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Simple groups of conjugate type rank 5

By

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1. Introduction

Let $\mathfrak{G}$ be a finite group, $I(\mathfrak{G})$ the set of indices of centralizers of non-central elements of $\mathfrak{G}$ in $\mathfrak{G}$, and r the number of elements in $I(\mathfrak{G})$. r is called the conjugate type rank of $\mathfrak{G}$. We introduce an ordering in $I(\mathfrak{G})$ as follows: let a and b be two elements of $I(\mathfrak{G})$. Then $a>b$ if and only if a divides b. Let k be the number of maximal elements in $I(\mathfrak{G})$. Then $\mathfrak{G}$ is called k-headed. We form a graph $C(\mathfrak{G})$ of $\mathfrak{G}$ as follows: the points of $C(\mathfrak{G})$ are the elements of $I(\mathfrak{G})$. The (oriented) edge ab of $C(\mathfrak{G})$ exists, where a and b are points of $C(\mathfrak{G})$, if and only if $a>b$. We denote the edge ab by $\begin{matrix} a \\ \uparrow \\ b \end{matrix}$. $C(\mathfrak{G})$ is called the conjugate type graph of $\mathfrak{G}$. The centralizer of any non-central element of $\mathfrak{G}$ in $\mathfrak{G}$ corresponding to an isolated point of $C(\mathfrak{G})$ is called free.

An obvious problem is as follows: Let r be a given positive integer. Then classify all (simple) groups $\mathfrak{G}$ such that conjugate type rank of $\mathfrak{G}$ are equal to r. When r increases, this problem probably will become more difficult with exponential growth rate. If, however, the shape of $C(\mathfrak{G})$ is given and coincident with that of the conjugate type graph of some known simple group, then the problem will become considerably tractable.

In previous papers we proved the following theorems:

(I) [7] A finite group $\mathfrak{G}$ is a simple group of the conjugate type

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rank 3 if and only if $\mathfrak{G}$ is isomorphic with some $LF(2, 2^m)$, $m \geqq 2$.

(II) [8] A finite group $\mathfrak{G}$ is a simple group of the conjugate type rank 4 if and only if $\mathfrak{G}$ is isomorphic with some $LF(2, q)$, where $q \geqq 7$ is odd.

It the present paper we prove the following theorem:

Theorem. *A simple group of conjugate type rank 5 and not of 3-headed is isomorphic with some* $Sz(l)$, $l = 2^{2n+1}$, $n \geqq 1$, *or* $LF(3, 4)$.

Remark. The 3-headed case is still open.

Notation and definition. Let $\mathfrak{X}$ be a finite group. $\boldsymbol{Z}(\mathfrak{X})$ is the center of $\mathfrak{X}$. If $\mathfrak{X}$ is solvable, then $\boldsymbol{F}(\mathfrak{X})$ is the Fitting subgroup of $\mathfrak{X}$. Let $\mathfrak{Y}$ be a subset of $\mathfrak{X}$. $|\mathfrak{Y}|$ is the number of elements in $\mathfrak{Y}$. $\pi(\mathfrak{X})$ is the set of prime divisors of $|\mathfrak{X}|$. If $\mathfrak{Y}$ is nonempty, then $\boldsymbol{Cs}\mathfrak{Y}$ is the centralizer of $\mathfrak{Y}$ in $\mathfrak{X}$. If $\mathfrak{Y} = \{Y\}$, $\boldsymbol{Cs}\mathfrak{Y} = \boldsymbol{Cs}\,Y$. $\boldsymbol{Ns}\mathfrak{Y}$ is the normalizer of $\mathfrak{Y}$ in $\mathfrak{X}$. $\langle\mathfrak{Y}\rangle$ is the subgroup generated by $\mathfrak{Y}$. If $\mathfrak{Y} = \{Y\}$, $\langle\mathfrak{Y}\rangle = \langle Y\rangle$. Let $\mathfrak{Z}$ be a subset of $\mathfrak{X}$. Then $[\mathfrak{Y}, \mathfrak{Z}]$ is the subset of $\mathfrak{X}$ consisting of $Y^{-1}Z^{-1}YZ$, where Y and Z are elements of $\mathfrak{Y}$ and $\mathfrak{Z}$, respectively. *A* proper subgroup $\mathfrak{F}$ of $\mathfrak{X}$ is called fundamental, if there exists an element X of $\mathfrak{X}$ such that $\mathfrak{F} = \boldsymbol{Cs}X$. $\mathfrak{F}$ is called maximal, if $\mathfrak{F}$ is containd in no other fundamental subgroups of $\mathfrak{X}$. $\mathfrak{F}$ is called minimal, if $\mathfrak{F}$ contains no other fundamental subgroup of $\mathfrak{X}$. $\mathfrak{F}$ is free, if $\mathfrak{F}$ is maximal and minimal.

2. 2-headed case

The purpose of this section is to show that this case does not occur.

Let $\mathfrak{G}$ be a simple group of conjugate type rank 5 and of 2-headed. Let n_i be maximal elements of $I(\mathfrak{G})$ $(i = 1, 2)$. Let A_i be an element of $\mathfrak{G}$ such that $\mathfrak{G} : \boldsymbol{Cs}A_i = n_i$ $(i = 1, 2)$. Then the class equation implies that $(n_1, n_2) = 1$. In particular, $\mathfrak{G} = \boldsymbol{Cs}A_1\boldsymbol{Cs}A_2$.

(2.1) Both $\boldsymbol{Cs}A_1$ and $\boldsymbol{Cs}A_2$ are not free.

Proof. See the proof of (2.2) in [8].

(2. 2) We may assume that $|\langle A_i\rangle| = p_i$ is a prime $(i=1,2)$.

Then $p_1 \neq p_2$, A_i is p_i-central, namely A_i belongs to the center of some Sylow p_i-subgroup of $\mathfrak{G}$, $n_2 \equiv 0 \pmod{p_1}$ and $n_1 \equiv 0 \pmod{p_2}$.

Proof. See the proof of (2. 3) in [8].

(2. 3) We have that either $|CsA_1| \not\equiv 0 \pmod{p_2}$ or $|CsA_2| \not\equiv 0 \pmod{p_1}$.

Proof. Assume the contrary that both $|CsA_1| \equiv 0 \pmod{p_2}$ and $|CsA_2| \equiv 0 \pmod{p_1}$.

Let $A_2'(\neq E)$ be an element of the center of a Sylow p_2-subgroup of CsA_1. We may assume that A_2' belongs to CsA_2. If $|CsA_2'| = |CsA_2|$, then $\mathfrak{G} = CsA_1 CsA_2'$. Since $A_1A_2' = A_2'A_1$, this implies that $\mathfrak{G}$ is not simple. If $|CsA_2'| = |CsA_1|$, then $\mathfrak{G} = CsA_2' CsA_2$. Since $A_2'A_2 = A_2A_2'$, this implies that $\mathfrak{G}$ is not simple. If $CsA_1A_2' = CsA_2'$, then A_1 belongs to $Z(CsA_2')$. Hence $A_1A_2 = A_2A_1$. Then $\mathfrak{G}$ is not simple.

Now $CsA_1 : CsA_1A_2'$ is prime to p_2. Let $\mathfrak{P}_2$ be a Sylow p_2-subgroup of CsA_2. Then we may assume that CsA_2' contains $Z(\mathfrak{P}_2)$. Since we may assume that A_2 does not belong to CsA_1, we may assume that CsA_1A_2' contains no conjugates of $Z(\mathfrak{P}_2)$. Thus we have that $CsA_2' : CsA_1A_2' \equiv 0 \pmod{p_2}$. Hence $|CsA_2'|$ does not divide $|CsA_1|$, but $|CsA_2'|$ is a proper divisor of $|CsA_2|$. Therefore a part of $C(\mathfrak{G})$ has the shape $\begin{array}{cc} n_1 & n_2 \\ \uparrow & \uparrow \\ \cdot & \nearrow \end{array}$. Now by symmetry we can conclude that

$C(\mathfrak{G})$ has the shape $\begin{array}{ccc} n_1 & & n_2 \\ \uparrow & & \uparrow \\ n_4 & & n_3 \\ & \nwarrow \ \nearrow & \\ & n_5 & \end{array}$.

Now assume that there exists a prime divisor q of $|\mathfrak{G}|$ such that q is prime to n_5. Then for every element X of $\mathfrak{G}$ CsX contains a Sylow q-subgroup $\mathfrak{Q} \neq \mathfrak{E}$ of $\mathfrak{G}$. Hence $Cs\mathfrak{Q}$ and its conjugates exhaust $\mathfrak{G}$. This implies that $\mathfrak{G} = Cs\mathfrak{Q}$. Hence $\mathfrak{G}$ is not simple. By a theorem of Burnside [5, p. 451] n_1 is not a prime power. Let p_2^* be a prime divisor of n_1 distinct from p_2. Let $\mathfrak{P}_2^*$ be a Sylow

p_2^*-subgroup of $\mathfrak{G}$ contained in CsA_2. Now assume that $\mathfrak{G}:CsZ\neq n_2$ for every element $Z\neq E$ of $Z(\mathfrak{P}_2^*)$. Let $A_2^*\neq E$ be an element of $Z(\mathfrak{P}_2^*)$. Then $CsA_2^*=CsA_2A_2^*$. Thus $\mathfrak{G}:CsA_2^*=n_4$. On the other hand, we may assume that CsA_2' contains $\mathfrak{P}_2^*$. Otherwise, replace A_2^* and A_2 by their appropriate conjugates. Then $CsA_2'=CsA_2'A_2^*$. Thus $CsA_2'=CsA_2^*$. Since A_2 belongs to $Z(CsA_2^*)$ and since A_1 belongs to CsA_2', $A_1A_2=A_2A_1$. Hence $\mathfrak{G}$ is not simple. Thus there exists a p_2^*-element $A_2^*\neq E$ such that $CsA_2=CsA_2^*$.

