), g^{-1}h^{-1}) \\ = \Delta_G(g)^{1/2} W_{h_1}(\omega(\mathfrak{D})) V_{h_1}(\omega(\mathfrak{D})) J_H^{\mathfrak{D}} W_{g^{-1}h^{-1}g}(g^{-1}(\omega)) \\ \times f(g^{-1}(\omega), g^{-1}).$$

Here h_1 is given by

$$(7.56) \quad h_1^{-1}g_1hg = h_1^{-1}(g_1hg_1^{-1})g_1g = h_0^{-1}g_1g \in E,$$

i. e.,

$$(7.57) \quad h_1 = (g_1hg_1^{-1})h_0.$$

Since,

$$(7.58) \quad g^{-1}(\omega) = (g^{-1}g_1^{-1})(\omega(\mathfrak{D})) = (h_0^{-1}g_1g)^{-1}(\omega(\mathfrak{D})),$$

and

$$(7.59) \quad J_H^{\mathfrak{D}}W_h(\omega)J_H^{\mathfrak{D}} = V_h(\omega),$$

$$(7.60) \quad \begin{aligned} J_H^{\mathfrak{D}}W_{g^{-1}h^{-1}g}(g^{-1}(\omega))J_H^{\mathfrak{D}}W_{g^{-1}(h^{-1})}((h_0^{-1}g_1g)^{-1}(\omega(\mathfrak{D}))) \\ = J_H^{\mathfrak{D}}W_{(h_0^{-1}g_1gg^{-1})(h^{-1})}(\omega(\mathfrak{D})) = (J_H^{\mathfrak{D}}W_{h_0^{-1}g_1(h^{-1})}(\omega(\mathfrak{D})))J_H^{\mathfrak{D}}J_H^{\mathfrak{D}} \\ = V_{h_0^{-1}g_1h^{-1}g^{-1}h_0}(\omega(\mathfrak{D}))J_H^{\mathfrak{D}} = V_{h_1^{-1}h_0}(\omega(\mathfrak{D}))J_H^{\mathfrak{D}}. \end{aligned}$$

Therefore,

$$(7.61) \quad \begin{aligned} (J_{\mathfrak{D}}\mathbf{f})(\omega, hg) &= \Delta_G(g)^{1/2}W_{g_1(h)}(\omega(\mathfrak{D}))W_{h_0}(\omega(\mathfrak{D})) \\ &\quad \times V_{h_0}(\omega(\mathfrak{D}))J_H^{\mathfrak{D}}\mathbf{f}(\omega, g) \\ &= W_{g_1(h)}(\omega(\mathfrak{D}))\{(J_{\mathfrak{D}}\mathbf{f})(\omega, g)\} = W_h(g_1^{-1}(\omega(\mathfrak{D})))\{(J_{\mathfrak{D}}\mathbf{f})(\omega, g)\} \\ &= W_h(\omega)(J_{\mathfrak{D}}\mathbf{f})(\omega, g). \end{aligned}$$

$$(7.62) \quad \begin{aligned} \|J_{\mathfrak{D}}\mathbf{f}\|^2 &= \iint_{\mathfrak{D} \times H \setminus G} \|(J_{\mathfrak{D}}\mathbf{f})(\omega, g)\|^2 d\mu_{\mathfrak{D}}(\omega) d\tilde{g} \\ &= \iint_{\mathfrak{D} \times H \setminus G} \|\Delta_G(g)^{1/2}W_{h_0}(\omega(\mathfrak{D}))V_{h_0}(\omega(\mathfrak{D}))J_H^{\mathfrak{D}}\mathbf{f}(g^{-1}(\omega), g^{-1})\|^2 \\ &\quad \times d\mu_{\mathfrak{D}}(\omega) d\tilde{g} \\ &= \iint_{\mathfrak{D} \times H \setminus G} \|J_H^{\mathfrak{D}}(\mathbf{f}(\omega, g^{-1}))\|^2 \Delta_G(g) d\mu_{\mathfrak{D}}(g(\omega)) d\tilde{g} \end{aligned}$$

$$\begin{aligned} &= \iint_{\mathfrak{D} \times H \backslash G} \|\mathbf{f}(\omega, g^{-1})\|^2 d\mu_{\mathfrak{D}}(\omega) d\tilde{g} \\ &= \iint_{\mathfrak{D} \times H \backslash G} \|\mathbf{f}(\omega, g)\|^2 d\mu_{\mathfrak{D}}(\omega) d\tilde{g} = \|\mathbf{f}\|_{\mathfrak{D}}^2 < +\infty. \end{aligned}$$

Analogous relations are valid for $S_{\mathfrak{D}}$'s.

This completes the proof.

Lemma 7.5. $\{\tilde{\mathfrak{F}}^{\mathfrak{D}}, \hat{W}_{g_1}(\mathfrak{D}), \hat{V}_{g_2}(\mathfrak{D}), J_{\mathfrak{D}}\}$ is a double representation of G . Here,

$$(7.63) \quad (\hat{W}_{g_1}(\mathfrak{D})\mathbf{f})(\omega, g) = \mathbf{f}(\omega, gg_1),$$

$$(7.64) \quad (\hat{V}_{g_2}(\mathfrak{D})\mathbf{f})(\omega, g) = \Delta_G(g_2)^{1/2} W_{h_0}(\omega(\mathfrak{D})) V_{h_0}(\omega(\mathfrak{D})) \mathbf{f}(g_2^{-1}\omega, g_2^{-1}g),$$

$$\omega = g_1^{-1}(\omega(\mathfrak{D})) \quad (g_1 \in E), h_0^{-1}g_1g_2 \in E.$$

Proof. It is easy to see that $\{\tilde{\mathfrak{F}}(\mathfrak{D}), \hat{W}_{g_1}(\mathfrak{D})\}, \{\tilde{\mathfrak{F}}(\mathfrak{D}), \hat{V}_{g_2}(\mathfrak{D})\}$ give unitary representations of G , and

$$(7.65) \quad \hat{W}_{g_1}(\mathfrak{D})\hat{V}_{g_2}(\mathfrak{D}) = \hat{V}_{g_2}(\mathfrak{D})\hat{W}_{g_1}(\mathfrak{D}), \quad \text{for any } g_1, g_2 \text{ in } G.$$

So that, it is sufficient to show that

$$(7.66) \quad J_{\mathfrak{D}}\hat{W}_g(\mathfrak{D})J_{\mathfrak{D}} = \hat{V}_g(\mathfrak{D}).$$

$$(7.67) \quad \begin{aligned} (J_{\mathfrak{D}}\hat{W}_{g_0}(\mathfrak{D})J_{\mathfrak{D}}\mathbf{f})(\omega, g) &= \Delta_G(g)^{1/2} W_{h_1}(\omega(\mathfrak{D})) V_{h_1}(\omega(\mathfrak{D})) \\ &\quad \times J_H^{\mathfrak{D}}(\hat{W}_{g_0}(\mathfrak{D})J_{\mathfrak{D}}\mathbf{f})(g^{-1}\omega, g^{-1}) \\ &= \Delta_G(g)^{1/2} W_{h_1}(\omega(\mathfrak{D})) V_{h_1}(\omega(\mathfrak{D})) J_H^{\mathfrak{D}}(J_{\mathfrak{D}}\mathbf{f})(g^{-1}\omega, g^{-1}g_0) \\ &= \Delta_G(g)^{1/2} W_{h_1}(\omega(\mathfrak{D})) V_{h_1}(\omega(\mathfrak{D})) J_H^{\mathfrak{D}}\{\Delta_G(g^{-1}g_0)^{1/2} W_{h_2}(\omega(\mathfrak{D})) \\ &\quad \times V_{h_2}(\omega(\mathfrak{D})) J_H^{\mathfrak{D}}\mathbf{f}((g_0^{-1}g)(g^{-1}\omega), g_0^{-1}g)\} \\ &= \Delta_G(g)^{1/2} W_{h_1}(\omega(\mathfrak{D})) W_{h_2}(\omega(\mathfrak{D})) V_{h_1}(\omega(\mathfrak{D})) V_{h_2}(\omega(\mathfrak{D})) \\ &\quad \times \mathbf{f}(g_0^{-1}\omega, g_0^{-1}g). \end{aligned}$$

Here $\omega = g_1^{-1}(\omega(\mathfrak{D}))$ ($g_1 \ni E$), $h_1^{-1}g_1g \in E$, and $g^{-1}(\omega) = g_2^{-1}(\omega(\mathfrak{D}))$ ($g_2 \in E$), $h_2^{-1}g_2g^{-1}g_0 \in E$. Thus, $g^{-1}g_1^{-1}(\omega(\mathfrak{D})) = g_2^{-1}(\omega(\mathfrak{D}))$, and

$$(7.68) \quad g_2 = h_1^{-1}g_1g, \quad h_2^{-1}h_1^{-1}g_1g_0 \in E.$$

The relation (7.68) shows, $h_0 = h_1h_2$, and

$$(7.69) \quad \begin{aligned} (J_{\mathfrak{D}}W_{g_0}(\mathfrak{D})J_{\mathfrak{D}}f)(\omega, g) &= \Delta_G(g_0)^{1/2}W_{h_0}(\omega(\mathfrak{D})) \\ &\quad \times V_{h_0}(\omega(\mathfrak{D}))f(g_0^{-1}(\omega), g_0^{-1}g) \\ &= (\hat{V}_{g_0}(\mathfrak{D})f)(\omega, g). \end{aligned}$$

This completes the proof.

