J. Math. Kyoto Univ. (JMKYAZ)
14-2 (1974) 319-340

On uniqueness in some characteristic Cauchy problem for first order systems

By

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(Received August 1, 1973)

1. Introduction.

In this paper we shall treat the following system;

$$A(t, x)\frac{\partial \vec{u}}{\partial t}=\sum_{j=1}^{n} B_j(t, x)\frac{\partial \vec{u}}{\partial x_j}+C(t, x)\vec{u}, \tag{1.1}$$

where $A(t, x)$, $B_j(t, x)$ $(j=1,\dots, n)$ and $C(t, x)$ are $N\times N$ matrices whose entries are all analytic in a neighborhood of $(t, x)=(0, 0)$, and $\vec{u}=\vec{u}(t, x)$ is the unknown of N-vector valued function. We consider the Cauchy problem for (1.1) with initial data on the hyperplane $t=0$, and are concerned only with the solution which is analytic in a neighborhood of $(t, x)=(0, 0)$, therefore we use the term "solution" only for analytic solution in what follows.

We assume that the initial plane $t=0$ is characteristic for (1.1), say, $A(t, x)$ is singular at $t=0$. Roughly speaking, the situation will be divided into two cases; one is where $\det A(t, x)$ vanishes only at $t=0$, and another where $\det A(t, x)$ vanishes identically in a neighborhood of the origin.

As for the former case, Y. Hasegawa [4], defining the notion of double characteristic, was mainly concerned with the existence of the analytic solution for single equations. And M. Miyake [5] showed that her method was applicable to some first order systems.

Our interest here is concentrated to the latter case, and we are concerned only with the uniqueness of solution. In our case we can not classify, in general, the characteristic of the initial surface in the manner of [5]. Therefore, we introduce a different classification of the system, since we have no approach to completely general ones.

Definition 1.1. The system (1.1) is said to be of type (p, q) in a neighborhood of the origin $\mathscr{V}$, if and only if the following conditions are fulfiled;

(1.2) the rank of $A(t, x)$ is constantly p in $\mathscr{V}$,

(1.3) the degree of $\det(\tau A(t, x)-C(t, x))$, as a polynomial in τ, is constantly q in $\mathscr{V}$.

Of cource we see $q \leqq p \leqq N$, and in our case $p<N$.

Section 2 is devoted to the preliminaries from ordinary differential equations which will suggest why we give Definition 1.1. In the third section, we shall give a necessary and sufficient condition for the solution of the system of type (p, p) with constant coefficients to be unique. In the fourth and final sections, we shall treat the system of type $(N-1, N-1)$ with variable coefficients.

2. Preliminaries from ordinary differential equations.

In this section we treat the following Cauchy's probelm for the sake of preparation for the succeeding sections;

$$A\frac{d\vec{u}}{dt}=B\vec{u},\ \vec{u}(0)=\vec{\phi}\,, \tag{2.1}$$

where A and B are $N\times N$ constant matrices and $\vec{u}=\vec{u}(t)$ and $\vec{\phi}$ are N-vectors.

As for the uniquness of the solution of (2.1), we have

Theorem 2.1. *A necessary and sufficient condition for the solution of* (2.1) *to be unique is that* $F(\tau)=\det(\tau A-B)$ *does not vanish identically as a polynomial in* τ.

Proof. (i) Sufficiency; let $F(\tau)\not\equiv 0$, then as is well known, each component of $\vec{u}$ must satisfy

$$(2.2) \qquad F(d/dt)v=0, \qquad v(0)=0.$$

Thus we can apply the Laplace transformation to the solution, and denoting its Laplace image by $\vec{U}(\tau)$, we have

$$(2.3) \qquad (\tau A-B)\vec{U}(\tau)=0,$$

hence by the analyticity of $\vec{U}(\tau)$ we obtain $\vec{U}(\tau)\equiv 0$ and consequently $\vec{u}(t)\equiv 0$.

