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On a generalization of $\bar{\partial}_b$

By

Masatake KURANISHI

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By meane of the complex of $\bar{\partial}$ exterior derivatives J. J. Kohn and H. Rossi introduced a complex of differential operators $\bar{\partial}_b$ on the smooth boundary of a relatively compact open submanifold in a complex manifold [3]. The subellipticity of $\bar{\partial}$ complex on the part of positive degrees implies the finite dimensionality of the positive degree cohomology of $\bar{\partial}_b$ complex over the closure of the open submanifold. The purpose of the present note is to generalize the above to more general cases of complex of differential operators of degree one, say $(\mathcal{E})$. Firstly we note that the construction of $\bar{\partial}_b$ complex is a special case of a construction valid for any $(\mathcal{E})$ provided the smooth boundary is non-characteristic (with respect to $(\mathcal{E})$). Secondly we show that the subelliptic estimate of $(\mathcal{E})_b$ complex on the part of positive degrees implies the finite dimensionality of the positive degree cohomology of $(\mathcal{E})$ over the closure of the interior of the smooth compact boundary, provided $(\mathcal{E})$ is elliptic. Since it might be useful to generalize the finite dimensionality theorem of cohomology of elliptic complex, we state the exact conditions needed in the proof of the theorem.

It is noted that this generalization of $\bar{\partial}_b$ is also considered by Sweeney in [5].

1. Let Y be a relatively compact open submanifold of a manifold $Y^{\#}$. We assume that Y has the smooth boundary M. Choose once for all a tubular neighborhood N of M in $Y^{\#}$ together with a diffeomorphism

(1) $$h: N \to M \times [-\varepsilon_1, \varepsilon_1] \qquad (\varepsilon_1 > 0)$$

such that $N \cap Y$ is mapped onto $M \times [-\varepsilon_1, 0)$. We denote by t the composition of the projection to $[-\varepsilon_1, \varepsilon_1]$ with h. t will be referred as the distance function to M, dt as the normal cotangent vector field. Let $E^j (j=0, 1, \cdots, a)$ be a sequence of vector bundles over $Y^{\#}$. Set $E^{-1} = E^{a+1} = \{0\}$, the zero vector bundle over $Y^{\#}$. Assume that we are given differential operators of order one

(2) $$D: C^\infty(Y^{\#}, E^j) \to C^\infty(Y^{\#}, E^{j+1}) \qquad (j=-1, 2, \cdots, a)$$

so that they form a complex $(\mathcal{E})$ of differential operators. When there is no possibility of confusion, we omit indexes j. For a cotangent vector ξ to $Y^{\#}$ we denote by $\sigma^j(\xi)$ the principal symbol of D^j. For a point x of $Y^{\#}$, E_x denotes the fiber over x of E. Thus, for ξ at x, $\sigma(\xi)$ is a linear map of E_x^j to E_x^{j+1}. When ξ is a cotangent vector field over a subset G of $Y^{\#}$, we may regard $\sigma(\xi)$ as a map of $E^j | G$ into $E^{j+1} | G$, as well as a linear map of $C^\infty(G, E^j)$ into $C^\infty(G, E^{j+1})$ in the case G is open.

Lemma 1. $\quad D \circ \sigma(dt) + \sigma(dt) \circ D = 0.$

Proof. It is enough to prove the equality on a neighborhood of each point in N. Pick a pair of a chart $(x_1, \cdots, x_n)$ of N such that $x_n = t$ and a local trivialization E^j over the domain of the chart. In terms of these, write $D^j = a_1^j(x)(-i)\dfrac{\partial}{\partial x_1} + \cdots + a_n^j(x)(-i)\dfrac{\partial}{\partial x_n} + a_j^0(x)$. Then $a_n^j(x) = \sigma(dt)$. By writing down the equality $D^{j+1} \circ D^j = 0$ and noting that $\sigma_n^{j+1} \circ \sigma_n^j(x) = 0$, we easily verify our equality. q. e. d.

Assume that M is non-characteristic with respect to $(\mathcal{E})$. This means by definition that for each x in N the symbol sequence at normal contangent

(3) $$\sigma(dt): E_x^j \to E_x^{j+1} \qquad (j=-1, 0, \cdots, a)$$

is exact. Therefore, if we denote by K_x^j the image by $\sigma(dt)$ of E^{j-1},

$K^j=\bigcap\{K^j_x; x\in N\}$ is a vector sub-bundle of $E^j|N$ (provided we schrink N if necessary). The assumption of M being non-charactristic and the notation K will be kept throughout the remainder of this paper.