Lemma 7.6. $\{\tilde{\mathfrak{H}}(\mathfrak{D}), \hat{W}_g(\mathfrak{D})\}$ is equivalent to $\{\tilde{\mathfrak{H}}(\mathfrak{D}), \hat{V}_g(\mathfrak{D})\}$.

Proof. Indeed, similar arguments as in the proof of Lemma 7.5 shows

$$(7.70) \quad S_{\mathfrak{D}}\hat{W}_g(\mathfrak{D})S_{\mathfrak{D}} = \hat{V}_g(\mathfrak{D}).$$

This gives the unitary equivalence between above two representations.

Proposition 7.1. The double representation $\{L^2(G), R_{g_1}, L_{g_2}, J\}$ is decomposed by (5.10) as follows.

$$(7.71) \quad \{L^2(G), R_{g_1}, L_{g_2}, J\} \cong \int_X \{\tilde{\mathfrak{H}}(\mathfrak{D}), \hat{W}_{g_1}(\mathfrak{D}), \hat{V}_{g_2}(\mathfrak{D})\} d\bar{\mu}(\mathfrak{D}).$$

Proof. This is a summarized result of Lemmata 7.3–7.5.

Lemma 7.7. By the central decomposition (5.10), \mathfrak{d} is decomposed as,

$$(7.72) \quad \mathfrak{d} \sim \int_X \hat{T}_{\mathfrak{d}}(\mathfrak{D}) d\mu(\mathfrak{D}).$$

Here

$$(7.73) \quad (\hat{T}_\delta(\mathfrak{D})f)(\omega, g) = \delta(g)f(\omega, g).$$

$\hat{T}_\delta(\mathfrak{D})$ is an operator on the space $\tilde{\mathfrak{H}}(\mathfrak{D})$ of induced representation $\text{Ind}_{H \uparrow G} \left(\int_{\mathfrak{D}} \omega d\mu_{\mathfrak{D}}(\omega) \right)$, as is given in §2.

Proof. This is evident from (7.46) and the definition of $\hat{T}_\delta(\mathfrak{D})$.

q. e. d.

We use the notations $F_\delta(t)$ and $\tilde{\mathfrak{H}}_{\delta,t}(\mathfrak{D})$ given by (2.28), (2.29) for the induced representation \mathfrak{D} . Analogously denote,

$$(7.74) \quad L^2(G, t, \delta) \equiv \{f \in L^2(G) : [\text{Support of } f] \subset F_\delta(t)\},$$

$$(7.75) \quad C_0(G, t, \delta) \equiv \{f \in C_0(G) : [\text{Support of } f] \subset F_\delta(t)\},$$

$$(7.76) \quad \tilde{\mathcal{A}}(\mathfrak{D}, t, \delta) \equiv \{\text{components on } \mathfrak{D} \text{ of } f \text{ in } C_0(G, t, \delta)\},$$

By the reason of Lemma 7.7, the following Lemma is deduced soon.

Lemma 7.8. For any $t(>0)$ and for $\tilde{\mu}$ -almost all \mathfrak{D} ,

$$(7.77) \quad \tilde{\mathfrak{H}}_{\delta,t}(\mathfrak{D}) = \{\text{components on } \mathfrak{D} \text{ of } f \text{ in } L^2(G, t, \delta)\}.$$

Lemma 7.9. For any $t(>0)$ and for $\tilde{\mu}$ -almost all \mathfrak{D} , $\tilde{\mathcal{A}}(\mathfrak{D}, t, \delta)$ are dense in $\tilde{\mathfrak{H}}_{\delta,t}(\mathfrak{D})$.

Proof. Because of Lemma 7.8., $\tilde{\mathcal{A}}(\mathfrak{D}, t, \delta)$ are contained in $\tilde{\mathfrak{H}}_{\delta,t}(\mathfrak{D})$ for $\tilde{\mu}$ -almost all \mathfrak{D} . And the density of $\tilde{\mathcal{A}}(\mathfrak{D}, t, \delta)$ in $\tilde{\mathfrak{H}}_{\delta,t}(\mathfrak{D})$ follows from the density of $C_0(G, t, \delta)$ in $L^2(G, t, \delta)$. q. e. d.

Now we consider the structure of a quasi-Hilbert algebra defined on $C_0(G)$, as in the begining of this §. We can transfer this structure onto the ring of operators $\{R_f\}$ by the map

$$(7.78) \quad f \rightarrow R_f \equiv \int_G f(g)R_g d_r g.$$

And

$$(7.79) \quad \text{i) } R_{f_1} \cdot R_{f_2} = R_{f_1 * f_2} \quad (\text{product}),$$

$$(7.80) \quad \text{ii) } \langle R_{f_1}, R_{f_2} \rangle \equiv \langle f_1, f_2 \rangle \quad (\text{scalar product}),$$

$$(7.81) \quad \text{iii) } (R_f)^* \equiv R_{f^*} \quad (\text{involution}),$$

$$(7.82) \quad \text{iv) } (R_f)^\wedge \equiv R_{f^\wedge} \quad (\text{bijective linear map}).$$

It must be remarked that $(R_f)^*$ is different from the ordinary adjoint operator of R_f .

By the decomposition (7.71), the operator R_f corresponds to

$$(7.83) \quad \left\{ \widehat{W}_f(\mathfrak{D}) \equiv \int_G f(g) \widehat{W}_g(\mathfrak{D}) d_\tau g \right\}_{\mathfrak{D} \in X}.$$

Lemma 7.10. *By the decomposition (7.71), the operations on $C_0(G)$ are transferred on the operations on the ring $\{\widehat{W}_f(\mathfrak{D})\}$ for $\bar{\mu}$ -almost all \mathfrak{D} as follows.*

$$(7.84) \quad \text{i) } f_1 * f_2 \rightarrow R_{f_1} \cdot R_{f_2} \rightarrow \widehat{W}_{f_1}(\mathfrak{D}) \widehat{W}_{f_2}(\mathfrak{D}),$$

$$(7.85) \quad \text{ii) } \langle f_1, f_2 \rangle = \langle R_{f_1}, R_{f_2} \rangle \rightarrow \tau_{\mathfrak{D}}((T(\mathfrak{D}) \widehat{W}_{f_2}(\mathfrak{D}))^* T(\mathfrak{D}) \widehat{W}_{f_1}(\mathfrak{D})) \\ \equiv \langle \widehat{W}_{f_1}(\mathfrak{D}), \widehat{W}_{f_2}(\mathfrak{D}) \rangle,$$

$$(7.86) \quad \text{iii) } f^* \rightarrow (R_f)^* \rightarrow \widehat{W}_{f^*}(\mathfrak{D}) \equiv (\widehat{W}_f(\mathfrak{D}))^*,$$

$$(7.87) \quad \text{iv) } f^\wedge \rightarrow (R_f)^\wedge \rightarrow \widehat{W}_{f^\wedge}(\mathfrak{D}) = (T(\mathfrak{D}))^{-1} \widehat{W}_f(\mathfrak{D}) T(\mathfrak{D}) \equiv (\widehat{W}_f(\mathfrak{D}))^\wedge.$$

Proof. i) is trivial.

ii) follows from the Plancherel formula (6.2) immediately.

iii), iv) are considered as the definitions of $(\widehat{W}_f(\mathfrak{D}))^*$ and $(\widehat{W}_f(\mathfrak{D}))^\wedge$.

q. e. d.

Lemma 7.11. *By the above operations, $\{\widehat{W}_f(\mathfrak{D}); f \in C_0\}$ becomes*

a quasi-Hilbert algebra, for $\bar{\mu}$ -almost all \mathfrak{D} .

Proof. We must check the definition of a quasi-Hilbert algebra in the Dixmier's book [2].

$$\begin{aligned}
 (7.88) \quad (i) \quad & \langle \hat{W}_{f_1}(\mathfrak{D}), \hat{W}_{f_2}(\mathfrak{D}) \rangle = \tau_{\mathfrak{D}}((T(\mathfrak{D}) \hat{W}_{f_2}(\mathfrak{D}))^* T(\mathfrak{D}) \hat{W}_{f_1}(\mathfrak{D})) \\
 & = \tau_{\mathfrak{D}}(T(\mathfrak{D}) W_{f_1}(\mathfrak{D}) (T(\mathfrak{D}) W_{f_2}(\mathfrak{D}))^*), \text{ (cf. Lemma 2.9.)} \\
 & = \tau_{\mathfrak{D}}((\hat{W}_{f_1^*}(\mathfrak{D}) T(\mathfrak{D}))^* \hat{W}_{f_2^*}(\mathfrak{D}) T(\mathfrak{D})) \\
 & = \tau_{\mathfrak{D}}((T(\mathfrak{D}) \hat{W}_{f_1^* \Delta^{1/2}}(\mathfrak{D}))^* (T(\mathfrak{D}) \hat{W}_{f_2^* \Delta^{1/2}}(\mathfrak{D}))) \\
 & = (\tau_{\mathfrak{D}}((T(\mathfrak{D}) \hat{W}_{f_1^*}(\mathfrak{D}))^* T(\mathfrak{D}) \hat{W}_{f_2^*}(\mathfrak{D}))) \\
 & = \langle (W_{f_1}(\mathfrak{D}))^*, (W_{f_2}(\mathfrak{D}))^* \rangle.
 \end{aligned}$$

