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A role of Fourier transform in the theory of infinite dimensional unitary group

By

Takeyuki Hida

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§ 0. Introduction

In the investigation of the sample function properties of the complex Brownian motion, the Fourier analysis plays a dominant role to discuss generalized harmonic analysis or frequency problems. For our purpose, we prefer the complex white noise rather than the complex Brownian motion, since probabilistic properties in question of the former are somewhat simpler than those of the latter. Thus, we are naturally led to discuss the Fourier transform of sample functions of the complex white noise which are, of course, generalized functions.

Our discussion, therefore, starts with the general set-up of the *complex white noise* (§1.). Let E be a σ-Hilbert nuclear space which is included densely in $L^2(R^1)$, and let E_c be the complexification of E. The complex white noise gives a probability measure ν, which is Gaussian, on the space E_c^* the conjugate space of E_c. We then, in § 1, come to the *infinite dimensional unitary group* $U(E_c)$ which is the collection of all the linear transformations on E_c leaving the L^2-norm invariant. Here, it should be noted that the basic space E_c must be chosen so that the Fourier transform is a linear isomorphism of the space E_c; namely the Fourier transform is a member of $U(E_c)$. We also wish to topologize the space E_c by using the differential operator D which is invariant (up to multiplicative constant) under the

Fourier transform. Indeed, this is possible. Take $E=\mathcal{S}$ the Schwartz space of C^∞-functions which are rapidly decreasing at infinity.

Intimate connection arises between the unitary group $U(\mathcal{S}_c)$ and the measure ν of the complex white noise. It is proved that each g^* the conjugate of $g\in U(\mathcal{S}_c)$ acts on $\mathcal{S}_c^*$ and ν is invariant under the action g^*. Further detailed probabilistic approach will be reported by the present author [3].

Coming back to the unitary group $U(\mathcal{S}_c)$, we can not set aside the shift, the conjugate of which determines the flow of the complex Brownian motion. The shift forms a one-parameter subgroup of $U(\mathcal{S}_c)$. Another one-parameter subgroup can be obtained by observing the Fourier transform of fractional order which has been developed by N. Wiener [1]. We are interested in the finite dimensional Lie subgroup of $U(\mathcal{S}_c)$ which contains the above one-parameter subgroup and has nice algebraic properties. Such a subgroup does exist as is prescribed in § 2.

The last section (§ 3) is devoted to a heuristic approach to find a finite dimensional Lie subgroup of $U(\mathcal{S}_c)$ together with some remarks for other developments.

§ 1. Complex white noise and unitary group $U(E_c)$.

Let E be a real σ-Hilbert nuclear space which is included densely in $L^2(R^1)$ and let E^* be the dual space of E. On the space E^* we can introduce a probability measure μ_σ such that the characteristic functional is given by

$$\int_{E^*} e^{i\langle x,\xi\rangle} d\mu_\sigma(x)=\exp\left[-\frac{\sigma^2}{2}\|\xi\|^2\right],\ \xi\in E, \tag{1}$$

where $\langle x,\xi\rangle$, $x\in E^*$, $\xi\in E$, is the canonical bilinear form and $\|\ \|$ denotes the $L^2(R^1)$-norm. The measure μ_σ is called the measure of (real) *white noise* with variance σ^2.

We now complexify E and E^* in a usual manner:

$$E_c=E+iE,$$
$$E_c^*=E^*+iE^*.$$

Obviously E_c is a vector subspace of the complex Hilbert space $L_c^2(R^1)$, and $\zeta=\xi+i\eta\in E_c(\xi,\eta\in E)$ and $z=x+iy\in E_c^*$ $(x,y\in E^*)$ are linked by the following canonical form

(2) $$\langle z,\zeta\rangle=(\langle x,\xi\rangle+\langle y,\eta\rangle)+i(-\langle x,\eta\rangle+\langle y,\xi\rangle).$$

Now we are ready to introduce the measure ν of a complex white noise. Let μ and μ' be the measure of real white noise with variance 1/2. Then the product measure $\nu=\mu\times\mu'$ can be introduced on the space E_c^*. Thus obtained measure space $(E_c^*, \boldsymbol{B}, \nu)$ or simply denoted by (E_c^*, ν) is called the (standard) *complex white noise*, where $\boldsymbol{B}$ is the σ-field generated by all the cylinder subsets of E_c^*. With this measuse ν on E_c^* each element of E_c^* can be thought of as a sample function of the derivative of the standard complex Brownian motion. As is expected, we can see that the systems $\{\langle z,\zeta\rangle;\ \zeta\in E_c\}$ and $\{\overline{\langle z,\zeta\rangle};\ \zeta\in E_c\}$ of random variables on the space (E_c^*, ν) are both complex Gaussian systems.