Now clearly $|CsA_1|\equiv 0 \pmod{p_2^*}$. Arguing with p_2^* instead of p_2, we obtain that n_5/n_1 is prime to p_2^* and that n_5/n_4 is divisible by p_2^*. Let $A_2^{*\prime}\neq E$ be a p_2^*-element of CsA_1A_2'. Then we may assume that $CsA_1A_2'=CsA_1A_2^{*\prime}$. Hence, since CsA_1A_2' is minimal, CsA_1A_2' is nilpotent. Let $\mathfrak{P}_2^{*\prime}$ be a Sylow p_2^*-subgroup of CsA_2'. Let $A_2^{*\prime\prime}\neq E$ be an element of $Z(\mathfrak{P}_2^{*\prime})$. Then $CsA_2'=CsA_2'A_2^{*\prime\prime}$. Thus $A_2^{*\prime\prime}A_1=A_1A_2^{*\prime\prime}$ and $A_2^{*\prime\prime}$ belongs to CsA_1A_2'. If $|CsA_2^{*\prime\prime}|=|CsA_2|$ then $\mathfrak{G}$ is not simple. We may assume that $|CsA_2^{*\prime\prime}|=|CsA_2'|$. If CsA_2' is not maximal, there exists an element $A\neq E$ of $\mathfrak{G}$ such that $|CsA|=|CsA_2|$ and $AA_1=A_1A$. Then $\mathfrak{G}$ is not simple. So we may assume that CsA_2' is maximal. Now in the theorem of Camina [**2**] we may put $\pi=\pi(Z(CsA_2'))$. Then since π contains at least two prime numbers we obtain that CsA_2' is nilpotent. Then clearly $A_1A_2=A_2A_1$. Thus $\mathfrak{G}$ is not simple.

(**2.4**) We have that both $|CsA_1|\not\equiv 0 \pmod{p_2}$ and $|CsA_2|\not\equiv 0 \pmod{p_1}$.

Proof. Assume that $|CsA_1|\equiv 0 \pmod{p_2}$. Then by (2.3) $|CsA_2|\not\equiv 0 \pmod{p_1}$. Let $A_2'\neq E$ be an element of the center of a Sylow p_2-subgroup of CsA_1. Then as in the beginning of the proof of (2.3) we obtain that $|CsA_2'|\neq|CsA_1|$, $|CsA_2|$ and that $CsA_2'\neq CsA_1A_2'$. Anyway $|CsA_1A_2'|\equiv 0 \pmod{p_1}$. Further we see that as in the second part of the proof of (2.3) $|CsA_2'|$ divides $|CsA_2|$. This is a contradiction.

(2.5) CsA_1 and CsA_2 are Hall subgroups of $\mathfrak{G}$.

Proof. See the proof of (2.7) in [8].

Now we see that $C(\mathfrak{G})$ has either the shape $\begin{array}{cccc} & n_1 & & n_2 \\ \nearrow & & \nwarrow & \uparrow \\ n_3 & & n_4 & n_5 \end{array}$ or the shape $\begin{array}{cc} n_1 & n_2 \\ \uparrow & \uparrow \\ n_3 & n_5 \\ \uparrow & \\ n_4 & \end{array}$.

(2.6) CsA_2 is not nilpotent.

Proof. Assume that CsA_2 is nilpotent. Since CsA_2 is not free, CsA_2 is obviously not abelian. We may assume that the Sylow p_2-subgroup $\mathfrak{P}_2$ of CsA_2 is not abelian. Then the Sylow p_2-complement $\mathfrak{U}$ of CsA_2 is abelian. By a theorem of Burnside [5, p. 491] $\mathfrak{U}\neq\mathfrak{E}$. Let $X\neq E$ be a primary element of CsA_2. If X belongs to $\mathfrak{U}$, then $CsX=CsA_2$. Let X belong to $\mathfrak{P}_2$ and let CsX be not contained in CsA_2. By a theorem of Wielandt [5, p. 285] CsX is nilpotent. Hence $CsX\subseteqq Cs\mathfrak{U}=CsA_2$. This is a contradiction. Hence CsA_2 is centralizer-closed. This contradicts [9].

Let B_5 be an element of $\mathfrak{G}$ such that $CsA_2\supseteqq CsB_5$ and such that $\mathfrak{G}:CsB_5=n_5$. Then by a theorem of Camina [2] n_5/n_2 is a power of p_2 and $Z(CsA_2)$ is a p_2-group.

(2.7) The Sylow p_2-complement $\mathfrak{U}$ of CsB_5 and moreover CsB_5 itself are abelian.

Proof. First we show that $\mathfrak{U}$ is abelian. If $\pi(\mathfrak{U})$ contains at least two prime numbers, this is obvious. So let us assume that $\mathfrak{U}$ is a q-group, where q is a prime. Let $B\neq E$ be an element of $\mathfrak{U}$. Then $CsB\subseteqq CsA_2$. In fact, otherwise, $|CsB|=|CsA_2|$. Then by a theorem of Camina [2] CsB is nilpotent. Then by a theorem of Wielandt [5, p. 285] CsA_2 is nilpotent against (2.6). Hence CsB is a conjugate of $\mathfrak{U}$ in CsA_2. By a theorem of Burnside [5, p. 492] CsA_2 is solvable. Thus a theorem of Fitting [5, p. 277] implies that $\mathfrak{U}$ is abelian. The rest is obvious.

(2.8) $|\boldsymbol{CsA}_2|$ is odd.

Proof. Assume that $|\boldsymbol{CsA}_2|$ is even. By a theorem of Walter [**13**] and by (2.7) $p_2=2$. By the proofs of (4.5) and (4.6) in [8] there exists a 2-element B such that $|\boldsymbol{CsB}|<|\boldsymbol{CsA}_2|$. By the proof of (2.7) $\boldsymbol{CsB}$ is abelian. Therefore we may assume that $B=B_5$ and that $\boldsymbol{CsB}$ is contained in $\boldsymbol{CsA}_2$.

Since $\boldsymbol{CsB}_5$ is nilpotent and since $\boldsymbol{CsA}_2=\mathfrak{P}_2\boldsymbol{CsB}_5$, $\boldsymbol{CsA}_2$ is solvable [**5**, p. 674]. Let $\mathfrak{P}_2^*$ be the Sylow 2-subgroup of $\boldsymbol{CsB}_5$. If $\boldsymbol{F}(\boldsymbol{CsA}_2)$ is a 2-group, then by a theorem of Fitting [**5**, p. 277] $\boldsymbol{F}(\boldsymbol{CsA}_2)\neq\mathfrak{P}_2^*$. Now $(\boldsymbol{F}(\boldsymbol{CsA}_2)\cap\boldsymbol{Ns}\mathfrak{P}_2^*)/\mathfrak{P}_2^*$ is the kernel of a Frobenius group $(\boldsymbol{F}(\boldsymbol{CsA}_2)\cap\boldsymbol{Ns}\mathfrak{P}_2^*)\mathfrak{U}/\mathfrak{P}_2^*$. Let A be an element of $\boldsymbol{F}(\boldsymbol{CsA}_2)\cap\boldsymbol{Ns}\mathfrak{P}_2^*$ outside $\mathfrak{P}_2^*$. If $A^{-1}\mathfrak{U}A\neq\mathfrak{U}$, then $\boldsymbol{CsB}_5$ contains $A^{-1}\mathfrak{U}A$. This is a contradiction. If $A^{-1}\mathfrak{U}A=\mathfrak{U}$, then $[A,\mathfrak{U}]$ is contained in $\mathfrak{U}\cap\boldsymbol{F}(\boldsymbol{CsA}_2)=\mathfrak{E}$. This is a contradiction. Hence $\boldsymbol{F}(\boldsymbol{CsA}_2)=\boldsymbol{CsB}_5$. Then $\boldsymbol{CsA}_2/\mathfrak{P}_2^*$ is a Frobenius group with $\boldsymbol{CsB}_5/\mathfrak{P}_2^*$ the kernel. Hence $\mathfrak{P}_2/\mathfrak{P}_2^*$ is cyclic or generalized quaternion.