(ii) By Lemma 2.10.,

$$\begin{aligned}
 (7.89) \quad & \langle \hat{W}_k(\mathfrak{D}) \hat{W}_{f_1}(\mathfrak{D}), \hat{W}_{f_2}(\mathfrak{D}) \rangle = \tau_{\mathfrak{D}}((T(\mathfrak{D}) \hat{W}_{f_2}(\mathfrak{D}))^* T(\mathfrak{D}) \hat{W}_{k^* f_1}(\mathfrak{D})) \\
 & = \tau_{\mathfrak{D}}((T(\mathfrak{D}) \hat{W}_{(k^* \Delta \Delta^{-1/2})^* f_2}(\mathfrak{D}))^* T(\mathfrak{D}) \hat{W}_{f_1}(\mathfrak{D})) \\
 & = \tau_{\mathfrak{D}}((T(\mathfrak{D}) \hat{W}_{k^* \wedge}(\mathfrak{D}) \hat{W}_{f_2}(\mathfrak{D}))^* T(\mathfrak{D}) \hat{W}_{f_1}(\mathfrak{D})) \\
 & = \langle \hat{W}_{f_1}(\mathfrak{D}), (\hat{W}_k(\mathfrak{D}))^* \wedge \hat{W}_{f_2}(\mathfrak{D}) \rangle.
 \end{aligned}$$

(iii) From Lemma 2.12, for fixed k in $C_0(G)$, the followings are valid.

$$\begin{aligned}
 (7.90) \quad & | \langle \hat{W}_k(\mathfrak{D}) \hat{W}_{f_1}(\mathfrak{D}), \hat{W}_{f_2}(\mathfrak{D}) \rangle | = | \tau_{\mathfrak{D}}((T(\mathfrak{D}) \hat{W}_{f_2}(\mathfrak{D}))^* \\
 & \qquad \qquad \qquad \times T(\mathfrak{D}) \hat{W}_k(\mathfrak{D}) \hat{W}_{f_1}(\mathfrak{D})) | \\
 & \leq c_k \{ \tau_{\mathfrak{D}}((T(\mathfrak{D}) \hat{W}_{f_1}(\mathfrak{D}))^* T(\mathfrak{D}) \hat{W}_{f_1}(\mathfrak{D})) \cdot \\
 & \qquad \qquad \qquad \times \tau_{\mathfrak{D}}((T(\mathfrak{D}) \hat{W}_{f_2}(\mathfrak{D}))^* T(\mathfrak{D}) \hat{W}_{f_2}(\mathfrak{D})) \}^{1/2} \\
 & = c_k \{ \langle \hat{W}_{f_1}(\mathfrak{D}), \hat{W}_{f_1}(\mathfrak{D}) \rangle \langle \hat{W}_{f_2}(\mathfrak{D}), \hat{W}_{f_2}(\mathfrak{D}) \rangle \}^{1/2}.
 \end{aligned}$$

This shows that for fixed k in $C_0(G)$, the map

$$(7.91) \quad \widehat{W}_f(\mathfrak{D}) \rightarrow \widehat{W}_k(\mathfrak{D}) \widehat{W}_f(\mathfrak{D})$$

is continuous with respect to the above pre-Hilbertian topology on $C_0(G)$.

(iv) Let $\mathfrak{H}(\mathfrak{D})$ be the Hilbert space, the completion of

$$(7.92) \quad \mathcal{A}(\mathfrak{D}) \equiv \{ \widehat{W}_f(\mathfrak{D}); f \in C_0(G) \}$$

with respect to the above scalar product.

By (7.27), for any f in $C_0(G)$, the norm of components $f(\mathfrak{D})$ of f with respect to the decomposition (5.10) are given by

$$(7.93) \quad \|f(\mathfrak{D})\|_2^2 = \tau_{\mathfrak{D}}((T(\mathfrak{D}) \widehat{W}_f(\mathfrak{D}))^* T(\mathfrak{D}) \widehat{W}_f(\mathfrak{D}))$$

for $\bar{\mu}$ -almost all \mathfrak{D} . Therefore, from the separability of G , the map,

$$(7.94) \quad U^0(\mathfrak{D}); \widehat{W}_f(\mathfrak{D}) \rightarrow f(\mathfrak{D})$$

gives an isometric linear map from a dense subspace of $\mathfrak{H}(\mathfrak{D})$ onto a dense subspace of $\tilde{\mathfrak{H}}(\mathfrak{D})$, for $\bar{\mu}$ -almost all \mathfrak{D} . As the unique bounded extension of $U^0(\mathfrak{D})$, we obtain an isometric operator $U(\mathfrak{D})$ from $\mathfrak{H}(\mathfrak{D})$ onto $\tilde{\mathfrak{H}}(\mathfrak{D})$.

On the other hand, from general theory of $L^2(G)$, the space

$$(7.95) \quad \mathcal{A}_0 \equiv \{k * f; k, f \in C_0(G)\}$$

is dense in $L^2(G)$, hence, for $\bar{\mu}$ -almost all \mathfrak{D} , the set $\tilde{\mathcal{A}}_0(\mathfrak{D})$ of components of functions in \mathcal{A}_0 with respect to the decomposition (5.10) is dense in $\tilde{\mathfrak{H}}(\mathfrak{D})$. And as the inverse image of dense set $\tilde{\mathcal{A}}_0(\mathfrak{D})$ by $U(\mathfrak{D})$, the set

$$(7.96) \quad \mathcal{A}^0(\mathfrak{D}) \equiv \{ \widehat{W}_k(\mathfrak{D}) \widehat{W}_f(\mathfrak{D}) = \widehat{W}_{k*f}(\mathfrak{D}); k, f \in C_0(G) \}$$

is dense in $\mathfrak{H}(\mathfrak{D})$, especially, dense in $\mathcal{A}(\mathfrak{D})$, for $\bar{\mu}$ -almost all \mathfrak{D} .

(v) At first we define a linear map T on $L^2(G)$ by

$$(7.97) \quad (Tf)(g) \equiv f^\wedge(g) \quad (= (\Delta_G(g))^{1/2} f(g)).$$

Obviously, T is a bijective self-adjoint positive definite operator and by Lemma 7.7, T is decomposed under the central decomposition (5.10) as

$$(7.98) \quad T \sim \int_X T(\mathfrak{D}) d\tilde{\mu}(\mathfrak{D}).$$

Here $T(\mathfrak{D}) \equiv T_{\mathcal{A}'_G}(\mathfrak{D})$ are bijective self-adjoint positive definite operators on $\mathfrak{F}(\mathfrak{D})$. Therefore, clearly, the operator

$$(7.99) \quad T_0(\mathfrak{D}) \equiv (U(\mathfrak{D}))^{-1} T(\mathfrak{D}) U(\mathfrak{D})$$

is also a bijective self-adjoint positive definite operator on $\mathfrak{H}(\mathfrak{D})$ for $\tilde{\mu}$ -almost all \mathfrak{D} .

But, by (7.98) it is easy to see that for $\tilde{\mu}$ -almost all \mathfrak{D} ,

$$(7.100) \quad T_0(\mathfrak{D})(\hat{W}_{f_1}(\mathfrak{D})) = \hat{W}_{f_1^\wedge}(\mathfrak{D}) = (\hat{W}_{f_1}(\mathfrak{D}))^\wedge.$$

Let a and b be two elements in $\mathfrak{H}(\mathfrak{D})$ such that for any f_1 and f_2 in $C_0(G)$,

$$(7.101) \quad \begin{aligned} \langle a, \hat{W}_{f_1}(\mathfrak{D}) \hat{W}_{f_2}(\mathfrak{D}) \rangle &= \langle b, (\hat{W}_{f_1}(\mathfrak{D}))^\wedge (\hat{W}_{f_2}(\mathfrak{D}))^\wedge \rangle \\ &= \langle b, (\hat{W}_{f_1}(\mathfrak{D}) \hat{W}_{f_2}(\mathfrak{D}))^\wedge \rangle = \langle b, T_0(\mathfrak{D})(\hat{W}_{f_1}(\mathfrak{D}) \hat{W}_{f_2}(\mathfrak{D})) \rangle \end{aligned}$$

Then (7.101) means that b is in the domain $\mathfrak{D}(T_0(\mathfrak{D}))$ of $(T_0(\mathfrak{D}))^* = T_0(\mathfrak{D})$, and

$$(7.102) \quad a = T_0(\mathfrak{D})b.$$

From the definition of $T_0(\mathfrak{D})$, $\mathcal{A}(\mathfrak{D})$ is contained in $\mathfrak{D}(T_0(\mathfrak{D}))$.

