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A note on Malliavin's result

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In this note, we show that by a similar method to that used in the author's [4], we can give a direct proof and a slightly preciser statement for M. P. Malliavin's result on modules of differentials of localities of complete intersection in [2].

Let A be an affine ring over a field k. Let $\mathfrak{p}$ be a prime ideal of A. We put $R=A_{\mathfrak{p}}$ and we denote by $\Omega^1_{R/k}$ the module of differentials of R over k. Then, the equivalence of 1^0 and 3^0 in Proposition 6 in [2] amounts to saying the following.

Theorem. *Assume that k is perfect, and R is a complete intersection and reduced. Let q be a non-negative integer. Then the following two statements are equivalent.*

(1) *If $\mathfrak{Q}$ is a prime ideal of R such that Krull-*$\dim R_{\mathfrak{Q}}\leqslant q$, *$R_{\mathfrak{Q}}$ is a regular local ring.*

(2) *Every R-sequence with q-elements or less is an $\Omega^1_{R/k}$-sequence.*

In case $q=1$, this is J. Lipman and the author's result in [1] and [4], "*With the same assumption as above, $\Omega^1_{R/k}$ is torsion free if and only if R is integrally closed*" and in case $q=2$, it is J. Lipman's result in [1], "*Under the same assumption as above, $\Omega^1_{R/k}$ is reflexive if and only if R has no singularities of codimension 1 and 2*".

Malliavin has given a proof of the above theorem, using an argument on q-torsion but we will give here an elementary proof, in the course of our proof, we will find out the following preciser statement.

Corollary. *With the same assumption as in the theorem, let* $\mathfrak{Q}$ *be a prime ideal of* R *such that Krull-*$\dim R_{\mathfrak{Q}}=s$. *Assume that* $R_{\mathfrak{Q}}$ *is not a regular local ring. Then, every R-sequence* $t_1, \cdots, t_r$ *in* $\mathfrak{Q}$ *with* $r<s$ *can be imbedded in an R-sequence* $t_1, \cdots, t_r, t_{r+1}, \cdots, t_s$ *in* $\mathfrak{Q}$ *which is not an* $\Omega^1_{R/k}$*-sequence.*

As in [4], the above theorem and corollary are also valid, if k is not perfect but we take a differential constant field k^* of R in the sense of E. Kunz [3] instead of k and Ω^1_{R/k^*} instead of $\Omega^1_{R/k}$, respectively. However, for the simplicity we will treat only the case where k is perfect. We can easily extend our proof to that for the general case, referring to the argument in [4].

Lemma. *Let* S *be a noetherian local ring. Let* $\mathfrak{N}$ *be a prime ideal of* S *such that height* $\mathfrak{N}=$ *depth* $\mathfrak{N}=1$. *We denote by* $\mathfrak{N}^{-1}$ *the set of elements* x *in the total ring of quotients of* S *such that* $x\mathfrak{N}\subset S$. *Then it holds that* $\mathfrak{N}^{-1}\supsetneqq S$. *Actually, we can find elements* $a, b\in S$ *such that* a *is a non-zero divisor in* S, $b/a\in\mathfrak{N}^{-1}$, $b/a\notin S$ *and* $sb\notin(a)$ *for every element* $s\in S-\mathfrak{N}$.

Proof. By our assumption, there exists a non-zero-divisor a in $\mathfrak{N}$. Let $(a)=r_0\cap r_1\cap\cdots\cap r_k$ be an irredundant primary decomposition of (a) such that the associated prime ideal of r_0 is $\mathfrak{N}$. Then, there exists a positive integer t such that $(a)\supset\mathfrak{N}^t\cap r_1\cap\cdots\cap r_k$ and $(a)\not\supset\mathfrak{N}^{t-1}\cap r_1\cap\cdots\cap r_k$. Take an element b of S such that $b\in\mathfrak{N}^{t-1}\cap r_1\cap\cdots\cap r_k$ and $b\notin(a)$. Then, it holds that $b/a\notin S$ and $b/a\in\mathfrak{N}^{-1}$ because $b\mathfrak{N}\subset(a)$. Let s be an element in $S-\mathfrak{N}$. Assume that $sb\in(a)$. Then we have $b\in aS_{\mathfrak{N}}\cap S=r_0$. Hence $b\in r_0\cap r_1\cap\cdots\cap r_k=(a)$, which is a contradiction. Hence we get our lemma.