First assume that $\mathfrak{P}_2/\mathfrak{P}_2^*$ is a generalized quaternion group of order 2^a. Then there exist elements Q and R of $\mathfrak{P}_2$ and S, T, U and V of $\mathfrak{P}_2^*$ such that $R^{-1}QR=Q^{-1}S$, $Q^{2^{a-2}}=R^2T$, $Q^{2^{a-1}}=U$, $R^4=V$ and $\mathfrak{P}_2/\mathfrak{P}_2^*=\langle Q,R\rangle\mathfrak{P}_2^*/\mathfrak{P}_2^*$. Now suppose that $\mathfrak{P}_2^*$ is not cyclic. Let $\mathfrak{W}$ be a normal subgroup of type (2.2) of $\mathfrak{P}_2$ contained in $\mathfrak{P}_2^*$. Then $\boldsymbol{CsR}^2$ contains $\mathfrak{W}$. If $|\boldsymbol{CsR}^2|=|\boldsymbol{CsB}_5|$, then by (2.7) $\boldsymbol{CsR}^2$ is abelian. This implies that $\boldsymbol{CsR}^2\subseteqq\boldsymbol{CsA}_2$ and that R^2 belongs to $\mathfrak{P}_2^*$. This is a contradiction. Hence $|\boldsymbol{CsR}^2|=|\boldsymbol{CsA}_2|$. If $\boldsymbol{F}(\boldsymbol{CsR}^2)$ is a 2-group, then we have that $\mathfrak{P}_2:\mathfrak{P}_2^*>|\mathfrak{U}|$. This is a contradiction. Hence $\boldsymbol{F}(\boldsymbol{CsR}^2)$ is not a 2-group. Let $\overline{\mathfrak{P}}_2$ and $\overline{\mathfrak{U}}$ be the Sylow 2-complement of $\boldsymbol{F}(\boldsymbol{CsR}^2)$, respectively. Let $\hat{\mathfrak{P}}_2$ be a Sylow 2-subgroup of $\boldsymbol{CsR}^2$. Then $\hat{\mathfrak{P}}_2/\overline{\mathfrak{P}}_2$ is cyclic or generalized quaternion. This implies that $\mathfrak{W}\cap\overline{\mathfrak{P}}_2\neq\mathfrak{E}$. Take an element $W(\neq E)$ of $\mathfrak{W}\cap\overline{\mathfrak{P}}_2$. Then $\boldsymbol{CsW}$ contains $\overline{\mathfrak{U}}$ and $\overline{\mathfrak{U}}$. This implies that $\mathfrak{U}=\overline{\mathfrak{U}}$. This is a contradiction. Therefore $\mathfrak{P}_2^*$ is cyclic. Hence $\mathfrak{P}_2\cap\boldsymbol{Cs}\mathfrak{P}_2^*\neq\mathfrak{P}_2^*$. Thus $\boldsymbol{CsQ}^{2^{a-2}}$

contains $\mathfrak{P}_2^*$. This implies that $|\boldsymbol{CsQ}^{2^{a-2}}|=|\boldsymbol{CsA}_2|$. If $\boldsymbol{F}(\boldsymbol{CsQ}^{2^{a-2}})$ is a 2-group, then we have that $\mathfrak{P}_2:\mathfrak{P}_2^*>|\mathfrak{U}|$. This contradiction shows that $\boldsymbol{F}(\boldsymbol{CsQ}^{2^{a-2}})$ is not a 2-group. If $\boldsymbol{CsQ}\neq\boldsymbol{CsQ}^{2^{a-2}}$, then $\boldsymbol{CsQ}=\boldsymbol{F}(\boldsymbol{CsQ}^{2^{a-2}})$. Let $\mathfrak{P}_2^{\#}$ be the Sylow 2-subgroup of $\boldsymbol{CsQ}$. Then $[Q,\mathfrak{P}_2^*]\subseteqq\mathfrak{P}_2^*\cap\mathfrak{P}_2^{\#}=\mathfrak{E}$. Since $|\boldsymbol{CsQ}|=|\boldsymbol{CsB}_5|$, this is a contradiction. Hence $\boldsymbol{CsQ}=\boldsymbol{CsQ}^{2^{a-2}}$. Similarly we obtain that $\boldsymbol{CsR}=\boldsymbol{CsR}^2$. Since $Q^{2^{a-2}}$ and R^2 commute, this implies that Q and R commute. This is a contradiction. Therefore $\mathfrak{P}_2/\mathfrak{P}_2^*$ is cyclic.

Let $\mathfrak{P}_2/\mathfrak{P}_2^*$ be of order 2^a and $P\mathfrak{P}_2^*$ a generator of $\mathfrak{P}_2/\mathfrak{P}_2^*$. Assume that $a\geqq 2$. As above, we obtain that $\mathfrak{P}_2^*$ is cyclic. Therefore, $\mathfrak{P}_2$ is metacyclic. Then by a theorem of Mazurov [**10**] $\mathfrak{P}_2$ is of type (2. 2) or of maximal class. This is a contradiction. Hence we obtain that $a=1$. Now we show that $\boldsymbol{Z}(\mathfrak{P}_2)$ is of order 2. Assume the contrary. If $|\boldsymbol{CsP}|=|\boldsymbol{CsB}_5|$, then by (2. 7) $\boldsymbol{CsP}$ is abelian and $\boldsymbol{CsP}\cap\mathfrak{U}=\mathfrak{E}$. Let $\mathfrak{U}^{\#}$ be the Sylow 2-complement of $\boldsymbol{CsP}$. Then $\mathfrak{U}\cap\mathfrak{U}^{\#}=\mathfrak{E}$. But since $\boldsymbol{CsP}$ contains $\boldsymbol{Z}(\mathfrak{P}_2)$, this is a contradiction. If $|\boldsymbol{CsP}|=|\boldsymbol{CsA}_2|$, then let $\widehat{\mathfrak{P}}_2$ and $\widehat{\mathfrak{U}}$ be the Sylow 2-subgroup and Sylow 2-complement of $\boldsymbol{F}(\boldsymbol{CsP})$. Then $\mathfrak{P}_2^*\cap\widehat{\mathfrak{P}}_2\neq\mathfrak{E}$ by assumption. Let $Z(\neq E)$ be an element of $\mathfrak{P}_2^*\cap\widehat{\mathfrak{P}}_2$. $\boldsymbol{CsZ}$ contains $\mathfrak{U}$ and $\widehat{\mathfrak{U}}$. Since $\mathfrak{U}\cap\widehat{\mathfrak{U}}=\mathfrak{E}$, and since $\boldsymbol{F}(\boldsymbol{CsZ})$ contains $\mathfrak{U}$ and $\widehat{\mathfrak{U}}$, this is a contradiction. Hence $|\boldsymbol{Z}(\mathfrak{P}_2)|=2$. Then by a lemma of Suzuki [**11**] $\mathfrak{P}_2$ is of type (2, 2) or of maximal class. Then by a theorem of Wong [**14**] we get a contradiction.

(2. 9) $\boldsymbol{F}(\boldsymbol{CsA}_2)$ is a p_2-group.

Proof. Assume the contrary. Then $\boldsymbol{F}(\boldsymbol{CsA}_2)=\boldsymbol{CsB}_5=\mathfrak{P}_2^*\times\mathfrak{U}$. Since $\boldsymbol{F}(\boldsymbol{CsA}_2)/\mathfrak{P}_2^*$ is the kernel of a Frobenius group $\boldsymbol{CsA}_2/\mathfrak{P}_2^*$, $\mathfrak{P}_2/\mathfrak{P}_2^*$ is cyclic by (2. 8). Let $\mathfrak{P}_2/\mathfrak{P}_2^*$ be of order p_2^a and $P\mathfrak{P}_2^*$ a generator of $\mathfrak{P}_2/\mathfrak{P}_2^*$. Assume that $a\geqq 2$. Then as in the proof of (2. 8) we obtain that $\mathfrak{P}_2^*$ is cyclic. Therefore $\mathfrak{P}_2$ is metacyclic. If $\mathfrak{P}_2$ is not abelian, then by a theorem of Huppert [**5**, p. 452] $\mathfrak{G}$ is not simple. Hence $\mathfrak{P}_2$ is abelian. Since $\langle P\rangle\cap\mathfrak{P}_2^*=\mathfrak{E}$, we obtain that $\mathfrak{P}_2=\mathfrak{P}_2^*\times\langle P\rangle$ is of type (p_2^a, p_2^a).

Now the set of elements X of $\mathfrak{G}$ such that $\mathfrak{G}:\boldsymbol{Cs}X=n_2$ coincides with the set of p_2-elements $\neq E$ in $\mathfrak{G}$. Every p_2-element $\neq E$ belongs to exactly one conjugate of $\mathfrak{P}_2^*$. Now $\boldsymbol{Ns}\mathfrak{P}_2^*=\boldsymbol{Cs}A_2$. In fact, otherwise, since $\boldsymbol{Cs}\mathfrak{P}_2^*=\boldsymbol{Cs}A_2$, by a theorem of Thompson [**5**, p. 499] we obtain that $\boldsymbol{Cs}A_2$ is nilpotent contradicting (2.6). Let e be the number of conjugacy classes of elements X of $\mathfrak{G}$ such that $\mathfrak{G}:\boldsymbol{Cs}X=n_2$. Then we obtain that

$$en_2=n_2(p_2^a-1).$$

Hence $e=p_2^a-1$. On the other hand, by a theorem of Burnside [**5**, p. 418] any two elements of $\mathfrak{P}_2$ which are conjugate in $\mathfrak{G}$ are conjugate in $\boldsymbol{Ns}\mathfrak{P}_2$. Since $\boldsymbol{Cs}\mathfrak{P}_2=\mathfrak{P}_2$, we obtain that $\boldsymbol{Ns}\mathfrak{P}_2:\mathfrak{P}_2=p_2^a+1$. In particular, there exists an involution J in $\boldsymbol{Ns}\mathfrak{P}_2$ such that J inverts A_2. Then by a theorem of Thompson [**5**, p. 499] we obtain that $\boldsymbol{Cs}A_2$ is nilpotent contradicting (2.6) Hence we obtain that $a=1$.