We have to say the existence of a sequence $\{\hat{W}_{f_n}(\mathfrak{D})\}$ in $\mathcal{A}(\mathfrak{D})$ which converges to b and $\{(\hat{W}_{f_n}(\mathfrak{D}))^\wedge = T_0(\mathfrak{D})(\hat{W}_{f_n}(\mathfrak{D}))\}$ converges to a . Transferring the problem onto $\mathfrak{F}(\mathfrak{D})$ by $U(\mathfrak{D})$, it is sufficient to say the existence of a sequence $\{f_n(\mathfrak{D})\}$ of the components on \mathfrak{D} of functions $\{f_n\}$ in $C_0(G)$, which converges to $b_1 = U(\mathfrak{D})b$ and $\{\hat{T}(\mathfrak{D})f(\mathfrak{D})\}$

converges to $a_1 = U(\mathfrak{D})a = U(\mathfrak{D})T_0(\mathfrak{D})b = T(\mathfrak{D})b_1$. Here we remember the form of $\tilde{\mathfrak{F}}(\mathfrak{D})$ and $T(\mathfrak{D})$. That is, $\tilde{\mathfrak{F}}(\mathfrak{D})$ is the space of $\Sigma \oplus \mathcal{K}^{\mathfrak{D}}$ -valued functions $\mathbf{v}(g)$ on G , which satisfy,

$$(7.103) \quad \mathbf{v}(hg) = W_h(\omega)\mathbf{v}(g),$$

$$(7.104) \quad \int_{H \setminus G} \|\mathbf{v}(g)\|^2 dg < +\infty.$$

Moreover

$$(7.105) \quad T(\mathfrak{D})\mathbf{v}(g) = (\Delta_G(g))^{1/2}\mathbf{v}(g).$$

Hence, $b_1(g)$ belongs to $\mathfrak{D}(T(\mathfrak{D}))$, if and only if

$$(7.106) \quad \int_{H \setminus G} \|b_1(g)\|^2 d\tilde{g} < +\infty,$$

and

$$(7.107) \quad \int_{H \setminus G} \Delta_G(g) \|b_1(g)\|^2 d\tilde{g} < +\infty.$$

Now we take a sequence $\{t(n)\}$ of positive numbers, such a way that

$$(7.108) \quad \int_{(F_{t(n)}(\mathcal{A}^{1/2}))^c} \|b_1(g)\|^2 d\tilde{g} < \frac{1}{2n},$$

$$(7.109) \quad \int_{(F_{t(n)}(\mathcal{A}^{1/2}))^c} \Delta_G(g) \|b_1(g)\|^2 d\tilde{g} < \frac{1}{2n}.$$

Here $(F_{t(n)}(\mathcal{A}^{1/2}))^c$ shows the complement in $H \setminus G$ of the image of the set $F_{t(n)}(\mathcal{A}^{1/2})$ (cf. (2.28)) by the canonical map from G onto $H \setminus G$.

As is shown in Lemma 2.8, $T(\mathfrak{D})$ leaves invariant $\mathfrak{F}_t(\mathfrak{D}) (\equiv \mathfrak{F}_{\mathcal{A}^{1/2}, t}(\mathfrak{D}))$ and its restriction $T^t(\mathfrak{D})$ on $\mathfrak{F}_t(\mathfrak{D})$ satisfies,

$$(7.110) \quad \|T^t(\mathfrak{D})\| = t.$$

Therefore, by Lemma 7.9., we can take an element $\mathbf{v}_n(\equiv \mathbf{v}_n(g))$ in $\tilde{\mathcal{A}}(\mathfrak{D}, t(n), \mathcal{A}_G^{1/2})$, such that

$$(7.111) \quad \int_{(F_{t(n)}(\mathcal{A}^{1/2}))} \|b_1(g) - \mathbf{v}_n(g)\|^2 d\tilde{g} < \frac{1}{2n},$$

$$(7.112) \quad \int_{(F_{t(n)}(\mathcal{A}^{1/2}))} \mathcal{A}_G(g) \|b_1(g) - \mathbf{v}_n(g)\|^2 dg = \\ \int_{(F_{t(n)}(\mathcal{A}^{1/2}))} \|(T(\mathfrak{D})b_1)(g) - T^t(\mathfrak{D})\mathbf{v}_n(g)\|^2 d\tilde{g} < \frac{1}{2n}.$$

That is,

$$(7.113) \quad \|b_1 - \mathbf{v}_n\|^2 < \frac{1}{n},$$

$$(7.114) \quad \|T(\mathfrak{D})b_1 - T(\mathfrak{D})\mathbf{v}_n\|^2 < \frac{1}{n}.$$

We obtained a sequence $\{\mathbf{v}_n\}$ in $\tilde{\mathcal{A}}(\mathfrak{D})(\supset \cup \tilde{\mathcal{A}}(\mathfrak{D}, t, \mathcal{A}_G^{1/2}))$ which converges to b_1 and $\{T(\mathfrak{D})\mathbf{v}_n\}$ converges to $T(\mathfrak{D})b_1 = a_1$.

This completes the proof.

Summarizing the above mentioned lemmas, we obtain the decomposition of the regular quasi-Hilbert algebra on G .

Proposition 7.2. *The map,*

$$(7.115) \quad f \rightarrow \int_X \hat{W}_f(\mathfrak{D}) d\tilde{\mu}(\mathfrak{D})$$

gives a decomposition of the regular quasi-Hilbert algebra $C_0(G)$ on G . And the correspondences of the operations on this algebras are given in (7.84)~(7.87).

§8. Invariance of the Plancherel measure under the operations of Kronecker product

In the previous paper [14], we proved an invariance of the Plan-

cherel measure under the Kronecker product operations for unimodular locally compact groups of type I. Here we shall extend this property to the case of more general locally compact groups, and show the uniqueness of measure satisfying such an invariance.

Let $\mathfrak{D}_0 = \{\mathfrak{H}_0, U_g^0\}$ be a given fixed unitary representation of G . We fix a complete orthonormal basis $\{v_j\}$ in \mathfrak{H}_0 , and take a vector v in \mathfrak{H}_0 . For a general unitary representation $\mathfrak{D} = \{\mathfrak{H}, U_g\}$, we consider the Kronecker product $\mathfrak{D}_0 \otimes \mathfrak{D}$. By the map

$$(8.1) \quad \varphi(\mathfrak{D}, v); u \rightarrow v \otimes u, \quad (u \in \mathfrak{H}),$$

\mathfrak{H} is mapped into $\mathfrak{H}_0 \otimes \mathfrak{H}$. The image of $\varphi(\mathfrak{D}, v)$ is the closed subspace $v \otimes \mathfrak{H}$ of $\mathfrak{H}_0 \otimes \mathfrak{H}$.

On the other hand, for any function f in $L^1(G)$, we can consider a bounded operator on $\mathfrak{H}_0 \otimes \mathfrak{H}$ as

$$(8.2) \quad U_f(\mathfrak{D}_0 \otimes \mathfrak{D}) \equiv \int_G f(g) U_g(\mathfrak{D}_0 \otimes \mathfrak{D}) d_\tau g.$$

And define a bounded map from \mathfrak{H} to $\mathfrak{H}_0 \otimes \mathfrak{H}$ by

$$(8.3) \quad U_f(\mathfrak{D}_0, \mathfrak{D}, v) \equiv U_f(\mathfrak{D}_0 \otimes \mathfrak{D}) \circ \varphi(\mathfrak{D}, v).$$

Lemma 8.1. *For any f, k in $L^1(G)$ and any v, w in \mathfrak{H}_0 ,*

$$(8.4) \quad (U_f(\mathfrak{D}_0, \mathfrak{D}, v))^* U_k(\mathfrak{D}_0, \mathfrak{D}, w) = \sum_j (U_{f\psi(j,v)}(\mathfrak{D}))^* U_{k\psi(j,w)}(\mathfrak{D}).$$

Here

$$(8.5) \quad \psi(j, v)(g) \equiv \langle U_g(\mathfrak{D}_0)v, v_j \rangle, \quad \psi(j, w)(g) \equiv \langle U_g(\mathfrak{D}_0)w, v_j \rangle,$$

which are equal to zero for fixed v and w except for countably many v_j 's, even if \mathfrak{H}_0 is non-countably infinite dimensional. And the summation is taken under uniform topology of operators on \mathfrak{H} .

Proof. At first, since the series $\sum_j |\langle U_g(\mathfrak{D}_0)v, v_j \rangle|^2$ converges

uniformly on any compact subset of G , for any f in $L^1(G)$, there exists a number N such that, for any u in \mathfrak{D}_0 ,

$$\begin{aligned}
 (8.6) \quad & \sum_{j \geq N} \|U_{f\psi(j,v)}(\mathfrak{D})u\|^2 = \sum_{j \geq N} \langle U_{f\psi(j,v)}(\mathfrak{D})u, U_{f\psi(j,v)}(\mathfrak{D})u \rangle \\
 & \leq \iint_{G \times G} |f(g_1)| |f(g_2)| \times \\
 & \quad \times \left| \sum_{j \geq N} \langle U_{g_1}(\mathfrak{D}_0)v, v_j \rangle \overline{\langle U_{g_2}(\mathfrak{D}_0)v, v_j \rangle} \|u\|^2 d_r g_1 d_r g_2 \right. \\
 & \leq \varepsilon \|u\|^2.
 \end{aligned}$$

But,

$$\begin{aligned}
 (8.7) \quad & \left\| \sum_{j=N}^M (U_{f\psi(j,v)}(\mathfrak{D}))^* U_{k\psi(j,w)}(\mathfrak{D}) \right\| \\
 & = \sup_{\substack{\|u\| \leq 1 \\ \|u'\| \leq 1}} \left| \sum_{j=N}^M \langle U_{k\psi(j,w)}(\mathfrak{D})u, U_{f\psi(j,v)}(\mathfrak{D})u' \rangle \right| \\
 & \leq \sup_{\substack{\|u\| \leq 1 \\ \|u'\| \leq 1}} \left(\sum_{j=N}^M \|U_{k\psi(j,w)}(\mathfrak{D})u\|^2 \cdot \sum_{j=N}^M \|U_{f\psi(j,v)}(\mathfrak{D})u'\|^2 \right)^{1/2}.
 \end{aligned}$$

By the reason of (8.6), there exists a number N such that the right hand side of (8.7) is bounded by ε , independently on M . This shows that the uniform convergence of the operator $\sum_j (U_{f\psi(j,v)}(\mathfrak{D}))^* U_{k\psi(j,w)}(\mathfrak{D})$ in the right hand side of (8.4).