(ii) Necessity; if $F(\tau)\equiv 0$, by considering suitable linear combinations of the row and column vectors of $(\tau A-B)$, that is, by considering $P(\tau A-B)Q$ for regular matirces P and Q (which are independent of τ), our equation is reduced to

$$(2.4) \qquad d\vec{u}_p/dt=K\vec{u}_p+L\vec{u}_r,\ \vec{u}_q=0, \quad M\vec{u}_p=0,$$

for some non-negative integers p, q and r with $p<N$ and $p+q+r=N$, where $\vec{u}_k$ denotes the k-vector.

If the column vectors $\{l_1,\dots, l_r\}$ of L are linearly dependent, then we may assume $l_r=0$. Thus we have a null solution $\vec{u}_p=0$, $\vec{u}_q=0$ and $\vec{u}_r={}^t(0,\dots, 0, \alpha(t))$ with $\alpha(0)=0$.

When $\{l_1,\dots, l_r\}$ are linearly independent, we can see easily Ker $M\neq\{0\}$, and then for an arbitrary non-zero vector $\vec{c}$ in Ker$M$, put $\vec{u}_p=\beta(t)\vec{c}$ with $\beta(0)=\beta'(0)=0$. Thus we have $L\vec{u}_r=(\beta'(t)-\beta(t)K)\vec{c}$, and by the assumption on L there exists a matrix L_0 such that L_0L becomes the $r\times r$ unit matrix. Hence we have a null solution $\vec{u}_p=\beta(t)\vec{c}$, $\vec{u}_q=0$ and $\vec{u}_r=L_0(\beta'(t)-\beta(t)K)\vec{c}$.

Since $F(\tau)\equiv 0$ it follows $r\neq 0$, hence the proof is completed. Q.E.D.

Now let us introduce the general solution of the equation $Ad\vec{u}/dt=B\vec{u}$.

Definition 2.1. Let the degree of $F(\tau)=\det(\tau A-B)$ be m. An N-vector valued function $\vec{u}(t)$ satisfying the equation $Ad\vec{u}/dt=B\vec{u}$ is said to be a general solution if and only if $\vec{u}(t)$ contains m arbitrary parameters, namely $\vec{u}(t)=\vec{u}(t;\, c_1,\dots, c_m)$.

Theorem 2.2. *Let $F(\tau)\not\equiv 0$, then the general solution $\vec{u}(t)$ is given by*

$$\vec{u}(t)=\frac{1}{2\pi i}\oint_{\Gamma}(\tau A-B)^{-1}e^{\tau t}\vec{c}\,d\tau\,, \tag{2.5}$$

where Γ is a Jordan curve enclosing the all roots of $F(\tau)=0$ and $\vec{c}$ is an arbitrary vector in $\boldsymbol{C}^N$.

Before proving above theorem, we prepare

Proposition 2.1. *Let us define a matrix P by*

$$P\vec{c}=\frac{1}{2\pi i}\oint_{\Gamma}(\tau A-B)^{-1}\vec{c}\,d\tau\,, \tag{2.6}$$

where Γ is a Jordan curve enclosing the all roots of $F(\tau)=0$ and $\vec{c}$ is in C^N, then the rank of P equals to the degree of $F(\tau)$.

For the proof of this proposition, we prepare some lemmas.

Lemma 2.1. *Let $\tau_1,\dots,\tau_k$ be arbitrary complex numbers, then*

$$\oint_{\Gamma}\prod_{j}(\tau-\tau_j)^{-\alpha_j}d\tau=0\,,$$

for any k-tuples of non-negative integers $\alpha=(\alpha_1,\dots,\alpha_k)$ with $|\alpha|\geqq 2$, where Γ is a Jordan curve enclosing $\tau_1,\dots,\tau_k$.

Proof. As is well-known, we can write

$$\prod_{j}(\tau-\tau_j)^{-\alpha_j}=\sum_{j=1}^{k}\sum_{\beta_j=1}^{\alpha_j}A^j_{\beta_j}(\tau-\tau_j)^{-\beta_j},$$

and since $|\alpha|\geqq 2$, we have

$$\sum_{j=1}^{k}A^j_1=0\,.$$

Thus by the well-known Cauchy's theorem, we can prove our assertion. Q.E.D.