If $|\boldsymbol{Cs}P|=|\boldsymbol{Cs}B_5|$, then by (2.7) $\boldsymbol{Cs}P$ is abelian. Then $\boldsymbol{Cs}P$ is contained in $\boldsymbol{Cs}A_2$. This is a contradiction. Hence $|\boldsymbol{Cs}P|=|\boldsymbol{Cs}A_2|$. Let $\hat{\mathfrak{P}}_2$ be the Sylow p_2-subgroup of $\boldsymbol{F}(\boldsymbol{Cs}P)$. Since $\hat{\mathfrak{P}}_2\cap\mathfrak{P}_2^*=\mathfrak{E}$, we have that $|\mathfrak{P}_2^*\cap\boldsymbol{Cs}P|=p_2$. If $\mathfrak{P}_2$ is abelian, we get a contradiction as above. So we may assume that $\mathfrak{P}_2$ is not abelian. Hence we have that $|\mathfrak{P}_2|=p_2^3$. By the transfer theorem of Wielandt [**5**, p. 447] $\boldsymbol{Ns}\mathfrak{P}_2\neq\mathfrak{P}_2$. Since $\boldsymbol{Z}(\mathfrak{P}_2)=\langle A_2\rangle$ we have that $\boldsymbol{Cs}A_2\neq\boldsymbol{Ns}\langle A_2\rangle$. Then by a theorem of Thompson [**5**, p. 499] $\boldsymbol{Cs}A_2$ is nilpotent against (2.6).

Remark. The proof of (2.10) of [8] is incomplete, because it leaves open the case where $\mathfrak{P}_2$ is abelian but not cyclic. The proof of (2.10) of [8] can be completed as above. But meanwhile Camina [**2**[has found an essentially simpler proof to kill the 2-headed case for the conjugate type rank 4 simple groups.

(2.10) Let $X\neq E$ be a p_2-element of $\mathfrak{G}$. Then $|\boldsymbol{Cs}X|=|\boldsymbol{Cs}A_2|$.

Proof. Assume that $|\boldsymbol{Cs}X|\neq|\boldsymbol{Cs}A_2|$. By (2.7) $\boldsymbol{Cs}X$ is abelian.

Hence we may assume that $\boldsymbol{Cs}X \subseteqq \boldsymbol{Cs}A_2$. Let $\widehat{\mathfrak{P}}_2$ and $\widehat{\mathfrak{U}}$ be the Sylow p_2-subgroup and Sylow p_2-complement of $\boldsymbol{Cs}X$, respectively. By (2.9) $\boldsymbol{F}(\boldsymbol{Cs}A_2) \neq \widehat{\mathfrak{P}}_2$. Hence $\boldsymbol{F}(\boldsymbol{Cs}A_2) \cap \boldsymbol{Ns}\widehat{\mathfrak{P}}_2 \neq \widehat{\mathfrak{P}}_2$. Let X_1 be an element of $\boldsymbol{F}(\boldsymbol{Cs}A_2) \cap \boldsymbol{Ns}\widehat{\mathfrak{P}}_2$ outside $\widehat{\mathfrak{P}}_2$. Then $[\widehat{\mathfrak{P}}_2, X_1^{-1}\widehat{\mathfrak{U}}X_1] = \mathfrak{E}$. If $X_1^{-1}\widehat{\mathfrak{U}}X_1 = \widehat{\mathfrak{U}}$, then $[X_1, \widehat{\mathfrak{U}}] = \boldsymbol{F}(\boldsymbol{Cs}A_2) \cap \widehat{\mathfrak{U}} = \mathfrak{E}$. This is a contradiction.

(2.11) $\mathfrak{P}_2$ is of exponent p_2.

Proof. Assume that $\mathfrak{P}_2$ is of exponent p_2^a, where $a \geqq 2$. Then by (2.10) we may assume that $\boldsymbol{Z}(\boldsymbol{Cs}A_2)$ contains an element C of order p_2^2. Let X be an element of $\boldsymbol{Cs}A_2$ of order p_2. Then $\boldsymbol{Cs}CX = \boldsymbol{Cs}C^{p_2} = \boldsymbol{Cs}A_2$. Hence all elements of $\boldsymbol{Cs}A_2$ of order p_2 belong to $\boldsymbol{Z}(\boldsymbol{Cs}A_2)$. This implies that $\mathfrak{P}_2 = \boldsymbol{Z}(\boldsymbol{Cs}A_2)$. Then by (2.9) $\boldsymbol{F}(\boldsymbol{Cs}A_2) = \mathfrak{P}_2$. Hence $\boldsymbol{Cs}A_2 \cap \boldsymbol{Ns}\mathfrak{U} = \boldsymbol{Cs}\mathfrak{U}$. If $\boldsymbol{Ns}\mathfrak{U} = \boldsymbol{Cs}\mathfrak{U}$, then by the transfer theorem of Burnside $\mathfrak{G}$ is not simple. Hence $\boldsymbol{Ns}\mathfrak{U} \neq \boldsymbol{Cs}\mathfrak{U}$. Let V be an element of $\boldsymbol{Ns}\mathfrak{U}$ outside $\boldsymbol{Cs}\mathfrak{U}$. Since $\boldsymbol{Cs}\mathfrak{U} = \mathfrak{P}_2^* \times \mathfrak{U}$, V normalizes $\mathfrak{P}_2^*$. Since $\boldsymbol{Cs}\mathfrak{P}_2^* = \boldsymbol{Cs}A_2$, V belongs to $\boldsymbol{Ns}(\boldsymbol{Cs}A_2)$, but not to $\boldsymbol{Cs}A_2$. Hence by a theorem of Thompson [5, p. 499] $\boldsymbol{Cs}A_2$ is nilpotent. This is a contradiction.

(2.12) $\pi(\boldsymbol{Cs}A_2) = \pi(\boldsymbol{Ns}\mathfrak{U})$.

Proof. If s is a prime of $\pi(\boldsymbol{Cs}\mathfrak{U})$ not belonging to $\pi(\boldsymbol{Cs}A_2)$, then let $S \neq E$ be an s-element of $\boldsymbol{Ns}\mathfrak{U}$. Then S normalizes $\mathfrak{P}_2^*$ and hence $\boldsymbol{Cs}\mathfrak{P}_2^*$. $\langle S \rangle \boldsymbol{Cs}\mathfrak{P}_2^*$ is a Frobenius group with $\boldsymbol{Cs}\mathfrak{P}_2^*$ the kernel. By a theorem of Thompson [5, p. 499], $\boldsymbol{Cs}\mathfrak{P}_2^*$ is nilpotent. By the proof of (2.10) $\boldsymbol{Cs}\mathfrak{P}_2^*$ contains $\boldsymbol{Cs}\mathfrak{U}$ properly. This is a contradiction. If p_2 does not belong to $\pi(\boldsymbol{Ns}\mathfrak{U})$ then by the transfer theorem of Burnside $\mathfrak{G}$ is not simple.

Now we get a desired contradiction as follows.

Let $\widehat{\mathfrak{P}}$ be a Sylow p_2-subgroup of $\boldsymbol{Ns}\mathfrak{U}$. Then $\boldsymbol{Ns}\mathfrak{U} = \widehat{\mathfrak{P}}\mathfrak{U}$ and $\widehat{\mathfrak{P}} \neq \mathfrak{E}$ by (2.12). Notice that $\boldsymbol{Cs}\mathfrak{U} = \mathfrak{P}_2^* \times \mathfrak{U}$, where $\mathfrak{P}_2^*$ contains A_2. Thus $\mathfrak{P}_2^* \cap Z(\widehat{\mathfrak{P}}) \neq \mathfrak{E}$. Let $A' \neq E$ be an element of $\mathfrak{P}_2^* \cap \boldsymbol{Z}(\widehat{\mathfrak{P}})$. Then $\boldsymbol{Cs}A'$ contains $\boldsymbol{Ns}\mathfrak{U}$. Let $\overline{\mathfrak{P}}$ be a Sylow p_2-subgroup of $\boldsymbol{Cs}A'$. Since