Thus, it is sufficient to show that for any u_1, u_2 in \mathfrak{D} ,

$$\begin{aligned}
 (8.8) \quad & \langle (U_f(\mathfrak{D}_0, \mathfrak{D}, v))^* U_k(\mathfrak{D}_0, \mathfrak{D}, w)u_1, u_2 \rangle \\
 & = \sum_j \langle (U_{f\psi(j,v)}(\mathfrak{D}))^* U_{k\psi(j,w)}(\mathfrak{D})u_1, u_2 \rangle.
 \end{aligned}$$

And this is shown by direct calculations as follows.

$$\begin{aligned}
 (8.9) \quad & \langle (U_f(\mathfrak{D}_0, \mathfrak{D}, v))^* U_k(\mathfrak{D}_0, \mathfrak{D}, w)u_1, u_2 \rangle \\
 & = \langle U_k(\mathfrak{D}_0, \mathfrak{D}, w)u_1, U_f(\mathfrak{D}_0, \mathfrak{D}, v)u_2 \rangle \\
 & = \langle U_k(\mathfrak{D}_0 \otimes \mathfrak{D}) \circ \varphi(\mathfrak{D}, w)u_1, U_f(\mathfrak{D}_0 \otimes \mathfrak{D}) \circ \varphi(\mathfrak{D}, v)u_2 \rangle
 \end{aligned}$$

$$\begin{aligned}
&= \langle U_k(\mathfrak{D}_0 \otimes \mathfrak{D})(w \otimes u_1), U_f(\mathfrak{D}_0 \otimes \mathfrak{D})(v \otimes u_2) \rangle \\
&= \iint_{G \times G} k(g_1) \overline{f(g_2)} \langle U_{g_1}(\mathfrak{D}_0 \otimes \mathfrak{D})(w \otimes u_1), U_{g_2}(\mathfrak{D}_0 \otimes \mathfrak{D}) \\
&\quad \times (v \otimes u_2) \rangle d_r g_1 d_r g_2 \\
&= \iint_{G \times G} k(g_1) \overline{f(g_2)} \langle U_{g_1}(\mathfrak{D}_0)w, U_{g_2}(\mathfrak{D}_0)v \rangle \\
&\quad \times \langle U_{g_1}(\mathfrak{D})u_1, U_{g_2}(\mathfrak{D})u_2 \rangle d_r g_1 d_r g_2 \\
&= \iint_{G \times G} k(g_1) \overline{f(g_2)} \sum_j \langle U_{g_1}(\mathfrak{D}_0)w, v_j \rangle \overline{\langle U_{g_2}(\mathfrak{D}_0)v, v_j \rangle} \\
&\quad \times \langle U_{g_1}(\mathfrak{D})u_1, U_{g_2}(\mathfrak{D})u_2 \rangle d_r g_1 d_r g_2 \\
&= \sum_j \iint_{G \times G} k(g_1) \psi(j, w)(g_1) \overline{f(g_2) \psi(j, v)(g_2)} \\
&\quad \times \langle U_{g_1}(\mathfrak{D})u_1, U_{g_2}(\mathfrak{D})u_2 \rangle d_r g_1 d_r g_2^*) \\
&= \sum_j \langle U_{k\psi(j, w)}(\mathfrak{D})u_1, U_{f\psi(j, v)}(\mathfrak{D})u_2 \rangle \\
&= \sum_j \langle (U_{f\psi(j, v)}(\mathfrak{D}))^* U_{k\psi(j, w)}(\mathfrak{D})u_1, u_2 \rangle.
\end{aligned}$$

q.e.d.

Now, we obtain the followings.

Proposition 8.1. For any f, k in $C_0(G)$ and any v, w in \mathfrak{S}_0 ,

$$\begin{aligned}
(8.10) \quad & \int_X \tau_{\mathfrak{D}}(T(\mathfrak{D})(\hat{W}_f(\mathfrak{D}_0, \mathfrak{D}, v))^* \hat{W}_k(\mathfrak{D}_0, \mathfrak{D}, w) T(\mathfrak{D})) d\tilde{\mu}(\mathfrak{D}) \\
&= \int_X \tau_{\mathfrak{D}}(T(\mathfrak{D})(\hat{W}_f(\mathfrak{D}))^* \hat{W}_k(\mathfrak{D}) T(\mathfrak{D})) d\tilde{\mu}(\mathfrak{D}) \langle w, v \rangle.
\end{aligned}$$

Here

$$\begin{aligned}
(8.11) \quad & \hat{W}_f(\mathfrak{D}_0, \mathfrak{D}, v) \equiv \hat{W}_f(\mathfrak{D}_0 \otimes \mathfrak{D}) \circ \varphi(\mathfrak{D}, v), \\
& \hat{W}_k(\mathfrak{D}_0, \mathfrak{D}, w) \equiv \hat{W}_k(\mathfrak{D}_0 \otimes \mathfrak{D}) \circ \varphi(\mathfrak{D}, w).
\end{aligned}$$

*) For the convergence of this integral, cf. [14].

And we extend the linear form $\tau_{\mathfrak{D}}$, to the operators of the form

$$(8.12) \quad T(\mathfrak{D})A_1^*A_2T(\mathfrak{D}) \equiv T(\mathfrak{D})\sum_j(\widehat{W}_{f_j}(\mathfrak{D}))^*\sum_j\widehat{W}_{k_j}(\mathfrak{D})T(\mathfrak{D}),$$

by

$$(8.13) \quad \tau_{\mathfrak{D}}(T(\mathfrak{D})A_1^*A_2T(\mathfrak{D})) = \sum_{j,l} \tau_{\mathfrak{D}}(T(\mathfrak{D})(\widehat{W}_{f_j}(\mathfrak{D}))^*\widehat{W}_{k_l}(\mathfrak{D})T(\mathfrak{D})).$$

And if H has the reduced dual of type I,

$$(8.14) \quad \int_X T_r(T(\mathfrak{D}^0)(\widehat{W}_f(\mathfrak{D}_0, \mathfrak{D}^0, v))^*\widehat{W}_k(\mathfrak{D}_0, \mathfrak{D}^0, w)T(\mathfrak{D}^0))d\bar{\mu}(\mathfrak{D}) \\ = \int_X T_r(T(\mathfrak{D}^0)(\widehat{W}_f(\mathfrak{D}^0))^*\widehat{W}_k(\mathfrak{D}^0)T(\mathfrak{D}^0))d\bar{\mu}(\mathfrak{D}) \langle w, v \rangle.$$

Proof. By the extended Plancherel formula (6.3),

$$(8.15) \quad \int_X \tau_{\mathfrak{D}}(T(\mathfrak{D})(\widehat{W}_f(\mathfrak{D}))^*\widehat{W}_k(\mathfrak{D})T(\mathfrak{D}))d\bar{\mu}(\mathfrak{D}) = \langle k\Delta_G^{1/2}, f\Delta_G^{1/2} \rangle.$$

Thus, using the result of Lemma 8.1.,

$$(8.16) \quad \int_X \tau_{\mathfrak{D}}(T(\mathfrak{D})(\widehat{W}_f(\mathfrak{D}_0, \mathfrak{D}, v))^*\widehat{W}_k(\mathfrak{D}_0, \mathfrak{D}, w)T(\mathfrak{D}))d\bar{\mu}(\mathfrak{D}) \\ = \sum_j \int_X \tau_{\mathfrak{D}}(T(\mathfrak{D})(\widehat{W}_{f\psi(j,v)}(\mathfrak{D}))^*(\widehat{W}_{k\psi(j,w)}(\mathfrak{D}))T(\mathfrak{D}))d\bar{\mu}(\mathfrak{D}) \\ = \sum_j \langle k\psi(j, w)\Delta_G^{1/2}, f\psi(j, v)\Delta_G^{1/2} \rangle \\ = \sum_j \int_G k(g) \langle U_g(\mathfrak{D}_0)w, v_j \rangle (\Delta_G(g))^{1/2} \overline{f(g) \langle U_g(\mathfrak{D}_0)v, v_j \rangle} \\ \quad \times (\Delta_G(g))^{1/2} d_r g \\ = \int_G k(g) \overline{f(g)} \sum_j \langle U_g(\mathfrak{D}_0)w, v_j \rangle \langle v_j, U_g(\mathfrak{D}_0)v \rangle \Delta_G(g) d_r g \\ = \int_G k(g) \overline{f(g)} \langle U_g(\mathfrak{D}_0)w, U_g(\mathfrak{D}_0)v \rangle \Delta_G(g) d_r g \\ = \langle k\Delta_G^{1/2}, f\Delta_G^{1/2} \rangle \langle w, v \rangle$$

$$= \int_X \tau_{\mathfrak{D}}(T(\mathfrak{D})(\widehat{W}_f(\mathfrak{D}))^* \widehat{W}_k(\mathfrak{D}) T(\mathfrak{D})) d\bar{\mu}(\mathfrak{D}) \langle w, v \rangle.$$

The proof of (8.14) is given just analogously. q. e. d.

Now we consider a n -dimensional subspace $\mathfrak{H}_n (n=1, 2, \dots)$ of \mathfrak{H}_0 . And take a complete orthonormal basis $\{u_j: 1 \leq j \leq n\}$ in \mathfrak{H}_n . Denote by $\varphi(\mathfrak{D}, \mathfrak{H}_n)$, the imbedding map of $\mathfrak{H}_n \otimes \mathfrak{H}$ into $\mathfrak{H}_0 \otimes \mathfrak{H}$.