We then define the infinite dimensional unitary group. Consider the collection $U(E_c)$ of all linear transformations on E_c satisfying the following two conditions:

i) g is a homeomorphism of E_c,

ii) $\|g\zeta\|=\|\zeta\|$ for every $\zeta\in E_c$.

It can easily be proved that $U(E_c)$ becomes a group under the product

$$(g_1g_2)\zeta=g_1(g_2\zeta).$$

Definition. The group $U(E_c)$ is called the *infinite dimensional unitary group*. If no confusion occurs, we call $U(E_c)$ simply the *unitary group*.

For any g in $U(E_c)$ we can define the adjoint g^* through the canonical form in such a way that

(3) $$\langle z, g\zeta\rangle=\langle g^*z,\zeta\rangle \text{ for every } z\in E_c^*,\ \zeta\in E_c.$$

The operator g^* defines a linear isomorphism of E_c^*. The collection

$\{g^*; g\in U(E_c)\}$ again forms a group, call it $U^*(E_c^*)$. For the g^* we have

$$g_1^* g_2^* = (g_2 g_1)^*,$$

which asserts that $U^*(E_c^*)$ is anti-isomorphic to the group $U(E_c)$.

We now come to a basic theorem to illustrate a relation between the complex white noise and the unitary group $U(E_c)$.

Theorem 1. *For every $g\in U(E_c)$, it holds that*

$$g^* \circ \nu = \nu. \tag{4}$$

Proof. Since the σ-field $\boldsymbol{B}$ is generated by the cylinder subsets of E_c^*, it suffices to show that

$$\{\langle z, \zeta\rangle ; \zeta\in E_c\} \text{ and } \{\langle z, g\zeta\rangle ; \zeta\in E_c\}$$

have the same probability distribution. In other words, it is enough to prove that for any finite number of ζ_k's, $1\leq k\leq n$, $(\langle z, \zeta_1\rangle, \cdots, \langle z, \zeta_n\rangle)$ and $(\langle z, g\zeta_1\rangle, \cdots, \langle z, g\zeta_n\rangle)$ have the same probability distribution on R^{2n}. This assertion is, however, an easy consequence of the definition of ν and of the expression (2).

Corollary *The operator U_g on $L^2(E_c^*, \nu)$ given by*

$$U_g\varphi(z) = \varphi(g^* z), \quad \varphi\in L^2(E_c^*, \nu) \tag{5}$$

is unitary. The collection $\mathcal{U}(E_c) = \{U_g; g\in U(E_c)\}$ is a group isomorphic to $U^*(E_c^*)$.

We can speak of a topology to be introduced to the unitary group $U(E_c)$ by the use of $\mathcal{U}(E_c)$. Another topologization of $U(E_c)$ may be possible by introducing the compact open topology. However, topology does not play any important role in this paper, so detailed discussions on this subject will be given in another paper.

§ 2. Subgroups of $U(\mathcal{S}_c)$.

As is emphasized in § 0, we have special interest in the Fourier transform $\mathcal{F}$. In order to apply $\mathcal{F}$ to a sample function of the comple white noise we first let $\mathcal{F}$ be in $U(E_c)$ (see expression (3)). Such a requirement that $\mathcal{F}\in U(E_c)$ may or may not be satisfied according as the choice of the basic space E. Another requirement arises when we discuss infinitesimal generators of one-parameter subgroup of $U(E_c)$, where we are given differential operators on E_c. Thus we require that the countable Hilbertian norms $\| \ \|_n$, $n\geq 1$, determining the topology of E and E_c must be defined in such a way that

$$\|D^n\xi\| = \|\xi\|_n, \quad \xi\in E, \tag{6}$$

with some second order differential operator $'D$. Finally, we expect that D and $\mathcal{F}$ are formally commutative:

$$D\mathcal{F}=\mathcal{F}D$$

Elementary observations lead us to take the Schwatz space $\mathcal{S}$ as the basic space E and

$$D=\frac{d^2}{du^2}-(u^2+1).$$

The countable norms defined by (6) with the above differential operator determines the usual topology of $\mathcal{S}$. Thus it is proved that