As an immediate consequence of Lemma 2.1, we obtain

Lemma 2.2. *Under the same conditions of Lemma.* 2.1,

$$\oint_{\Gamma} \tau^{\sigma} \prod_{j} (\tau-\tau_j)^{-\alpha_j} d\tau = 0\,,$$

for any non-negative integer σ with $\sigma \leqq |\alpha|-2$.

The proof is carried out by Lemma 2.2 and the induction in σ.

Now let $\operatorname{rank} A=p$. When $p=0$ or $p=N$, the matrix P in (2.6) is clearly zero matrix or E_N, the unit matrix of degree N, therefore we assume $0<p<N$. Moreover, we may assume

$$A=\begin{pmatrix} E_p & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix} B_0 & K \\ L & B_1 \end{pmatrix},$$

where E_p denotes the $p \times p$ unit matrix.

Lemma 2.3. *If the $(N-p)\times(N-p)$ prinicpal minor B_1 is regular, then the assertion of Proposition* 2.1 *is valid.*

Proof. Since $\det B_1 \neq 0$, the degree of $F(\tau)$ is p and moreover we may assume $B_1=E_{N-p}$, $K=0$ and $L=0$. Thus we can easily see $P=E_p$ by Lemma 2.2. Q. E. D.

Lemma 2.4. *Let $p=N-1$ and $B_1=0$, then the assertion of Proposition* 2.1 *is valid.*

Proof. Let $L=(l_1,\dots, l_{N-1})$. Since $F(\tau)\not\equiv 0$, we may assume $l_{N-1}\neq 0$ without loss of generality. Then consider the matrix interchanged the $(N-1)$-th and the N-th vectors in $\tau A-B$;

$$\left(\begin{array}{ccc:c} \tau-* & & * & * \\ & \ddots & & \vdots \\ * & \tau-* & & * \\ & & * & \tau-* \\ \hdashline * \cdots\cdots * & & 0 & l_{N-1} \end{array}\right).$$

Thus, after a suitable linear combination of row vectors and column vectors, we may assume the N-th row vector is of the form; $(0,\dots, 0, l_{N-1})$. Proceeding this procedure in the $(N-1)$ principal minor, we may assume that $(\tau A-B)$ is as follows;

$$\left(\begin{array}{ccc|cccc}
\tau-* & & * & & & & \\
 & \ddots & & & & * & \\
* & & \tau-* & & & & \\
\hline
 & & & b_1 & \beta_1(\tau) & & * \\
 & & & & \ddots & \ddots & \\
 & 0 & & & & \ddots & \beta_k(\tau) \\
 & & & 0 & & & b_{k+1}
\end{array}\right).$$

for some integer k $(1 \leqq k \leqq N-1)$, where b_j $(j=1, 2,\dots, k+1)$ are non-zero constants and $\beta_j(\tau)=\tau+c_j$. Then, as is easily seen from Lemma 2.2, P is given by

$$\begin{pmatrix} E_{N-k-1} & * \\ 0 & 0 \end{pmatrix},$$

and since the degree of $(\tau A-B)$ is $(N-k-1)$, our lemma is proven.

Q.E.D.

Lemma 2.5. *Let* $p \leqq N-2$ *and* $B_1=0$, *then the assertion of proposition* 2.1 *is valid.*

Proof. We prove this by the induction in N. Of course, we may assume $N \geqq 2$, and when $N=2$, by Lemma 2.4 it is correct.

Now, let B_1 be $q \times q$ zero matrix, then we may assume that $\tau A-B$ is as follows;

$$\left(\begin{array}{ccc|c|ccc}
\tau-* & & * & & & & \\
 & \ddots & & * & & * & \\
* & & \tau-* & & & & \\
\hline
 & & & & \tau-* & & * \\
 & * & & B' & & \ddots & \\
 & & & & * & & \tau-* \\
\hline
 & 0 & & 0 & & E_q &
\end{array}\right),$$

and if B' is regular, then by Lemma 2.3 we accomplish the proof. When B' is singular, we may assume that B' is given by

$$\begin{pmatrix} 0 & 0 \\ 0 & E_m \end{pmatrix},$$

with some m, and thus, since it suffices to consider the first $(N-m-q)$-principal minor, by the assumption of induction we complete the proof.