On the other hand we can extend canonically the linear form $\tau_{\mathfrak{D}}$ on the space \mathfrak{A}_0^2 of operators on \mathfrak{H} onto $M_n \otimes \mathfrak{A}_0^2$ by defining

$$(8.17) \quad \tilde{\tau}_{\mathfrak{D}}(A \otimes B) \equiv T_r(A) \times \tau_{\mathfrak{D}}(B) \equiv \left(\sum_j^n \langle Au_j, u_j \rangle \right) \tau_{\mathfrak{D}}(B).$$

And for the identity operators I_n on \mathfrak{H}_n and I on \mathfrak{H}_0 , put

$$(8.18) \quad \hat{T}_n(\mathfrak{D}) \equiv I_n \otimes T(\mathfrak{D}), \text{ and } \hat{T}(\mathfrak{D}) \equiv I \otimes T(\mathfrak{D}).$$

Proposition 8.2. *For any f in $C_0(G)$ and any n -dimensional subspace \mathfrak{H}_n of \mathfrak{H}_0 .*

$$(8.19) \quad (n)^{-1} \int_X \tilde{\tau}_{\mathfrak{D}}(\hat{T}_n(\mathfrak{D})(\widehat{W}_f(\mathfrak{D}_0, \mathfrak{D}, \mathfrak{H}_n))^* \widehat{W}_f(\mathfrak{D}_0, \mathfrak{D}, \mathfrak{H}_n) \hat{T}_n(\mathfrak{D})) d\bar{\mu}(\mathfrak{D}) \\ = \int_X \tau_{\mathfrak{D}}(T(\mathfrak{D})(\widehat{W}_f(\mathfrak{D}))^* \widehat{W}_f(\mathfrak{D}) T(\mathfrak{D})) d\bar{\mu}(\mathfrak{D}).$$

Here,

$$(8.20) \quad \widehat{W}_f(\mathfrak{D}_0, \mathfrak{D}, \mathfrak{H}_n) \equiv \widehat{W}_f(\mathfrak{D}_0 \otimes \mathfrak{D}) \circ \varphi(\mathfrak{D}, \mathfrak{H}_n).$$

And if H has the reduced dual of type I,

$$(8.21) \quad (n)^{-1} \int_X \|\widehat{W}_f(\mathfrak{D}_0, \mathfrak{D}, \mathfrak{H}_n) \hat{T}_n(\mathfrak{D})\|^2 d\bar{\mu}(\mathfrak{D}) \\ = \int_X \|\widehat{W}_f(\mathfrak{D}) T(\mathfrak{D})\|^2 d\bar{\mu}(\mathfrak{D}).$$

Proof. Put P_j the projection on \mathfrak{H}_n , whose range is the space $\{Cu_j\}$ spanned by the vector u_j , and put

$$(8.22) \quad \tilde{P}_j \equiv P_j \otimes I.$$

From the definition (8.17) of $\tilde{\tau}_{\mathfrak{D}}$,

$$(8.23) \quad \tilde{\tau}_{\mathfrak{D}}(A \otimes B) = \sum_j \tilde{\tau}_{\mathfrak{D}}(\tilde{P}_j(A \otimes B)\tilde{P}_j),$$

and for an operator C on $\mathfrak{H}_0 \otimes \mathfrak{H}$,

$$(8.24) \quad \tilde{\tau}_{\mathfrak{D}}((\varphi(\mathfrak{D}, \mathfrak{H}_n))^* C \varphi(\mathfrak{D}, \mathfrak{H}_n)) = \sum_j \tau_{\mathfrak{D}}((\varphi(\mathfrak{D}, u_j))^* C \varphi(\mathfrak{D}, u_j)).$$

Therefore

$$\begin{aligned} (8.25) \quad & \tilde{\tau}_{\mathfrak{D}}(\hat{T}_n(\mathfrak{D})(\hat{W}_f(\mathfrak{D}_0, \mathfrak{D}, \mathfrak{H}_n))^* \hat{W}_f(\mathfrak{D}_0, \mathfrak{D}, \mathfrak{H}_n) \hat{T}_n(\mathfrak{D})) \\ &= \tilde{\tau}_{\mathfrak{D}}(\hat{T}_n(\mathfrak{D})(\varphi(\mathfrak{D}, \mathfrak{H}_n))^* (\hat{W}_f(\mathfrak{D}_0 \otimes \mathfrak{D}))^* \\ & \quad \times \hat{W}_f(\mathfrak{D}_0 \otimes \mathfrak{D}) \varphi(\mathfrak{D}, \mathfrak{H}_n) \hat{T}_n(\mathfrak{D})) \\ &= \tilde{\tau}_{\mathfrak{D}}((\varphi(\mathfrak{D}, \mathfrak{H}_n))^* \hat{T}(\mathfrak{D})(\hat{W}_f(\mathfrak{D}_0 \otimes \mathfrak{D}))^* \\ & \quad \times \hat{W}_f(\mathfrak{D}_0 \otimes \mathfrak{D}) \hat{T}(\mathfrak{D}) \varphi(\mathfrak{D}, \mathfrak{H}_n)) \\ &= \sum_j \tau_{\mathfrak{D}}((\varphi(\mathfrak{D}, u_j))^* \hat{T}(\mathfrak{D})(\hat{W}_f(\mathfrak{D}_0 \otimes \mathfrak{D}))^* \\ & \quad \times \hat{W}_f(\mathfrak{D}_0 \otimes \mathfrak{D}) \hat{T}(\mathfrak{D}) \varphi(\mathfrak{D}, u_j)) \\ &= \sum_j \tau_{\mathfrak{D}}(T(\mathfrak{D})(\hat{W}_f(\mathfrak{D}_0, \mathfrak{D}, u_j))^* \hat{W}_f(\mathfrak{D}_0, \mathfrak{D}, u_j) T(\mathfrak{D})). \end{aligned}$$

Thus (8.19) follows from Proposition 8.1 directly.

The equality (8.21) is obtained by analogous way from (8.14).

q. e. d.

Considering the Kronecker product operation as the product on the reduced quasi-dual X of G , the Plancherel measure has a property of the invariance under this product. This invariance is very analogous to the invariance of the Haar measure on groups under the group product operation. Indeed, in abelian case, the Plancherel measure is just the Haar measure on the dual group. Thus, following along the line

of the theory of Haar measures on groups, we shall discuss the uniqueness of invariant measure on X up to constant.

Definition 8.1. Let X_0 be the set of all equivalence classes of unitary representations of G , dimensions of which are at most countably infinite. Here, a subset \mathfrak{I} of X_0 is called an ideal, when

- 1) for any countable subset $\{\mathfrak{D}_j\}$ of \mathfrak{I} , $\Sigma \oplus \mathfrak{D}_j$ belongs to \mathfrak{I} ,
- 2) if \mathfrak{D} is in \mathfrak{I} , any subrepresentation of \mathfrak{D} is in \mathfrak{I} ,
- 3) for any \mathfrak{D} in \mathfrak{I} and any representation \mathfrak{D}_0 in X_0 , $\mathfrak{D}_0 \otimes \mathfrak{D}$ is in \mathfrak{I} .

Lemma 8.2. The set $\mathfrak{I}_{\mathfrak{R}}$ of all equivalence classes of subrepresentations of countable multiple $\Sigma \oplus \mathfrak{R}$ is the smallest non-empty ideal. Here, \mathfrak{R} shows the regular representation of G .

Proof. From the definition of $\mathfrak{I}_{\mathfrak{R}}$, 1) and 2) of Definition 8.1 are trivial. Moreover, as is well-known, for any representation \mathfrak{D}_0 in X_0 , $\mathfrak{D}_0 \otimes \mathfrak{R}$ is equivalent to the multiple $\Sigma \oplus \mathfrak{R}$ of \mathfrak{R} , with the multiplicity of dimension of \mathfrak{D}_0 , hence 3) is valid for $\mathfrak{I}_{\mathfrak{R}}$. Thus $\mathfrak{I}_{\mathfrak{R}}$ is a non-empty ideal.

Next, for any element \mathfrak{D}_0 in a given ideal, $\mathfrak{R} \otimes \mathfrak{D}_0$ must belong to \mathfrak{I} . But from the commutativity of the Kronecker products $\mathfrak{R} \otimes \mathfrak{D}_0$ is equivalent to $\mathfrak{D}_0 \otimes \mathfrak{R} \sim \Sigma \oplus \mathfrak{R}$. This means that $\Sigma \oplus \mathfrak{R}$ belongs to \mathfrak{I} , hence \mathfrak{R} is in \mathfrak{I} . Therefore, $\mathfrak{I}_{\mathfrak{R}}$ is contained in \mathfrak{I} . That is, $\mathfrak{I}_{\mathfrak{R}}$ is the smallest. q. e. d.

Now we denote the Borel structure on X given in §4, by \mathfrak{B} .