Proof of the theorem. By our assumption, there exists a polynomial ring $B=k[X_1, \cdots, X_n]$, a prime ideal $\mathfrak{P}$ of B and a reduced ideal $\mathfrak{S}$ of $B_{\mathfrak{P}}$ which is generated by r elements $f_1, f_2, \cdots, f_r$ in B with $r=$height $\mathfrak{S}$ such that $R\simeq B_{\mathfrak{P}}/\mathfrak{S}$. Let d be the canonical derivation of B into $\Omega^1_{B/k}$. $x_1, x_2, \cdots, x_n$ denote the classes of $X_1, X_2, \cdots, X_n$ modulo $\mathfrak{S}$. Then we have

$$\Omega^1_{R/k}\simeq R\otimes_B \Omega^1_{B/k}/\sum_{i=1}^{r} R(1\otimes df_i)$$

and

$$1\otimes df_i=\sum_{j=1}^{n}\frac{\partial f_i(x)}{\partial X_j}(1\otimes dX_j) \qquad (i=1,2,\cdots,r).$$

Let $\mathfrak{t}$ be an arbitrary minimal prime ideal of R. Then $R_{\mathfrak{t}}$ is a field and we have

$$\Omega^1_{R_{\mathfrak{t}}/k}\simeq R_{\mathfrak{t}}\otimes_R \Omega^1_{R/k}\simeq R_{\mathfrak{t}}\otimes_B \Omega^1_{B/k}/\sum_{i=1}^{r} R_{\mathfrak{t}}(1\otimes df_i).$$

$\Omega^1_{B/k}$ is a B-free module and $\Omega^1_{R_{\mathfrak{t}}/k}$ is an $R_{\mathfrak{t}}$-free module. Since $\mathrm{rank}_B \Omega^1_{B/k}-\mathrm{rank}_{R_{\mathfrak{t}}}\Omega^1_{R_{\mathfrak{t}}/k}=r$, $1\otimes df_1, \cdots 1\otimes df_r$ are linearly independent over $R_{\mathfrak{t}}$. Hence they are linearly independent modulo $\mathfrak{t}$. Since $\mathfrak{t}$ is arbitrary, $1\otimes df_1, \cdots, 1\otimes df_r$ are linearly independent over R.

(1)$\Rightarrow$(2). We use an induction on the number of elements in an R-sequence. Let $h_1, h_2, \cdots, h_s$ be an R-sequence with $s\leqq q$. By our induction assumption, $h_1, \cdots, h_{s-1}$ form an $\Omega^1_{R/k}$-sequence. We put $\bar{R}=R/(h_1, h_2, \cdots, h_{s-1})$ and $M=\Omega^1_{R/k}/(h_1, h_2, \cdots, h_{s-1})\Omega^1_{R/k}$. Then it holds that $M\simeq \bar{R}\otimes_B \Omega^1_{B/k}/\sum_{i=1}^{r}\bar{R}(1\otimes df_i)$. Let $\bar{h}_s$ be the class of h_s modulo $(h_1, \cdots, h_{s-1})$. We have only to prove that $\bar{h}_s$ is not a zero-divisor for M. Let $\bar{x}_1, \cdots, \bar{x}_n$ be the classes of $x_1, \cdots, x_n$ in $\bar{R}$. Assume that we have a relation

$$\bar{h}_s\left(\sum_{j=1}^{n}\bar{t}_j(1\otimes dX_j)\right)=\sum_{i=1}^{r}\bar{l}_i(1\otimes df_i)$$

in $\bar{R}\otimes_B \Omega^1_{B/k}$, where $\bar{t}_j, \bar{l}_i\in\bar{R}$. It means that

$$\bar{h}_s\left(\sum_{j=1}^{n}\bar{t}_j(1\otimes dX_j)\right)=\sum_{j=1}^{n}\left(\sum_{i=1}^{r}\bar{l}_i\frac{\partial f_i(\bar{x})}{\partial X_j}\right)(1\otimes dX_j).$$