Pick $s\in C^\infty(M, E^j)$. Extend s to a section u of E^j over N (schrinking N if necessary). Since $\sigma(dt)\circ\sigma(dt)=0$ we see immediately that $(\sigma(dt)Du)|M$ is independent of the way we extend s. Thus, if u' is another extension of s, $(Du-Du')|M$ is in the kernel of $\sigma(dt)$. Since M is non-characteristic, this means $(Du-Du')|M$ is a section of $K^{j+1}|M$. Hence, if we denote by θ the projection $E^{j+1}|M\to(E^{j+1}|M)/(K^{j+1}|M)$, $\theta\circ Du|M$ is independent of the way we extend s. Therefore, $s\to\theta\circ Du|M$ is a differential operator of order one $C^\infty(M, E^j)\to C^\infty(M, E^{j+1}/K^{j+1})$. By Lemma 1 it follows easily that for $s\in C^\infty(M, K^j)$ the image by the above map is zero. Then it is easy to see that it induces a differential operator of order one

(4) $$D^j_b: C^\infty(M, E^j/K^j)\to C^\infty(M, E^{j+1}/K^{j+1})$$

Since $D\circ D=0, \{D^j_b\}$ is a complex. The projection θ induces a linear map

(5) $$\theta: C^\infty(Y, E^j)\to C^\infty(M, E^j/K^j).$$

Pick a linear map

(6) $$\kappa: C^\infty(M, E^j/K^j)\to C(^\infty\overline{Y}, E^j)$$

such that $\theta\circ\kappa$ is the identity map. Then it follows that

(7) $$\sigma(dt)(\kappa\theta u-u)|M)=0 \qquad (u\in C^\infty(Y, E^j)).$$

Denote by r_M the restriction map to M. Then by the definition of D_b,

(8) $$D_b\theta=\theta D,$$

(9) $$\sigma(dt)r_M\kappa D_b=\sigma(dt)r_M D\kappa.$$

2. In order to find conditions to insure finite dimensionality of j-th cohomology over $\overline{Y}$ of the complex $(\mathcal{E})$, we assume the existence of the following linear maps. Let

(10) $$N^j: C^\infty(M, E^j/K^j) \to C^\infty(M, E^{j-1}/K^{j-1}) \qquad (j \geqslant 1)$$

be a linear map such that

(11) $$D_b N + D N_b = I + K_1$$

where I is the identity map and K_1 is of order $-\infty$ (with respect to Frechét space structures induced by Sobolev norms). Let

(12) $$W^j: C_0^\infty(Y^\#, E^j) \to C_0^\infty(Y^\#, E^{j-1}) \qquad (j \geqslant 1)$$

be a linear map such that

(13) $$r_Y D W + r_Y W D = r_Y + r_Y K_2$$

where r_Y is the restriction map to Y and K_2 is of order $-\infty$. Denote by $H_s(Y^\#, E^j)$ be a Sobolev s-norm completion of $C_0^\infty(Y^\#, E^j)$. We introduce the following:

Assumption 1. For each real number s there is an integer r such that W is a continuous map of $C_0^\infty(Y^\#, E^j)$ with Sobolev s-norm into $C_0^\infty(Y^\#, E^{j-1})$ with Sobolev r-norm.

Thus W induces a map of $H_s(Y^\#, E^j)$ into $H_r(Y^\#, E^{j-1})$. Denote by χ the characteristic function of Y. For each $u \in C^\infty(Y^\#, E^j)$, χu may be regarded as a L_2-section of E^j over $Y^\#$. Hence we may apply W to χu. Denote by δ the Dirac measure supported by M. Then, for any $u \in C^\infty(M, E^j)$, $\delta \otimes u$ can be regarded as an element of $H_{-1}(Y^\#, E^j)$, (for the detail cf. [4]). Therefore we may apply W to it.

Assumption 2. Besides Assumption 1 we require further that $W(\chi u)$ and $W(\delta \otimes u)$ are C^∞ on Y and that $r_Y W(\chi u)$ and $r_Y W(\delta \otimes u)$ are extendable to C^∞ sections over $Y^\#$. Thus we may regard them as elements of $C^\infty(\bar{Y}, E^{j-1})$.

When the complex $(\mathcal{E})$ is elliptic, a pseudo-differential operator W^j satisfying (13) exists. Therefore for such choice of W Assumption 1 and 2 are satisfied.