Definition 8.2. A standard positive measure $\tilde{\mu}_1$ on (X, \mathfrak{B}) is called admissible, when

- 1) the direct integral

$$(8.26) \quad \tilde{\mathfrak{D}} \equiv \int_X \mathfrak{D} d\tilde{\mu}_1(\mathfrak{D})$$

is central,

2) for $\tilde{\mu}_1$ -almost all \mathfrak{D} , the following unitary representation $\mathfrak{D}_1 = \{\mathfrak{H}(\mathfrak{D}), U_g(\mathfrak{D})\}$ is equivalent to \mathfrak{D} by the equivalence relation $U^0(\mathfrak{D})$ defined by (7.94). Here $\mathfrak{H}(\mathfrak{D})$ is the Hilbert space obtained by the completion of $\mathcal{A}(\mathfrak{D})$ in (7.92), with respect to the scalar product (7.85), and $U_g(\mathfrak{D})$ is the operator of continuous extension of the map

$$(8.27) \quad \widehat{W}_f(\mathfrak{D}) \rightarrow \widehat{W}_f(\mathfrak{D}) \widehat{W}_{g^{-1}}(\mathfrak{D})$$

on $\mathcal{A}(\mathfrak{D})$.

Moreover an admissible measure $\tilde{\mu}_1$ is called invariant, when for any f in $C_0(G)$ and for any \mathfrak{D}_0 in X_0 , for any v in the space \mathfrak{D}_0 of representation \mathfrak{D}_0 , the followings are valid.

$$(8.28) \quad \int_X \tau_{\mathfrak{D}}(T(\mathfrak{D})(\widehat{W}_f(\mathfrak{D}_0, \mathfrak{D}, v))^* \widehat{W}_f(\mathfrak{D}_0, \mathfrak{D}, v) T(\mathfrak{D})) d\tilde{\mu}_1(\mathfrak{D}) \\ = \int_X \tau_{\mathfrak{D}}(T(\mathfrak{D})(\widehat{W}_f(\mathfrak{D}))^* \widehat{W}_f(\mathfrak{D}) T(\mathfrak{D})) d\tilde{\mu}_1(\mathfrak{D}) \|v\|^2.$$

Here we must remark that (8.28) is equivalent to the following equation, for any f and k in $C_0(G)$ and for any \mathfrak{D}_0 , for any v, w in \mathfrak{D}_0 .

$$(8.29) \quad \int_X \tau_{\mathfrak{D}}(T(\mathfrak{D})(\widehat{W}_f(\mathfrak{D}_0, \mathfrak{D}, v))^* \widehat{W}_k(\mathfrak{D}_0, \mathfrak{D}, w) T(\mathfrak{D})) d\tilde{\mu}_1(\mathfrak{D}) \\ = \int_X \tau_{\mathfrak{D}}(T(\mathfrak{D})(\widehat{W}_f(\mathfrak{D}))^* \widehat{W}_k(\mathfrak{D}) T(\mathfrak{D})) d\tilde{\mu}_1(\mathfrak{D}) \langle w, v \rangle.$$

Indeed, for brevity, put the left hand side of (8.29) $\lambda(k, w; f, v)$ and put the right hand side of (8.29) $\lambda_0(k, f) \langle w, v \rangle$, then it is easy to see that the both sides are bilinear with respect to k, w and bi-skew linear with respect to f, v . And (8.28) shows,

$$(8.30) \quad \lambda(f, v; f, v) = \lambda_0(f, f) \langle v, v \rangle.$$

Now we fix a non-zero v in \mathfrak{D}_0 , and substitute f with $f \pm k, f \pm ik$.

Subtractions of both sides leads us to

$$(8.31) \quad \lambda(f, v; k, v) = \lambda_0(f, k) \langle v, v \rangle.$$

Next, substitute v with $v \pm w$ and $v \pm iw$, then in a similar way, we obtain

$$(8.32) \quad \lambda(f, v; k, w) = \lambda_0(f, k) \langle w, v \rangle.$$

This is just (8.29).

Lemma 8.3. *If $\tilde{\mu}_1$ is an invariant admissible measure on $\{X, \mathfrak{B}\}$, $\mathfrak{D}_0 \otimes \tilde{\mathfrak{D}}$ is equivalent to some subrepresentation of the multiple $\Sigma \oplus \tilde{\mathfrak{D}}$ of $\tilde{\mathfrak{D}}$ with the multiplicity of the dimension of \mathfrak{D}_0 , for any \mathfrak{D}_0 in X_0 . Here $\tilde{\mathfrak{D}}$ is the representation defined by (8.26).*

Proof. From 2) of Definition 8.2, the space $\mathfrak{H}(\tilde{\mathfrak{D}})$ of representation $\tilde{\mathfrak{D}}$ is considered as the space obtained by completion of the space of operator fields

$$(8.33) \quad \mathbf{v}_f \equiv \{\hat{W}_f(\mathfrak{D})\}$$

on X , with respect to the norm defined by

$$(8.34) \quad \|\mathbf{v}_f\|^2 = \int_X \tau_{\mathfrak{D}}((\hat{W}_f(\mathfrak{D}))^*(T(\mathfrak{D}))^2 \hat{W}_f(\mathfrak{D})) d\tilde{\mu}_1(\mathfrak{D}).$$

On the other hand, by Lemma 8.1,

$$(8.35) \quad \begin{aligned} \tau_{\mathfrak{D}}(T(\mathfrak{D})(\hat{W}_f(\mathfrak{D}_0, \mathfrak{D}, v))^* \hat{W}_k(\mathfrak{D}_0, \mathfrak{D}, w) T(\mathfrak{D})) \\ = \sum_j \tau_{\mathfrak{D}}(T(\mathfrak{D})(\hat{W}_{f\psi(j,v)}(\mathfrak{D}))^* \hat{W}_{k\psi(j,w)}(\mathfrak{D}) T(\mathfrak{D})). \end{aligned}$$

Therefore, (8.28), hence (8.29) are equivalent to

$$(8.36) \quad \sum_j \langle \mathbf{v}_{k\psi(j,w)}, \mathbf{v}_{f\psi(j,v)} \rangle = \langle \mathbf{v}_k, \mathbf{v}_f \rangle \langle w, v \rangle.$$

The left hand side of (8.36) is considered as the scalar product of the vectors $\{\mathbf{v}_{k\psi(j,w)}\}_j$ and $\{\mathbf{v}_{f\psi(j,v)}\}_j$ in the space $\sum_j \oplus \mathfrak{H}(\tilde{\mathfrak{D}})$. And the right hand side of (8.36) is the scalar product of the vectors $w \otimes \mathbf{v}_k$ and $v \otimes \mathbf{v}_f$ in the space $\mathfrak{H}_0 \otimes \mathfrak{H}(\tilde{\mathfrak{D}})$. Thus it is easy to see the map U defined by the followings gives an isometric map from $\mathfrak{H}_0 \otimes \mathfrak{H}(\tilde{\mathfrak{D}})$ into $\sum_j \oplus \mathfrak{H}(\tilde{\mathfrak{D}})$.

$$(8.37) \quad U(\sum_l v_l \otimes \mathbf{v}_{f_l}) \equiv \{\sum_l \mathbf{v}_{f_l \psi(j, v_l)}\}_j.$$

Because of (8.27) $U_g(\tilde{\mathfrak{D}})\mathbf{v}_f$, corresponds to the operator field $\{\hat{W}_f(\mathfrak{D})\hat{W}_{g^{-1}}(\mathfrak{D}) = W_{f*\delta_{g^{-1}}}(\mathfrak{D})\}$. Therefore,

$$(8.38) \quad U(U_g(\mathfrak{D}_0)v \otimes U_g(\tilde{\mathfrak{D}})\mathbf{v}_f) = \{\mathbf{v}_{(f*\delta_{g^{-1}})\psi(j, U_g(\mathfrak{D}_0)v)}\}_j.$$

Since,

$$(8.39) \quad \psi(j, U_g(\mathfrak{D}_0)v)(g_0) = \langle U_{g_0}(\mathfrak{D}_0)U_g(\mathfrak{D}_0)v, v_j \rangle = \psi(j, v)(g_0g),$$

$$(8.40) \quad U(U_g(\mathfrak{D}_0)v \otimes U_g(\tilde{\mathfrak{D}})\mathbf{v}_f) = \{\mathbf{v}_{(f\psi(j,v)*\delta_{g^{-1}})}\}_j = \{U_g(\tilde{\mathfrak{D}})\mathbf{v}_{f\psi(j,v)}\}_j.$$

This shows,

$$(8.41) \quad U(U_g(\mathfrak{D}_0) \otimes U_g(\tilde{\mathfrak{D}})) \subseteq \sum_j \oplus U_g(\tilde{\mathfrak{D}}).$$

This completes the proof.

Corollary 1. *The set of all equivalent classes of subrepresentations of countable multiple $\sum \oplus \tilde{\mathfrak{D}}$ of $\tilde{\mathfrak{D}}$ is an ideal.*

Proof. This is a direct result of Lemma 8.3 and the definition of an ideal. q. e. d.

Corollary 2. *The regular representation \mathfrak{R} is equivalent to a subrepresentation of $\sum \oplus \tilde{\mathfrak{D}}$.*

Proof. Trivial from Lemma 8.2 and Corollary 1 to Lemma 8.3.

q. e. d.

From 1) of Definition 8.2 and from the uniqueness of the central decomposition,

$$(8.42) \quad \sum_j \oplus \tilde{\mathfrak{D}} \simeq \int_X (\sum_j \oplus \mathfrak{D}) d\tilde{\mu}_1(\mathfrak{D})$$

gives the central decomposition of $\sum \oplus \tilde{\mathfrak{D}}$. Therefore, the central decomposition of any subrepresentation \mathfrak{D}_1 of $\sum \oplus \tilde{\mathfrak{D}}$ is given by

$$(8.43) \quad \mathfrak{D}_1 \simeq \int_X P(\mathfrak{D})(\sum \oplus \mathfrak{D}) d\tilde{\mu}_1(\mathfrak{D}).$$

Here $P(\mathfrak{D})$ is the projection on $\sum \oplus \mathfrak{D}$, which is defined by the decomposition of the projection P on $\mathfrak{D}(\tilde{\mathfrak{D}})$ to the space of subrepresentation \mathfrak{D}_1 as

$$(8.44) \quad P \sim \int_X P(\mathfrak{D}) d\tilde{\mu}_1(\mathfrak{D})$$

for $\tilde{\mu}_1$ -almost all \mathfrak{D} .