$\boldsymbol{Ns}\mathfrak{U}\neq\boldsymbol{Cs}\mathfrak{U}$, $\boldsymbol{F}(\boldsymbol{Cs}A')\neq\overline{\mathfrak{P}}$. $\boldsymbol{Ns}\mathfrak{U}/\mathfrak{P}_2^*$ is a Frobenius group with $\boldsymbol{Cs}\mathfrak{U}/\mathfrak{P}_2^*$ the kernel. Since $\boldsymbol{F}(\boldsymbol{Cs}A')\cap\boldsymbol{Ns}\mathfrak{U}=\boldsymbol{F}(\boldsymbol{Cs}A')\cap\boldsymbol{Cs}\mathfrak{U}$ and since $\overline{\mathfrak{P}}=\boldsymbol{F}(\boldsymbol{Cs}A')(\overline{\mathfrak{P}}\cap\boldsymbol{Ns}\mathfrak{U})$, $\overline{\mathfrak{P}}\cap\boldsymbol{Ns}\mathfrak{U}/\mathfrak{P}_2^*$ is cyclic. Hence $\overline{\mathfrak{P}}:\boldsymbol{F}(\boldsymbol{Cs}A')=p_2$. Put $\boldsymbol{Ns}\overline{\mathfrak{P}}\cap\boldsymbol{Cs}A'=\overline{\mathfrak{P}}\overline{\mathfrak{U}}$, where $\overline{\mathfrak{U}}$ is a subgroup of $\mathfrak{U}$. If $\overline{\mathfrak{U}}\neq\mathfrak{E}$, then let $X\neq E$ be an element of $\boldsymbol{Ns}\mathfrak{U}\cap\overline{\mathfrak{P}}$ outside $\mathfrak{P}_2^*$. Then $[X,\overline{\mathfrak{U}}]=\overline{\mathfrak{P}}\cap\overline{\mathfrak{U}}=\mathfrak{E}$. Since $\boldsymbol{Cs}\overline{\mathfrak{U}}=\boldsymbol{Cs}\mathfrak{U}=\mathfrak{P}_2^*\times\mathfrak{U}$, this is a contradiction. Hence $\boldsymbol{Ns}\overline{\mathfrak{P}}\cap\boldsymbol{Cs}A'=\overline{\mathfrak{P}}$. By the transfer theorem of Wielandt [**5**, p. 447] $\boldsymbol{Cs}A'$ is p_2-nilpotent. This is a contradiction.

3. 4-headed case

Let $\mathfrak{G}$ be a simple group of conjugate type rank 5 and of 4-headed. Let n_i be maximal elements of $I(\mathfrak{G})$ $(i=1, 2, 3, 4)$. Let A_i be an element of $\mathfrak{G}$ such that $\mathfrak{G}:\boldsymbol{Cs}A_i=n_i$ $(i=1, 2, 3, 4)$.

Part *A*. The purpose of this part is to prove that at least one of the $\boldsymbol{Cs}A_i$ $(i=1, 2, 3, 4)$ is free.

Assume the contrary. Then let X_i be an element of $\mathfrak{G}$ such that $\boldsymbol{Cs}X_i$ is properly contained in $\boldsymbol{Cs}A_i$ $(i=1, 2, 3, 4)$. Thus $\mathfrak{G}:\boldsymbol{Cs}X_i=n_5$ $(i=1, 2, 3, 4)$.

(**3A. 1**) $\boldsymbol{Cs}A_i$ is not nilpotent $(i=1, 2, 3, 4)$.

Proof. Assume that $\boldsymbol{Cs}A_1$ is nilpotent. Obviously there exists a nonabelian Sylow p_1-subgroup $\mathfrak{P}_1$ of $\boldsymbol{Cs}A_1$, where p_1 is a prime. We may assume that A_1 is an element of $\boldsymbol{Z}(\mathfrak{P}_1)$. Hence $\mathfrak{P}_1$ is a Sylow p_1-subgroup of $\mathfrak{G}$. Let $\mathfrak{U}$ be the Sylow p_1-complement of $\boldsymbol{Cs}A_1$. Clearly $\mathfrak{U}$ is abelian. Since $\boldsymbol{Cs}A_1$ is not a Hall subgroup of $\mathfrak{G}$, there exists a prime q in $\pi(\mathfrak{U})$ such that the Sylow q-subgroup $\mathfrak{Q}$ of $\boldsymbol{Cs}A_1$ is not a Sylow q-subgroup of $\mathfrak{G}$. Then there exists a q-element $Q\neq E$ of $\mathfrak{Q}$ such that a Sylow q-subgroup of $\boldsymbol{Cs}Q$ contains $\mathfrak{Q}$ properly. Since $\boldsymbol{Cs}A_1$ is contained in $\boldsymbol{Cs}Q$, this is a contradiction.

Now by a theorem of Camina [**2**] we obtain that $\boldsymbol{Cs}A_i:\boldsymbol{Cs}X_i=p_i^{e_i}$, where p_i is a prime, and that $\boldsymbol{Z}(\boldsymbol{Cs}A_i)$ is a p_i-group $(i=1, 2, 3, 4)$. By the choice of A_i the p_i are distinct.

(**3A. 2**) $\pi(\mathfrak{G})=\{p_1, p_2, p_3, p_4\}$.

Proof. Let q be a prime divisor of $|\mathfrak{G}|$ distinct from p_i ($i=$ 1. 2, 3, 4). We may assume that $\boldsymbol{Cs}A_1$ contains a Sylow q-subgroup $\mathfrak{Q}$ of $\mathfrak{G}$. Let $Q \neq E$ be an element of $\boldsymbol{Z}(\mathfrak{Q})$. Then we have that $\boldsymbol{Cs}A_1Q$ contains $\mathfrak{Q}$ and that $|\boldsymbol{Cs}A_1Q|=|\boldsymbol{Cs}X_1|$. This shows that $\boldsymbol{Cs}\mathfrak{Q}$ and its conjugates exhaust $\mathfrak{G}$. Hence $\mathfrak{G}=\boldsymbol{Cs}\mathfrak{Q}$. This contradicts the simplicity of $\mathfrak{G}$.

(**3A. 3**) Let $|\boldsymbol{Cs}X|=|\boldsymbol{Cs}X_1|$. Then $\boldsymbol{Cs}X$ is abelian.

Proof. This is obvious, since $p_1p_2p_3p_4$ divides $|\boldsymbol{Cs}X|$ and since $\boldsymbol{Cs}A_i : \boldsymbol{Cs}X_i$ is a power of p_i ($i=1, 2, 3, 4$)

(**3A. 4**) We may choose $X=X_1$ and A_i ($i=1, 2, 3, 4$) so that $\boldsymbol{Cs}X$ is contained in $\bigcap_{i=1}^{4} \boldsymbol{Cs}A_i$.

Proof. We show that $\boldsymbol{Cs}X$ contains a p_i-element A_i' ($i>1$) such that $\boldsymbol{Cs}X$ is contained in $\boldsymbol{Cs}A_i'$ and that $|\boldsymbol{Cs}A_i'|=|\boldsymbol{Cs}A_i|$. Let $A_i'' \neq E$ be any p_i-element of $\boldsymbol{Cs}X$. We may assume that A_i'' belongs to $\boldsymbol{Cs}A_i$. If $\boldsymbol{Cs}A_i''=\boldsymbol{Cs}X$, then $\boldsymbol{Cs}X$ contains A_i. Put $A_i'=A_i$. If $|\boldsymbol{Cs}A_i''|=|\boldsymbol{Cs}A_i|$, put $A_i'=A_i''$.

Let $\mathfrak{P}_i$ be a Sylow p_i-subgloup of $\boldsymbol{Cs}A_i$. Then by (3A. 4) $\boldsymbol{Cs}A_i = \mathfrak{P}_i\boldsymbol{Cs}X$. In particular, $\boldsymbol{Cs}A_i$ is solvable ($i=1, 2, 3, 4$) [**5**, p. 674].

(**3A. 5**) For at least one i, $\boldsymbol{F}(\boldsymbol{Cs}A_i)$ is a p_i-group.

Proof. Assume the contrary. Then $\boldsymbol{Cs}X=\boldsymbol{F}(\boldsymbol{Cs}A_i)$ ($i=1, 2, 3,$ 4). Hence $\mathfrak{G}=\boldsymbol{Ns}(\boldsymbol{Cs}X)$. This contradicts the simplicity of $\mathfrak{G}$.

We assume that $\boldsymbol{F}(\boldsymbol{Cs}A_1)$ is a p_1-group. Let $\mathfrak{P}_i^*$ be the Sylow p_i-subgroup of $\boldsymbol{Cs}X$ ($i=1, 2, 3, 4$).

(**3A. 6**) For at least three i's, $\boldsymbol{F}(\boldsymbol{Cs}A_i)$ is a p_i-group.

Proof. By a theorem of Fitting [**5**, p. 277] we have that $\boldsymbol{F}(\boldsymbol{Cs}A_1)$ contains $\mathfrak{P}_1^*$ properly. Then $(\boldsymbol{F}(\boldsymbol{Cs}A_1) \cap \boldsymbol{Ns}\mathfrak{P}_1^*)\mathfrak{P}_2^*\mathfrak{P}_3^*\mathfrak{P}_4^*/\mathfrak{P}_1^*$

is a Frobenius group with $F(CsA_1)\cap Ns\mathfrak{P}_1^*/\mathfrak{P}^k$ the kernel. Therefore $\mathfrak{P}_2^*\mathfrak{P}_3^*\mathfrak{P}_4^*$ is cyclic. Now assume that $F(CsA_i)$ is not a p_i-group for $i=3$, 4. Then $F(CsA_i)=CsX$ for $i=3$, 4. We may assume that $p_3>p_4$. Since $\mathfrak{P}_4^*$ is cyclic, we may assume that $\mathfrak{P}_3\mathfrak{P}_4^*$ is p_4-nilpotent. Hence $[\mathfrak{P}_3, \mathfrak{P}_4^*]\subseteqq\mathfrak{P}_3\cap\mathfrak{P}_4^*=\mathfrak{E}$. This is a contradiction.