Proposition 1. *The Fourier transform $\mathcal{F}$ is a member of $U(\mathcal{S}_c)$.*

With the choice of $E=\mathcal{S}$ we try to find interesting subgroups of $U(\mathcal{S}_c)$. As soon as the unitary group $U(\mathcal{S}_c)$ is defined, one may think that finite dimentional unitary group $U(n)$ is embedded by a suitable choice of a finite dimensional subspace of $\mathcal{S}_c$. Of course, this is true. However our interest centers entirely on the one-parameter subgroups of $U(\mathcal{S}_c)$ which are related with the Fourier transform.

Being inspired by the Wiener's work, we first consider the so-called Fourier-Mehler transform. Let $\mathcal{F}_\theta$ be defined by

(7) $$(\mathcal{F}_\theta\zeta)(u)=K_\theta\circ\zeta(u)\equiv\int K_\theta(u,v)\zeta(v)dv,$$
where
$$K_\theta(u,v)=\{\pi(1-e^{2i\theta})\}^{-1/2}\exp\left[-\frac{i(u^2+v^2)}{2\tan\theta}+\frac{iuv}{\sin\theta}\right].$$
The operator $\mathcal{F}_\theta$ is well defined except the cases $\theta\equiv 0, \pi/2, \pi, 3\pi/2$ (mod, 2π), respectively.

Proposition 2. *$\mathcal{F}_\theta$ extends to a continuous, periodic one-parameter subgroup of $U(\mathcal{S}_c)$ such that*

(8) $$\mathcal{F}_\theta\cdot\mathcal{F}_{\theta'}=\mathcal{F}_{\theta+\theta'}=\mathcal{F}_{\theta''},\ \theta''\equiv\theta+\theta(mod.\ 2\pi)$$
$$\mathcal{F}_\theta\to I\ (identity)\ as\ \theta\to 0.$$

Proof. Take a system of functions
$$\xi_n(u)=(2^n n!\sqrt{\pi})^{-1/2}H_n(u)\exp\left[-\frac{u^2}{2}\right],$$
H_n: the Hermite polynomial,
$n=0, 1, 2, \cdots\cdots$

Elementary computations prove that $\mathcal{F}_\theta$ can be applied to the functions ξ_n with the results

(9) $$(\mathcal{F}_\theta\xi_n)(u)=e^{in\theta}\xi_n(u),\ \theta\neq 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ n=0, 1, \cdots.$$

It is noted that each ξ_n belongs to $\mathcal{S}_c$ and that the ξ_n's form a complete orthonormal system in $L_c^2(R^1)$. Therefore, we first prove that, by using (9), $\mathcal{F}_\theta$ extends to a one-parameter group satisfying the relation (8) for every n and then prove that $\mathcal{F}_\theta$ acts on the entire space $\mathcal{S}_c$ homeomorphically. It is obvious that $\|\mathcal{F}_\theta\zeta\|=\|\zeta\|$ holds for every θ and ζ. Thus our assertisn is proved.

Sometimes, $\mathcal{F}_\theta$ is called the *Fourier-Mehler transform.* With a particular choices of $\theta=\pi/2$ and $\theta=3\pi/2$ we see that the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are embedded in our one-parameter group $\{\mathcal{F}_\theta\}$.

We are now ready to apply $\mathcal{F}_\theta^*$ to any sample function z of the complex white noise as follows:
$$\langle\mathcal{F}_\theta^* z, \zeta\rangle=\langle z, \mathcal{F}_\theta\zeta\rangle,$$

in particular

$$\langle \mathcal{F}_\theta^* z, \xi_n \rangle = e^{-in\theta} \langle z, \xi_n \rangle,$$

which tells us certain sample function property of the complex white noise (or the complex Brownian motion).