Q.E.D.

Lemma 2.6. *Let* $p \leqq N-2$ *and* $\det B_1 = 0$, *then the assertion of Proposition* 2.1 *is valid.*

The proof of this lemma is reduced to that of Lemma 2.5.

Combining from Lemma 2.3 to Lemma 2.6, we can obtain Proposition 2.1.

For the proof of Theorem 2.2, we prepare some lemmas further.

Lemma 2.7. (*Resolvent equation*) *Let* τ *and* σ *be arbitrary complex numbers which do not make* $F(\tau)$ *vanish, then it follows*

$$(\tau A - B)^{-1} - (\sigma A - B)^{-1} = (\sigma - \tau)(\tau A - B)^{-1} A (\sigma A - B)^{-1}. \tag{2.7}$$

Lemma 2.8. *Let* τ_j *be the root of* $F(\tau) = 0$ *and* Γ_j *be a Jordan curve encolsing only* τ_j, *and then define* P_j *by*

$$P_j \vec{c} = \frac{1}{2\pi i} \oint_{\Gamma_j} (\tau A - B)^{-1} \vec{c} \, d\tau, \tag{2.8}$$

then it follows

$$P = \sum_j P_j, \tag{2.9}$$

$$P_j A P_k = \delta_{jk} P_j, \tag{2.10}$$

where δ_{jk} *stands for the Kronecker's delta.*

Thus, if we set $Q = AP$ and $Q_j = AP_j$, we have

$$\operatorname{Im} Q = \sum_j \oplus \operatorname{Im} Q_j \qquad \text{(direct sum)}. \tag{2.11}$$

Proposition 2.2. *We have* $\operatorname{rank} Q = \operatorname{rank} P$.

Proof. At first we show that A is one to one on $\operatorname{Im} P$. Let $A\vec{\phi}$

$=0$ for some $\vec{\phi}$ in $\operatorname{Im} P$, say $\vec{\phi}=P\vec{\psi}$, then $PAP=P$ follows from Lemma 2.8, so we have $0=PA\vec{\phi}=PAP\vec{\psi}=P\vec{\psi}=\vec{\phi}$.

Now let $\operatorname{rank} P=p$ and $\{\vec{\omega}_1,\dots,\vec{\omega}_p\}$ be a base of $\operatorname{Im} P$ and $\sum_j c_j A\vec{\omega}_j=0$, then we have $\sum_j c_j\vec{\omega}_j=0$ since A is one to one, and have $c_j=0$ for each j, and this proves $\operatorname{rank} Q=\operatorname{rank} P$. Q.E.D.

Now, let us consider

$$S(t)\vec{c}=\frac{1}{2\pi i}\oint_{\Gamma} e^{\tau t}(\tau A-B)^{-1}\vec{c}\,d\tau \tag{2.12}$$

and

$$S_j(t)\vec{c}=\frac{1}{2\pi i}\oint_{\Gamma_j} e^{\tau t}(\tau A-B)^{-1}\vec{c}\,d\tau\,, \tag{2.13}$$

where Γ and Γ_j are Jordan curves enclosing the all roots of $F(\tau)=0$ and only the root τ_j respectively. Let the multiplicity of τ_j be α_j, and the cofactor of $\tau A-B$ be $\Delta(\tau)$, and expanding $e^{\tau t}$ into Taylor series about $\tau=\tau_j$, we have by Lemma 2.2,

$$S_j(t)\vec{c}=\frac{e^{\tau_j t}}{2\pi i}\oint_{\Gamma_j}\sum_{\kappa=0}^{\alpha_j-1}(\tau-\tau_j)^{\kappa}t^{\kappa}/\kappa!(\tau-\tau_j)^{-\alpha_j}\theta(\tau)\vec{c}\,d\tau, \tag{2.14}$$

where $\theta(\tau)$ is given by

$$\theta(\tau)=\prod_{k\neq j}(\tau-\tau_k)^{-\alpha_k}\Delta(\tau)\,. \tag{2.15}$$

Proof of Theorem 2.2 (I) (*when each* τ_j *is simple root*).