We assume that $F(CsA_i)$ is a p_i-group for $i=1$, 2, 3.

(**3A. 7**) If $F(CsA_4)$ is not a p_4-group, then $p_4<p_i$ ($i=1$, 2, 3).

Proof. If so, we have that $F(CsA_4)=CsX$. By the proof of (3A. 6) CsX is cyclic. Since $\mathfrak{P}_i^*\mathfrak{P}_4/\mathfrak{P}_4^*$ is a Frobenius group with $\mathfrak{P}_i^*\mathfrak{P}_4^*/\mathfrak{P}_4^*$ the kernel, $p_i>p_4$ ($i=1$, 2, 3)

Now we may assume that $p_1>p_2>p_3>p_4$. Then $F(CsA_1)=\mathfrak{P}_1$.

We show that $Ns\mathfrak{P}_1$ and its conjugates exhaust $\mathfrak{G}$. Let $G\neq E$ be any element of $\mathfrak{G}$. If $|CsG|=|CsA_1|$, then $\mathfrak{G}$ is a p_1-element. If $|CsG|=|CsA_i|$ for $i>1$, then G is a p_i-element. Since CsG is not free, there exists an element H in CsG such that CsH is properly contained in CsG. G belongs to CsH. By the proof of (3A. 4) there exists a p_1-element $A_1'\neq E$ such that CsH is contained in CsA_1' and that $|CsA_1'|=|CsA_1|$. Therefore, $Ns\mathfrak{P}_1=\mathfrak{G}$ and $\mathfrak{G}$ is not simple. This is a cntradiction.

Part B. We use the same notation as in Part A. By Part A. we may assume that CsA_4 is free. The purpose of this part is to prove that at least one of CsA_i ($i=1$, 2, 3) is also free.

Assume the contrary. Then let X_i be an element of $\mathfrak{G}$ such that CsX_i is properly contained in CsA_i ($i=1$, 2, 3). Then $\mathfrak{G}:CsX_i=n_5$ ($i=1$, 2, 3).

(**3B. 1**) CsA_i is not nilpotent ($i=1$, 2, 3).

Proof. See the proof of (3A. 1).

Now by a theorem of Camina [2] we obtain that $CsA_i:CsX_i=p_i^{a_i}$, where p_i is a prime, and that $Z(CsA_i)$ is a p_i-group ($i=1$, 2, 3). By the choice of A_i the p_i are distinct.

(**3B. 2**) $\pi(\boldsymbol{CsA}_i)=\{p_1, p_2, p_3\}$ $(i=1, 2, 3)$

Proof. Let q be a prime of $\pi(\boldsymbol{CsA}_1)$ distinct from p_i $(i=1, 2, 3)$. We may assume that $\boldsymbol{CsA}_1$ contains a Sylow q-subgroup $\mathfrak{Q}$ of $\mathfrak{G}$. Let $Q\neq E$ be an element of $\boldsymbol{Z}(\mathfrak{Q})$. Then we have that $\boldsymbol{CsA}_1Q$ contains $\mathfrak{Q}$ and $|\boldsymbol{CsA}_1Q|=|\boldsymbol{CsX}_1|$. This shows that $\mathfrak{G}$ is of isolated type and hence $\mathfrak{G}$ is not simple [**6**].

(**3B. 3**) Let $|\boldsymbol{CsX}|=|\boldsymbol{CsX}_1|$. Then $\boldsymbol{CsX}$ is abelian.

Proof. See the proof of (3A. 3)

(**3B. 4**) We may choose $X=X_1$ and A_i $(i=1, 2, 3)$ so that $\boldsymbol{CsX}$ is contained in $\bigcap_{i=1}^{3}\boldsymbol{CsA}_i$.

Proof. See the proof of (3A. 4).

Let $\mathfrak{P}_i$ be a Sylow p_i-subgroup of $\boldsymbol{CsA}_i$. Then by (3B. 3) $\boldsymbol{CsA}_i=\mathfrak{P}_i\boldsymbol{CsX}$. In particular, $\boldsymbol{CsA}_i$ is solvable $(i=1, 2, 3)$ [**5**, p. 674].

(**3B. 5**) $p_i=2$ for $i=1$ or 2 or 3.

Proof. Assume the contrary. Then by a theorem of Feit-Thompson [**3**] $\boldsymbol{CsA}_4$ is of even order. Since $\boldsymbol{CsA}_4$ is free, $\boldsymbol{CsA}_4$ is abelian [**6**]. In particular, a Sylow 2-subgroup of $\mathfrak{G}$ is abelian. Therefore, by a theorem of Walter [**13**] we get a contradiction.

We assume that $p_3=2$. Then $\mathfrak{P}_3$ is not abelian and, in particular, of exponent $\geqq 4$.

(**3B. 6**) There exists a 2-element Y such that $|\boldsymbol{CsY}|=|\boldsymbol{CsX}|$.

Proof. Assume the contrary. Let A_3' be an element of $\boldsymbol{Z}(\boldsymbol{CsA}_3)$ of order 4. Let A be any involution of $\boldsymbol{CsA}_3$. Then since $\boldsymbol{CsAA}_3'$ is contained in $\boldsymbol{Cs}(A_3')^2$, we obtain that $\boldsymbol{CsA}_3'A=\boldsymbol{CsA}=\boldsymbol{CsA}_3$. This implies that $\mathfrak{P}_3$ is abelian. This is a contradiction.

(**3B. 7**) We can take Y as in (3B. 4).

Proof. Since $\boldsymbol{Cs}Y$ is minimal, $\boldsymbol{Cs}Y$ is the direct product of the Sylow 2-subgroup and the abelian Sylow 2-complement. The rest is obvious.

(**3B.** 8) $\boldsymbol{F}(\boldsymbol{Cs}A_3)$ is not a 2-group.

Proof. Assume the contrary. By a theorem of Fitting [**5**, p. 277] we have that $\boldsymbol{F}(\boldsymbol{Cs}A_3)$ contains $\mathfrak{P}_3^*$ properly. Let A be an element of $\boldsymbol{F}(\boldsymbol{Cs}A_3)\cap\boldsymbol{Ns}\mathfrak{P}_3^*$ outside $\mathfrak{P}_3^*$. Then if A belongs to $\boldsymbol{Ns}\mathfrak{P}_1^*$, $[A, \mathfrak{P}_1^*]$ is contained in $\mathfrak{P}_1^*\cap\boldsymbol{F}(\boldsymbol{Cs}A_3)=\mathfrak{E}$. Since $\boldsymbol{Cs}\mathfrak{P}_1^*\cap\boldsymbol{Cs}A_3$ is contained in $\boldsymbol{Cs}X$, this is a contradiction. Therefore, $A^{-1}\mathfrak{P}_1^*A\neq\mathfrak{P}_1^*$ and $[\mathfrak{P}_3^*, A^{-1}\mathfrak{P}_1^*A]=\mathfrak{E}$. This shows that $|\boldsymbol{Cs}X|=|\boldsymbol{Cs}A_3|$. This contradicts (3B. 6).

(**3B. 9**) Let $|\boldsymbol{Cs}X'|=|\boldsymbol{Cs}X|$. Then $\boldsymbol{Cs}X'$ is conjugate with $\boldsymbol{Cs}X$ in $\mathfrak{G}$.

Proof. By (3B. 3) $\boldsymbol{Cs}X'$ is abelian. Since $\boldsymbol{Cs}X$ contains $\boldsymbol{Z}(\mathfrak{P}_3)$, we may assume that $\boldsymbol{Cs}X$ contains a 2-element A_3' of $\boldsymbol{Cs}X'$. Then $\boldsymbol{Cs}A_3'$ contains both $\boldsymbol{Cs}X$ and $\boldsymbol{Cs}X'$. Now by (3B. 8) $\boldsymbol{F}(\boldsymbol{Cs}A_3')$ is not a 2-group. This implies that $\boldsymbol{Cs}X=\boldsymbol{Cs}X'=\boldsymbol{Cs}(\mathfrak{P}_1^*\mathfrak{P}_2^*)$.

Now every element of $\mathfrak{G}$ is conjugate either to an element of $\boldsymbol{Cs}A_4$ or to an element of $\boldsymbol{Cs}X$. Since $\boldsymbol{Cs}X$ is normal in $\boldsymbol{Cs}A_3$, $\boldsymbol{Ns}(\boldsymbol{Cs}X)$ contains $\boldsymbol{Cs}X$ properly. Since $\boldsymbol{Cs}A_4$ is abelian or an p-group of exponent p, if $\boldsymbol{Ns}(\boldsymbol{Cs}A_4)=\boldsymbol{Cs}A_4$ then by the transfer theorem of Wielandt [**5**, p. 447] $\mathfrak{G}$ is not simple. Hence $\boldsymbol{Ns}(\boldsymbol{Cs}A_4)\neq\boldsymbol{Cs}A_4$. Therefore by counting the number of elements in $\mathfrak{G}$ we get a contradiction.