Recall that the importance of the flow $\{T_t\}$ of the complex Brownian motion on the measure space (E_c^*, ν) which comes from the *shift* $\{S_t\}$ defined by

(10) $$S_t\zeta(u) = \zeta(u-t), \quad -\infty < t < \infty.$$

Obviously $\{S_t\}$ is a one-parameter subgroup of $U(\mathcal{S}_c)$. The flow $\{T_t\}$ is obtained by taking the adjoint of S_t: $T_t = S_t^*$. By the use of the Fourier transform a third one-parameter subgroup of $U(\mathcal{S}_c)$ arises from the shift. Define π_t by

(11) $$\pi_t = \mathcal{F} S_t \mathcal{F}^{-1}, \quad -\infty < t < \infty,$$

namely

$$(\pi_t\zeta)(u) = e^{iut}\zeta(u).$$

Observing the Weyl commutation relation between $\{S_t\}$ and $\{\pi_t\}$, a fourth one-parameter subgroup $\{I_t\}$ is naturally introduced:

(12) $$(I_t\zeta)(u) = e^{it}\zeta(u), \quad -\infty < t < \infty,$$

and we have

(13) $$\pi_s S_t = I_{st} S_t \pi_s$$

So far we have discussed four one-parameter subgroups of $U(\mathcal{S}_c)$ with special emphasis on Fourier transform and the shift. For further heuristic approach it is convenient for us to use the infinitesimal generators of one-parameter subgroups. Such an approach will be discussed in the next section.

§ 3. Infinitesimal generators and their commutation relations

The infinitesimal generators of a one-parameter group $\{g_t\}$ is defined as $\frac{d}{dt} g_t |_{t=0}$. We now consider such generators of the one-parameser subgroup of $U(\mathcal{S}_c)$ which appeared in the last section.

	one-parameter group	generator
	$\mathcal{F}_\theta$	$if=-\frac{i}{2}\left(\frac{d^2}{du^2}-u^2+I\right)$
(14)	S_t	$s=-\frac{d}{du}$
	π_t	$i\pi=iu\cdot$
	I_t	iI

Let $[A, B]$ denote the commutator of A and B: $[A, B]=AB-BA$. Then we have

$$
\begin{aligned}
&[s, f]=-\pi \\
(15)\qquad &[\pi, f]=-s \\
&[\pi, s]=\mathrm{I}
\end{aligned}
$$

Observing the explicit expressions of the above generators in (14) and the commutation relations (15), we notice that there exists a differential operater τ of the first order which is invariant under the Fourier transform and is acted transversally by s and π:

$$\mathcal{F}\cdot\tau\mathcal{F}^{-1}=-\tau,$$

and

$$
\begin{aligned}
(16)\qquad &[\tau, s]=-s, \\
&[\tau, \pi]=\pi.
\end{aligned}
$$

The exact form of such τ given by

$$(17)\qquad \tau=u\frac{d}{du}+\frac{1}{2}I$$

(up to constant). It is interesting to note that τ has an important probabilistic meaning; more precisely the adjoint of $\exp[t\tau]$, $-\infty<t<\infty$, determines a flow on $(\mathcal{S}_c^*, \nu)$ induced by the Ornstein-Uhlenbeck Browoian motion.

The last infinitesimal generator can also be introduced from τ by using the generator of the Fourier-Mehler transform. Set

$$(18)\qquad \sigma=\frac{1}{2}[\tau, f],$$

then we are given new cnmmutation relations as follows:

$$
(19) \qquad \begin{aligned}
&[\sigma, f] = 2\tau, \\
&[\sigma, s] = \pi, \\
&[\sigma, \pi] = -s \\
&[\tau, \sigma] = 2f + I.
\end{aligned}
$$

The algebraic property for these generators are quite simple as is indicated in the following theorem. From the commutation relations (15), (16) and (19) we can prove

Theorem 2. *The vector space $\mathcal{A}$ spanned by $\{I, s, \pi, f, \tau, \sigma\}$ is closed under the product* [,], *that is, $\mathcal{A}$ forms a Lie algebra.* ii) *The radical of $\mathcal{A}$ is the ideal generated by $\{I, s, \pi\}$.*

Before closing this section, some concluding remarks are in order.

1. Although we have briefly mentioned probabilistic meaning on the one-parameter subgroup corresponding to each one of the generators, some further interpretation can be given from the point of view of probability theory (c. f. [3]).
2. We have discussed finite number of generators so that we can find a finite dimensional Lie subgroup of $U(\mathcal{S}_c)$, by stressing the importance of the Fourier transform and the shift. There are, of course, other approaches to find interesting subgroups (for example, see [4]).
3. A subgroup of $U(\mathcal{S}_c)$ consisting of all finite dimensional unitary transformations is also interesting to be investigated. For one thing, such a group has close connections with the infinite dimensional Laplacian operator or with the infinite dimensional harmonic function.

NAGOYA UNIVERSITY

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