That is, we have simultaneous linear equations

$$(1)\qquad \sum_{i=1}^{r} \frac{\partial f_i(\bar{x})}{\partial X_j} \bar{l}_i = \bar{h}_s \bar{t}_j \qquad (j=1, 2, \cdots, n),$$

with respect to $\bar{l}_1, \cdots, \bar{l}_r$. Let $\bar{\mathfrak{Q}}$ be an associated prime ideal of $(\bar{h}_s)$. Let $\mathfrak{Q}$ be the pre-image of $\bar{\mathfrak{Q}}$ in R, which is an associated prime ideal of $(h_1, \cdots, h_{s-1}, h_s)$. Then we have height $\mathfrak{Q}=s\leqslant q$. Therefore, by our assumption and the Jacobian criteria it holds that the rank of the matrix$\left(\frac{\partial f_i(\bar{x})}{\partial X_j}\right)_{\substack{i=1,2,\cdots,r\\ j=1,2,\cdots,n}}$ mod. $\bar{\mathfrak{Q}}$ equals r. Therefore, it follows from (1) that $\bar{l}_i \in \bar{h}_s \bar{R}_{\bar{\mathfrak{Q}}} \cap \bar{R}$. Since $\bar{\mathfrak{Q}}$ is arbitrary, we have $\bar{l}_i \in \bar{h}_s \bar{R}$ $(i=1, 2, \cdots, r)$. Hence, we can put $\bar{l}_i = \bar{h}_s \bar{l}_i'$ $(i=1, 2, \cdots, r)$, where $\bar{l}_i' \in \bar{R}$. Then we have

$$\bar{h}_s\Big(\sum_{j=1} \bar{t}_j (1\otimes dX_j) - \sum_{i=1}^{r} \bar{l}_i' (1\otimes df_i)\Big) = 0$$

and therefore

$$\sum_{j=1}^{n} \bar{t}_j (1\otimes dX_j) = \sum_{i=1}^{r} \bar{l}_i' (1\otimes df_i),$$

because $\bar{h}_s$ is not a zero-divisor for $\bar{R}\otimes_B \Omega^1_{B/k}$. Hence it follows that $\bar{h}_s$ is not a zero-divisor for M.

(2)⇒(1). First we show the following assertion by induction on s. "Let $h_1, \cdots, h_s$ be an R-sequence with $s\leqslant q$. Then $1\otimes df_1, \cdots, 1\otimes df_r$ are linearly independent modulo $(h_1, \cdots, h_s)$ in $R\otimes_B \Omega^1_{B/k}$." We already proved it in case $s=0$. Assume that $s\geqslant 1$. Let $\beta_1, \cdots, \beta_r$ be elements in R such that

$$\beta_1(1\otimes df_1) + \cdots + \beta_r(1\otimes df_r) \equiv 0 \text{ mod. } (h_1 \cdots, h_s) R\otimes_B \Omega^1_{B/k},$$

that is, there exist $m_1, \cdots, m_s \in R\otimes_B \Omega^1_{B/k}$ such that

$$\beta_1(1\otimes df_1) + \cdots + \beta_r(1\otimes df_r) = h_1 m_1 + \cdots + h_s m_s.$$

Since $h_1, \cdots, h_s$ are an $\Omega^1_{R/k}$-sequence, there exist $l_1, \cdots, l_{s-1} \in R\otimes_B \Omega^1_{B/k}$ and $\gamma_1, \cdots, \gamma_r \in R$ such that

$$m_s = h_1 l_1 + \cdots + h_{s-1} l_{s-1} + \gamma_1(1\otimes df_1) + \cdots + \gamma_r(1\otimes df_r).$$

Hence we have

$$(\beta_1 - h_s\gamma_1)(1\otimes df_1) + \cdots + (\beta_r - h_s\gamma_r)(1\otimes df_r) \equiv 0$$
$$\text{mod. } (h_1, \cdots, h_{s-1})R\otimes_B \Omega^1_{B/k}.$$

Therefore, by induction assumption we have

$$\beta_i - h_s\gamma_i \in (h_1, \cdots, h_{s-1}) \qquad (i=1, 2, \cdots, r).$$

Hence it holds that

$$\beta_i \in (h_1, \cdots, h_{s-1}, h_s) \qquad (i=1, 2, \cdots, r).$$

Hence $1\otimes df_1, \cdots, 1\otimes df_r$ are linearly independent modulo $(h_1, \cdots, h_s)$.