Part C. We use the same notation as in Part A. By Parts A and B we may assume that $\boldsymbol{Cs}A_3$ and $\boldsymbol{Cs}A_4$ are free. The purpose of this part is to prove that at least one of $\boldsymbol{Cs}A_i$ $(i=1, 2)$ is also free.

Assume the contrary. Then let X_i be an element of $\mathfrak{G}$ such that $\boldsymbol{Cs}X_i$ is properly contained in $\boldsymbol{Cs}A_i$ $(i=1, 2)$. Then $\mathfrak{G}:\boldsymbol{Cs}X_i=$

n_5 $(i=1, 2)$.

(3C. 1) $\boldsymbol{CsA_i}$ is not nilpotent $(i=1, 2)$.

Proof. See the proof of (3A. 1).

Now by a theorem of Camina [**2**] we obtain that $\boldsymbol{CsA_i}: \boldsymbol{CsX_i}=p_i^{e_i}$, where p_i is a prime, and that $\boldsymbol{Z}(\boldsymbol{CsA_i})$ is a p_i-group $(i=1, 2)$. By the choice of A_1 and A_2, p_1 and p_2 are distinct.

(3C. 2) p_1 or $p_2=2$.

Proof. See the proof of (3B. 5).

We assume that $p_2=2$. Then P_2 is not abelian, and, in particular, of exponent $\geqq 4$.

(3C. 3) There exists a 2-element X such that $|\boldsymbol{CsX}|=|\boldsymbol{CsX_1}|$. $\boldsymbol{CsX}$ is the direct product of the Sylow 2-subgsoup $\mathfrak{P}_2^*$, the abelian Sylow p_1-subgroup $\mathfrak{P}_1^*$ and the abelian Hall $\{2, p_1\}$ -complement $\mathfrak{A}$ of $\boldsymbol{CsX}$.

Proof. See the proof of (3B. 6).

(3C. 4) We may choose A_1 and A_2 so that $\boldsymbol{CsA_1}\cap\boldsymbol{CsA_2}=\boldsymbol{CsX}$.

Proof. Obvious.

Since $\boldsymbol{CsA_i}=\mathfrak{P}_i\boldsymbol{CsX}$, $\boldsymbol{CsA_i}$ is solvable $(i=1, 2)$ [**5**, p. 674].

(3C. 5) $\boldsymbol{F}(\boldsymbol{CsA_2})$ is not a 2-group.

Proof. See the proof of (3B. 8).

Therefore $\boldsymbol{F}(\boldsymbol{CsA_2})=\boldsymbol{CsX}=\mathfrak{P}_1^*\times\mathfrak{P}_2^*\times\mathfrak{A}$. Since $\mathfrak{P}_2\boldsymbol{CsX}/\mathfrak{P}_2^*$ is a Frobenius group with $\boldsymbol{CsX}/\mathfrak{P}_2^*$ the kernel, $\mathfrak{P}_2/\mathfrak{P}_2^*$ is cyclic or generalized quaternion. Let A_2' be an element of $\mathfrak{P}_2$ outside $\mathfrak{P}_2^*$. If $|\boldsymbol{CsA_2'}|=|\boldsymbol{CsX}|$, then A_2 commutes with a p_1-element not belonging to $\mathfrak{P}_1^*$. This is a contradiction. Hence $|\boldsymbol{CsA_2'}|=|\boldsymbol{CsA_2}|$. If $\boldsymbol{CsA_2'}$ contains a 2-element X' of $\boldsymbol{CsA_2}$ such that $\boldsymbol{CsX'}=\boldsymbol{CsX}$, then A_2' belongs to

$\boldsymbol{CsX}$. This is a contradiction. Hence $\boldsymbol{Cs}A_2'$ does not contain such an element. If $\langle A_2'\rangle \cap \mathfrak{P}_2^* \neq \mathfrak{E}$, then $\boldsymbol{Cs}A_2'$ contains $\mathfrak{P}_1^* \times \mathfrak{A}$. This is a contradiction. Hence $\langle A_2'\rangle \cap \mathfrak{P}_2^* = \mathfrak{E}$.

(**3C. 6**) $\boldsymbol{Z}(\mathfrak{P}_2)$ is elementary abelian.

Proof. First we show that $\boldsymbol{Z}(\boldsymbol{Cs}A_2)$ is elementary abelian. Otherwise, we may assume that A_2 is an element of order 4. Let A_2' be an involution of $\mathfrak{P}_2$ outside $\mathfrak{P}_2^*$. Then $\boldsymbol{Cs}A_2'A_2 = \boldsymbol{Cs}A_2^2 = \boldsymbol{Cs}A_2$. This shows that A_2' belongs to $\boldsymbol{Z}(\boldsymbol{Cs}A_2)$, and hence to $\boldsymbol{Cs}X$. This is a contradiction. Now assume that $\boldsymbol{Z}(\mathfrak{P}_2)$ is not elementary abelian. Let A_2'' be an element of $\boldsymbol{Z}(\mathfrak{P}_2)$ of order 4. Then A_2'' does not belong to $\mathfrak{P}_2^*$ by the first argument. But $A_2''X = XA_2''$. This is a contradiction.

(**3C. 7**) $\mathfrak{P}_2 : \mathfrak{P}_2^* = 2$.

Proof. This is obvious by (3C. 6) and the argument following (3C. 5).

(**3C. 8**) Let A_2' be an element of $\mathfrak{P}_2$ outside $\mathfrak{P}_2^*$. Then $\boldsymbol{Cs}A_2' \cap \mathfrak{P}_2^* = \langle A_2\rangle$.

Proof. Let G be an element of $\mathfrak{G}$ such that $\boldsymbol{Z}(G^{-1}\mathfrak{P}_2G)$ contains A_2'. Then $G^{-1}A_2G$ belongs to $\boldsymbol{Z}(G^{-1}\mathfrak{P}_2G)$, $\boldsymbol{Cs}G^{-1}A_2G : \boldsymbol{Cs}G^{-1}XG = 2$ and $\boldsymbol{Cs}G^{-1}XG = G^{-1}\mathfrak{P}_1^*G \times G^{-1}\mathfrak{P}_2^*G \times G^{-1}\mathfrak{A}G$. Then A_2' belongs to $G^{-1}\mathfrak{P}_2^*G$. Hence $\boldsymbol{Cs}A_2' = \boldsymbol{Cs}G^{-1}A_2G$. Now assume that $\boldsymbol{Cs}A_2' \cap \mathfrak{P}_2^*$ contains $\langle A_2\rangle$ properly. Then $\boldsymbol{Cs}G^{-1}XG$ contains an element $A_2'' \neq E$ of $\mathfrak{P}_2^*$. Then $\boldsymbol{Cs}A_2''$ contains $\mathfrak{P}_1^*$ and $G^{-1}\mathfrak{P}_1^*G$. The first argument shows that $\boldsymbol{F}(\boldsymbol{Cs}A_2'')$ is not a 2-group. Hence $\mathfrak{P}_1^* = G^{-1}\mathfrak{P}_1^*G$. This is a contradiction.

Now by a lemma of Suzuki [**11**] $\mathfrak{P}_2$ is dihedral or quasi-dihedral. Hence by a theorem of Gorenstein-Walter [**4**] or a theorem of Alperin-Brauer-Gorenstein [**1**] we get a contradiction.

Remark. The argument in ((c), p. 244) of [8] is incomplete, since the argument appeals to [**9**] which is not applicable in that case. One way to amend it is to follow the argument in Part C.

Part D. We use the same notation as in Part A. By Parts A, B and C we may assume that $\boldsymbol{CsA}_i$ $(i=1, 2, 3, 4)$ is free. The purpose of this part is to prove that $\mathfrak{G}$ is isomorphic with some $Sz(l)$, where $l=2^{2n+1}$, $n\geqq 1$, or $LF(3, 4)$. By [7] $\boldsymbol{CsA}_1$ is not free. Let X be an element of $\mathfrak{G}$ such that $\boldsymbol{CsA}_1 : \boldsymbol{CsX}=n_5$.

(**3D. 1**) $\boldsymbol{CsA}_1$ is a Hall subgroup of $\mathfrak{G}$. Furtheremore, $|\mathfrak{G}|=\prod_{i=1}^{4} |\boldsymbol{CsA}_i|$.

Proof. This is obvious.

(**3D. 2**) $\boldsymbol{CsA}_1$ is of even order.

Proof. See the proof of (3B. 5).

(**3D. 3**) We may assume that $\boldsymbol{CsA}_1$ is not nilpotent.

Proof. If $\boldsymbol{CsA}_1$ is nilpotent, then by a theorem of Wielandt [**5**, p. 285] all subgroups $\mathfrak{X}$ of $\mathfrak{G}$ with $|\mathfrak{X}|=|\boldsymbol{CsA}_1|$ are nilpotent. Hence, in particular, the centralizer of every involution of $\mathfrak{G}$ is 2-closed. Therefore by a theorem of Suzuki [**12**] we get the theorem. Hence we may assume that $\boldsymbol{CsA}_1$ is not nilpotent.

Now by a theorem of Camina [**2**] we obtain that $\boldsymbol{CsA}_1 : \boldsymbol{CsX}=p^a$, where p is a prime, and that $\boldsymbol{Z}(\boldsymbol{CsA}_1)$ is a p-group.