Lemma 8.4. *The Plancherel measure $\tilde{\mu}$ is absolutely continuous with respect to any invariant admissible measure $\tilde{\mu}_1$ on (X, \mathfrak{B}) .*

Proof. By the reason of Corollary 2 to Lemma 8.3, we can apply (8.43) to the case that $\mathfrak{R} \simeq \mathfrak{D}_1$. Thus the central decomposition of \mathfrak{R} must be given by

$$(8.45) \quad \mathfrak{R} \simeq \int_X P(\mathfrak{D})(\sum \oplus \mathfrak{D}) d\tilde{\mu}_1(\mathfrak{D}).$$

On the other hand, the arguments in §5 claim that the central decomposition of \mathfrak{R} is given by

$$(8.46) \quad \mathfrak{R} \simeq \int_X \mathfrak{D} d\tilde{\mu}(\mathfrak{D}).$$

Again from the uniqueness of the central decomposition, $\tilde{\mu}$ must be absolutely continuous with respect to $\tilde{\mu}_1$. q. e. d.

We can prove the converse assertion,

Lemma 8.5. *Any invariant admissible measure $\tilde{\mu}_1$ on (X, \mathfrak{B}) is absolutely continuous with respect to the Plancherel measure $\tilde{\mu}$.*

Proof. If not, we can write as

$$(8.47) \quad \tilde{\mu}_1 = \tilde{\nu}_1 + \tilde{\nu}_2.$$

Here $\tilde{\nu}_1$ and $\tilde{\mu}$ are mutually singular and $\tilde{\nu}_2$ is absolutely continuous with respect to $\tilde{\mu}$.

By Lemma 8.3, $\mathfrak{R} \otimes \tilde{\mathfrak{D}}$ is equivalent to a subrepresentation of $\Sigma \oplus \tilde{\mathfrak{D}}$. So we can apply (8.43) to the case $\tilde{\mathfrak{D}}_1 \sim \mathfrak{R} \otimes \tilde{\mathfrak{D}}$, and we obtain the central decomposition of the form

$$(8.48) \quad \mathfrak{R} \otimes \tilde{\mathfrak{D}} \simeq \int_X P_1(\mathfrak{D})(\Sigma \oplus \mathfrak{D}) d\tilde{\mu}_1(\mathfrak{D}).$$

On the other hand, $\mathfrak{R} \otimes \tilde{\mathfrak{D}}$ has the central decomposition

$$(8.49) \quad \mathfrak{R} \otimes \tilde{\mathfrak{D}} \simeq \Sigma \oplus \mathfrak{R} \simeq \int_X \Sigma \oplus \mathfrak{D} d\tilde{\mu}(\mathfrak{D}).$$

Comparing (8.48) and (8.49) and from the uniqueness of the central decomposition, we obtain that

$$(8.50) \quad P_1(\mathfrak{D}) = 0, \quad \text{for } \tilde{\nu}_1\text{-almost all } \mathfrak{D}.$$

And from (8.33) and (8.37), this is equivalent to that for any j and any v in $\mathfrak{H}_0(=L^2(G))$ and any f in $C_0(G)$,

$$(8.51) \quad \hat{W}_{f\psi(j,v)}(\mathfrak{D}) = 0, \quad \text{for } \tilde{\nu}_1\text{-almost all } \mathfrak{D}.$$

By the reason of the arbitrariness of the selection of the complete or-

thonormal basis $\{v_j\}$ in \mathfrak{H}_0 , we obtain that for any v, u in $L^2(G)$ and any f in $C_0(G)$,

$$(8.52) \quad \widehat{W}_{f\psi}(\mathfrak{D}) = 0, \quad \text{for } \nu_1\text{-almost all } \mathfrak{D}.$$

Here

$$(8.53) \quad \psi(g) \equiv \langle U_g v, u \rangle.$$

Since for any function k in $C_0(G)$, it is easy to find functions f, u, v in $C_0(G)$ such a way that

$$(8.54) \quad f(g) \langle U_g v, u \rangle = k(g),$$

The relation (8.52) means that for any f in $C_0(G)$

$$(8.55) \quad \widehat{W}_f(\mathfrak{D}) = 0, \quad \text{for } \nu_1\text{-almost all } \mathfrak{D}.$$

That is, $\mathcal{A}(\mathfrak{D})$, consequently, $\mathfrak{H}(\mathfrak{D})$ are trivial for ν_1 -almost all \mathfrak{D} . This contradicts to 2) of Definition 8.2. q. e. d.

Proposition 8.3. *Any invariant admissible measure $\tilde{\mu}_1$ on (X, \mathfrak{B}) is the form of*

$$(8.56) \quad d\tilde{\mu}_1(\mathfrak{D}) = c d\tilde{\mu}(\mathfrak{D}) \quad (c > 0).$$

Proof. Because of Lemmata 8.4 and 8.5, there exists a positive measurable function $w(\mathfrak{D})$ on X , such that

$$(8.57) \quad d\tilde{\mu}_1(\mathfrak{D}) = w(\mathfrak{D}) d\tilde{\mu}(\mathfrak{D}).$$

Consider the scalar operator $w(\mathfrak{D})I_{\mathfrak{D}}$ on each space $\mathfrak{H}(\mathfrak{D})$, and define a positive definite self-adjoint operator A on $L^2(G)$ by

$$(8.58) \quad A = \int_X w(\mathfrak{D}) I_{\mathfrak{D}} d\tilde{\mu}(\mathfrak{D}).$$

Since the direct integral (8.46) is the central decomposition of \mathfrak{K} , it is easy to see that

$$(8.59) \quad AR_g = R_g A, \quad AL_g = L_g A, \quad \text{for any } g \text{ in } G.$$

Moreover,

$$(8.60) \quad \begin{aligned} \langle Af, k \rangle &= \int_X \tau_{\mathfrak{D}}((T(\mathfrak{D}) \widehat{W}_k(\mathfrak{D}))^* T(\mathfrak{D}) \widehat{W}_f(\mathfrak{D})) w(\mathfrak{D}) d\bar{\mu}(\mathfrak{D}) \\ &= \int_X \tau_{\mathfrak{D}}((T(\mathfrak{D}) \widehat{W}_k(\mathfrak{D}))^* T(\mathfrak{D}) \widehat{W}_f(\mathfrak{D})) d\bar{\mu}_1(\mathfrak{D}). \end{aligned}$$

Now consider the decomposition

$$(8.61) \quad \mathfrak{D}_0 \otimes \mathfrak{K} \sim \Sigma \oplus \mathfrak{K}$$

which is obtained by the map U in (8.37) in the case $\bar{\mu}_1 = \bar{\mu}$. And, we get

$$(8.62) \quad U(v \otimes f) = \{f\psi(j, v)\}_j, \quad v \in \mathfrak{D}_0, \quad f \in L^2(G),$$

$$(8.63) \quad \psi(j, v)(g) = \langle U_g v, v_j \rangle.$$

And the invariance of $\bar{\mu}$ asserts

$$(8.64) \quad \langle v, w \rangle \langle f, k \rangle_{L^2(G)} = \sum_j \langle f\psi(j, v), k\psi(j, w) \rangle_{L^2(G)}.$$

Substituting Af for f ,

$$(8.65) \quad \langle v, w \rangle \langle Af, k \rangle_{L^2(G)} = \sum_j \langle (Af)\psi(j, v), k\psi(j, w) \rangle_{L^2(G)}.$$

On the other hand, from the invariance of $\bar{\mu}_1$,

$$(8.66) \quad \langle v, w \rangle \langle Af, k \rangle_{L^2(G)} = \sum_j \langle A(f\psi(j, v)), k\psi(j, w) \rangle_{L^2(G)}.$$

But the vectors $\{k\psi(j, w)\}_j$ span the space $\sum_j \oplus L^2(G)$, therefore the followings are valid.

$$(8.67) \quad \{(Af)\psi(j, v)\}_j = \{A(f\psi(j, v))\}_j.$$

Thus A must commute with the operators of multiplication of the

functions $\psi(j, v)$ and short arguments lead us to that there exists a positive measurable function $a(g)$ on G and

$$(8.68) \quad (Af)(g) = a(g)f(g), \quad \text{for almost all } g.$$

Lastly, by (8.59) we obtain that $a(g)$ is constant for almost all g . That is, for some positive constant c ,

$$(8.69) \quad A = c \cdot I.$$

Consequently,

$$(8.70) \quad w(\mathfrak{D}) = c, \quad \text{for } \tilde{\mu}\text{-almost all } \mathfrak{D}.$$

And

$$(8.71) \quad d\tilde{\mu}_1(\mathfrak{D}) = c d\tilde{\mu}(\mathfrak{D}).$$

This completes the proof.

(Added in proof, December 14, 1971) *Note.* The author owes motivation to study the theory of Plancherel formula for non-unimodular groups to the late Professor A. Kohari who died on November 20, 1971. The author wishes to express acknowledgement for his suggestions given in 1961.

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