Assume that (1) does not hold. Then, there exists a prime ideal $\mathfrak{Q}$ of R such that height $\mathfrak{Q} = s \leqslant q$ and $R_{\mathfrak{Q}}$ is not a regular local ring. Then $1\otimes df_1, \cdots, 1\otimes df_r$ are linearly dependent modulo $\mathfrak{Q}$ by the Jacobian criteria. Hence there exist $\xi_1, \cdots, \xi_r \in R$ such that

$$(\xi_1, \xi_2, \cdots, \xi_r) \not\equiv (0, 0, \cdots, 0) \text{ mod. } \mathfrak{Q}$$

and $\xi_1(1\otimes df_1) + \cdots + \xi_r(1\otimes df_r) \in \mathfrak{Q}R\otimes_B \Omega^1_{B/k}$. We may assume that $\xi_1 \notin \mathfrak{Q}$. Since R is a Cohen-Macaulay ring, there exists an R-sequence $t_1, \cdots, t_{s-1}$ in $\mathfrak{Q}$. We put $R' = R/(t_1, \cdots, t_{s-1})$ and $\mathfrak{Q}' = \mathfrak{Q}/(t_1, \cdots, t_{s-1})$. Then we have depth $\mathfrak{Q}' =$ height $\mathfrak{Q}' = 1$. Hence we can apply the lemma and find elements $e, h \in R$ whose residue classes modulo $(t_1, \cdots, t_{s-1})$ are e', h' such that h' is not a zero-divisor in R', $e'/h' \notin R'$, $(e'/h')\mathfrak{Q}' \subset R'$ and $s'e' \notin (h')$ for every $s' \in R' - \mathfrak{Q}'$. Then, $t_1, \cdots, t_{s-1}, h$ form an R-sequence and $e\mathfrak{Q} \subset (t_1, \cdots, t_{s-1}, h)$. Hence we have

$$e\sum_{i=1}^{r} \xi_i(1\otimes df_i) \in e\mathfrak{Q}R\otimes_B \Omega^1_{B/k} \subset (t_1, \cdots, t_{s-1}, h)R\otimes_B \Omega^1_{B/k}.$$

Therefore, there exist $c_1, \cdots, c_n \in R$ such that

$$e\sum_{i=1}^{r} \xi_i(1\otimes df_i) \equiv h\sum_{j=1}^{n} c_j(1\otimes dX_j) \text{ mod. } (t_1, \cdots, t_{s-1})R\otimes_B \Omega^1_{B/k}.$$

We shall prove that $\sum_{j=1}^{n} c_j(1\otimes dX_j)$ is not a linear combination of $1\otimes df_i$ $(i=1, 2, \cdots, r)$ over R modulo $(t_1, \cdots, t_{s-1})$. Assume the contrary. Then there exist $l_1, \cdots, l_r \in R$ such that

$$e\sum_{i=1}^{r} \xi_i(1\otimes df_i) \equiv h\sum_{i=1}^{r} l_i(1\otimes df_i) \text{ mod. } (t_1, \cdots, t_{s-1})R\otimes_B \Omega^1_{B/k}.$$

Since $1\otimes df_1, \cdots, 1\otimes df_r$ are linearly independent modulo $(t_1, \cdots, t_{s-1})$, it holds that

$$e\xi_1 \equiv hl_1 \text{ mod. } (t_1, \cdots, t_{s-1}).$$

We denote by ξ_1' the class of ξ_1 mod. $(t_1, \cdots, t_{s-1})$. Then we have $e'\xi_1' \in (h')$ with $\xi_1' \in R' - \mathfrak{Q}'$, which is a contradiction. Hence if follows that h' is a zero-divisor for $R'\otimes_R \Omega^1_{R/k}$, hence $t_1, \cdots, t_{s-1}, h$ are not an $\Omega^1_{R/k}$-sequence, contradicting our assumption.

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