(**3D. 4**) We may assume that $p=2$.

Proof. Otherwise, let J be an involution in $\boldsymbol{CsA}_1$. Then $\boldsymbol{CsJ}=\boldsymbol{CsA}_1 J$ is nilpotent. Hence, as in the proof of (3D. 3) we may assume that $p=2$.

Now, as before, $\mathfrak{P}_1$ is not abelian and, in particular, of exponent $\geqq 4$.

(**3D. 5**) There exists a 2-element Y such that $|\boldsymbol{CsY}|=|\boldsymbol{CsX}|$.

Proof. See the proof of (3B. 6).

(3D. 6) $\boldsymbol{F}(\boldsymbol{CsA}_1)$ is not a 2-group.

Proof. Since $\boldsymbol{CsA}_1=\mathfrak{P}_1\boldsymbol{CsY}$ and since $\boldsymbol{CsY}$ is nilpotent, $\boldsymbol{CsA}_1$ is solvable [**5**, p. 674]. Now see the proof of (3B. 8).

By (3D. 3) $|\pi(\boldsymbol{CsA}_1)|\geqq 2$. Let q be a prime of $\pi(\boldsymbol{CsA}_1)$ distinct from 2. Let $\mathfrak{Q}$ be a Sylow q-subgroup of $\boldsymbol{CsA}_1$. Let $X\neq E$ be an element of $\boldsymbol{Z}(\mathfrak{Q})$. Then $\boldsymbol{CsX}=\boldsymbol{CsXA}_1$ is the direct product of the abelian Sylow 2-subgroup $\mathfrak{P}_1^*$, the Sylow q-subgroup $\mathfrak{Q}$ and the abelian Hall $\{2, q\}$-complement $\mathfrak{A}$. By (3D. 5) $\mathfrak{Q}$ is also abelian. Thus $\boldsymbol{CsX}$ is abelian.

Now to complete the proof it suffices to prove the following proposition, which is incompatible with (3D. 6)

(3D. 7) $\boldsymbol{F}(\boldsymbol{CsA}_1)$ is a 2-group.

Proof. Assume the contrary. Then $\boldsymbol{F}(\boldsymbol{CsA}_1)=\boldsymbol{CsX}$. $\mathfrak{P}_1\boldsymbol{CsX}/\mathfrak{P}_1^*$ is a Frobenius group with $\boldsymbol{CsX}/\mathfrak{P}_1^*$ the kernel. Hence $\mathfrak{P}_1/\mathfrak{P}_1^*$ is cyclic or generalized quaternion [**5**, p. 502].

First we assume that $\mathfrak{P}_1/\mathfrak{P}_1^*$ is a generalized quaternion group of order 2^b. Then there exist elements Q and R of $\mathfrak{P}_1$ and S, T, U and V of $\mathfrak{P}_1^*$ such that $R^{-1}QR=Q^{-1}S$, $Q^{2^{b-2}}=R^2T$, $Q^{2^{b-1}}=U$, $R^4=V$ and $\mathfrak{P}_1/\mathfrak{P}_1^*=\langle Q, R\rangle\mathfrak{P}_1^*/\mathfrak{P}_1^*$. Now we further assume that $\mathfrak{P}_1^*$ is not cyclic. Let $\mathfrak{W}$ be a normal subgroup of $\mathfrak{P}_1$ of type (2, 2) contained in $\mathfrak{P}_1^*$. Then $\boldsymbol{CsR}^2$ contains $\mathfrak{W}$. If $|\boldsymbol{CsR}^2|=|\boldsymbol{CsX}|$, then by the remark just before (3D. 7) $\boldsymbol{CsR}^2$ is abelian. This implies that $\boldsymbol{CsR}^2$ is contained in $\boldsymbol{CsA}_1$. This is a contradiction. Hence $|\boldsymbol{CsR}^2|=|\boldsymbol{CsA}_1|$. If $\boldsymbol{F}(\boldsymbol{CsR}^2)$ is a 2-group, then we have that $\mathfrak{P}_1\colon\mathfrak{P}_1^*>|\mathfrak{Q}|$. This is a contradiction. Hence $\boldsymbol{F}(\boldsymbol{CsR}^2)$ is not a 2-group. Let $\mathfrak{P}_1$, $\mathfrak{P}_1^\#$, $\mathfrak{Q}^\#$ and $\mathfrak{A}^\#$ be a Sylow 2-subgroup of $\boldsymbol{CsR}^2$, the abelian Sylow 2-subgroup, the abelian Sylow q-subgroup and the abelian Hall $\{2, q\}$-complement of $\boldsymbol{F}(\boldsymbol{CsR}^2)$, respectively. Then $\mathfrak{P}_1/\mathfrak{P}_1^\#$ is cyclic or generalized quaternion.. This implies that $\mathfrak{W}\cap\mathfrak{P}_1^\#\neq\mathfrak{E}$. Take an element $W\neq E$ of

$\mathfrak{W}\cap\mathfrak{P}_1^\#$. Then CsW contains $\mathfrak{Q}^\#$ and $\mathfrak{Q}$. This implies that $\mathfrak{Q}^\#=\mathfrak{Q}$. This is a contradiction. Therefore $\mathfrak{P}_1^*$ is cyclic. Hence $\mathfrak{P}_1\cap Cs\mathfrak{P}_1^*$ contains $\mathfrak{P}_1^*$ properly. Thus $CsQ^{2^{b-2}}$ contains $\mathfrak{P}_1^*$. This implies that $|CsQ^{2^{b-2}}|=|CsA_1|$. As above, $F(CsQ^{2^{b-2}})$ is not a 2-group. If $|CsQ|=|CsX|$, then $CsQ=F(CsQ^{2^{b-2}})$. Let $\mathfrak{P}_1$ be the Sylow 2-subgroup of CsQ. Then $[Q, \mathfrak{P}_1^*]$ is contained in $\mathfrak{P}_1^*\cap\mathfrak{P}_1=\mathfrak{E}$. This is a contradiction. Hence $CsQ=CsQ^{2^{b-2}}$. Similarly we obtain that $CsR=CsR^2$. Since $Q^{2^{b-2}}$ and R^2 commute, this implies that Q and R commute. This is a contradiction. Therefore $\mathfrak{P}_1/\mathfrak{P}_1^*$ is cyclic.

Let $\mathfrak{P}_1/\mathfrak{P}_1^*$ be of order 2^b and $P\mathfrak{P}_1^*$ a generator of $\mathfrak{P}_1/\mathfrak{P}_1^*$. Assume that $b\geqq 2$. As above, we may assume that $\mathfrak{P}_1$ is cyclic. Therefore $\mathfrak{P}_1$ is metacyclic. Then by a theorem of Mazurov [**10**] $\mathfrak{P}_1$ is of type (2, 2) or of maximal class. This is a contradiction. Hence we obtain that $b=1$. Now we show that $Z(\mathfrak{P}_1)$ is of order 2. Assume the contrary. If $|CsP|=|CsX|$, then by the remark just before (3D. 7) CsP is abelian and $CsP\cap(\mathfrak{Q}\times\mathfrak{A})=\mathfrak{E}$. Let $\mathfrak{Q}^\#$ be the abelian Sylow q-subgroup of CsP. Then $\mathfrak{Q}^\#\cap\mathfrak{Q}=\mathfrak{E}$. But since CsP contains $Z(\mathfrak{P}_1)$, this is a contradiction. If $|CsP|=|CsA_1|$, then let $\mathfrak{P}_1^\#$ and $\mathfrak{Q}^\#$ be the abelian Sylow 2-subgroup and the abelian Sylow q-subgroup of $F(CsP)$. Then $\mathfrak{P}_1\cap\mathfrak{P}_1^\#\neq\mathfrak{E}$ by assumption. Let $Z\neq E$ be an element of $\mathfrak{P}_1\cap\mathfrak{P}_1^\#$. CsZ contains $\mathfrak{Q}$ and $\mathfrak{Q}^\#$. Since $\mathfrak{Q}\cap\mathfrak{Q}^\#=\mathfrak{E}$ and since $F(CsZ)$ contains $\mathfrak{Q}$ and $\mathfrak{Q}^\#$, this is a contradiction. Hence $|Z(\mathfrak{P}_1)|=2$. Since P is of order 2 (See the proof of (3C. 6)), by a lemma of Suzuki [11] $\mathfrak{P}_1$ is of type (2, 2) or of maximal class. Then by a theorem of Wong [14] we get a contradiction.

Remark. The argument of Part D, together with [8], shows that we obtain the followiog theorem. *Let $\mathfrak{G}$ be a simple group such that $C(\mathfrak{G})$ has the following shape* $\begin{matrix} n_1 & n_2 & \cdots & n_k \\ \downarrow & & & \\ n_{k+1} & & & \end{matrix}$. *Then* $k=3$ *or 4. If $k=3$, then $\mathfrak{G}$ is isomorphic with $LF(2, l)$, where l is an odd prime power bigger than 5. If $k=4$, then $\mathfrak{G}$ is isomorphic with $Sz(l)$, where $l=2^{2n+1}$, $n\geqq 1$, or $LF(3, 4)$.*

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