For $u \in C^\infty(Y, E^j)$ we find by (11)

$$\sigma(dt)r_M\kappa(D_bN\theta u+ND_b\theta u)=\sigma(dt)r_M\kappa(\theta u+K_1\theta u)$$
$$=\sigma(dt)r_Mu+\sigma(dt)r_MK_1\theta u.$$

Then by (9) and (8)

$$\sigma(dt)r_MD\kappa N\theta u+\sigma(dt)r_M\kappa N\theta Du=\sigma(dt)r_Mu+\sigma(dt)r_M\kappa K_1\theta u$$

Thus, by setting

(14) $\qquad S'=\kappa N\theta, \qquad K_3=r_M\kappa K_1\theta,$

we find that

(15) $\qquad \sigma(dt)r_M(DS'u+S'Du)=\sigma(dt)r_Mu+\sigma(dt)K_3u$

$$(u\in C_\infty(Y,E^j)).$$

Since $\sigma(dt)\sigma(dt)=0$, we find that

$$D(\chi u+i\delta\otimes\sigma(dt)r_MS'u)_M=\chi Du+i\delta\otimes\sigma(dt)r_Mu+i\delta\otimes r_MD\sigma(dt)S'u$$
$$=\chi Du+i\delta\otimes(\sigma(dt)r_Mu-\sigma(dt)r_MDS'u) \qquad \text{(by Lemma 1)}$$
$$=\chi Du+i\delta\otimes(\sigma(dt)r_MS'Du-\sigma(dt)K_3u) \qquad \text{(by 15)}$$

Therefore by (13)

$$-r_YD\,W(\chi u+i\delta\otimes\sigma(dt)r_MS'u)+I_r\chi u+r_YK_2(\chi u+i\delta\otimes\sigma(dt)r_MS'u)$$
$$=r_YW(\chi Du+i\delta\otimes\sigma(dt)r_MS'Du-i\delta\otimes\sigma(dt)K_3u)$$

If $v\in C(Y,E^j)$ is extendable to a C^∞ section over $Y^\sharp$, we may regard v as an element of $C^\infty(\overline{Y},E^j)$, which we denote by αv. Since each term of the above equality is extendable to a C^∞ section over $Y^\sharp$, we may set

$$Su=\alpha r_YW(\chi u+i\delta\otimes\sigma(dt)r_MS'u)$$
$$Ku=\alpha r_YK_2(\chi u+i\delta\otimes\sigma(dt)r_MS'u)+\alpha r_Y\,W(i\delta\otimes\sigma(dt)K_3u).$$

Then

$$S:C^\infty(\overline{Y},E^j)\to C_\infty(\overline{Y},E^{j-1}), \qquad K:C^\infty(\overline{Y},E^i)\to C^\infty(\overline{Y},E^i)$$

and

(16) $\qquad DSu+SDu=u+Ku \qquad (u\in C^\infty(\overline{Y},E^i)).$

Assumption 3. The map $u\to\alpha r_YW(i\delta\otimes\sigma(dt)K_3u)$ is of order $-\infty$. If W is a pesudo-differential operator, Assumption 3 is satisfied.

Theorem. *Assume that M is a non-characteristic submanifold* *of $Y^{\#}$ with respect to $(\mathcal{E})$. Assume further that N and W satisfying Assumptions* 1, 2, 3 *exist for $j\geqslant 1$. Then j-th cohomology of $(\mathcal{E})$ over $\overline{Y}$ is finite dimensional for $j\geqslant 1$.*

Proof. Denote by L the kernel of $D: C^{\infty}(\overline{Y}, E^{j})\rightarrow C_{\infty}(Y, E^{j+1})$. Then, for $u\in L$, $u+Ku=DSu\in L$. Then $I+K$ sends L into L. Since K is of order $-\infty$, $I+K: L\rightarrow L$ has finite dimensional cokernel. Since $I+K=DS$ on L, it follows immediately then that the quotient vector space of L by the image of $D: C^{\infty}(Y, E^{j-1})\rightarrow C^{\infty}(Y, E^{j})$ is finite dimensional. q. e. d.

By means of hermitian metrices on fibers of E^{j} depending smoothly on base points we introduce the formal adjoint D_{b}^{*} of D_{b}. Then by the theorem of Kohn-Nirenberg [3] we obtain the following:

Corollary. Assume that $(\mathcal{E})$ is elliptic and M is a non-characteristic submanifold of $Y^{\#}$ with respect to $(\mathcal{E})$. Assume further that for a constant $a>0$ we have a sub-elliptic estimate

$$||D_{b}u||^{2}+||D_{b}^{*}u||^{2}+||u||_{-1}\geqslant c||u||_{a}^{2} \quad (u\in C^{\infty}(M, E^{j}/K^{j}))$$

for $j\geqslant 1$. Then j-th cohomology of $(\mathcal{E})$ over Y is finite dimensional for $j\geqslant 1$.

COLUMBIA UNIVERSITY AND
UNIVERSITY OF WARWICK.

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