It suffices to show that the rank of $S(t)$ equals to the degree of $F(\tau)$ for any t, equivalently to the rank of P by Proposition 2.1. Obviously, $S(t)=\sum_j S_j(t)=\sum_j e^{\tau_j t}P_j$, and since $PAP_j=P_j$ by Lemma 2.8, we see $\operatorname{Im} S(t)\subset\operatorname{Im} P$, and consequently $\operatorname{rank} S(t)\leqq\operatorname{rank} P$. On the other hand, $AS(t)=\sum_j e^{\tau_j t}Q_j$, thus $\operatorname{rank} AS(t)=\sum_j\operatorname{rank} Q_j=\operatorname{rank} Q=\operatorname{rank} P$ by (2.11) and Proposition 2.2, hence $\operatorname{rank} S(t)\geqq\operatorname{rank} AS(t)=\operatorname{rank} P$. Q.E.D.

In order to prove our theorem when $F(\tau)=0$ has multiple roots, we introduce some notations. We define following matrices;

(2.16) $$P_{j,k}=\frac{1}{2\pi i}\oint_{\Gamma_j}(\tau-\tau_j)^k(\tau A-B)^{-1}d\tau\,,$$

and

(2.17) $$Q_{j,k}=AP_{j,k},$$

where k denotes a non-negative integer.

The following lemma is easy.

Lemma 2.9. *The following equations hold.*

(2.18) $$P_{j,k}AP_{m,n}=\delta_{jm}P_{j,k+n},$$

(2.19) $$Q_{j,k}Q_{m,n}=\delta_{jm}Q_{j,k+n},$$

(2.20) $$Q_{j,k}=Q^k_{j,1}.$$

Now denoting $R_j(t)=\sum_{p=0}^{\alpha_j-1}t^pQ^p_{j,1}$, we have

(2.21) $$AS_j(t)=e^{\tau_j t}R_j(t)Q_j.$$

Lemma 2.10. *For any t, $R_j(t)$ is one to one on* $\operatorname{Im}Q_j$, *and consequently* $\operatorname{rank}AS_j(t)=\operatorname{rank}Q_j$.

Proof. Let $R_j(t)Q_j\vec{c}=0$ for some $\vec{c}$ in $\boldsymbol{C}^N$. Multiplying $Q^{\alpha_j-1}_{j,1}$ to both sides and noting $Q^k_{j,1}=0$ $(k\geqq\alpha_j)$, we obtain $Q^{\alpha_j-1}_{j,1}Q_j\vec{c}=0$, and proceeding this procedure we have $Q^k_{j,1}Q_j\vec{c}=0$ for $k\geqq 1$. This shows $Q_j\vec{c}=0$. The latter part is obvious. Q.E.D.

Proof of Theorem 2.2 (II) (*when $F(\tau)=0$ has multiple roots*). Using Lemma 2.9 and Lemma 2.10 and observing $AS(t)=\sum_j AS_j(t)$, we can complete the proof as well as the case when $F(\tau)=0$ has only simple roots. Q.E.D.

Suming up the results obtained above, we have

Theorem 2.3. *The Cauchy problem* (2.1) *has a unique solution $\vec{u}(t)$, if and only if* $\det(\tau A-B)\not\equiv 0$ *as a polynomial in τ and $\vec{\phi}$ belongs* *to the range of P. Moreover, then $\vec{u}(t)$ is given by $S(t)A\vec{\phi}$ and it is contained in the range of P for any t.*

3. The case of constant coefficients.

In this section we consider the equation;

$$A\frac{\partial \vec{u}}{\partial t}=\sum_{j=1}^{n} B_j\frac{\partial \vec{u}}{\partial x_j}+C\vec{u}, \tag{3.1}$$

where A, B_j's and C are $N\times N$ constant matrices and $\vec{u}=\vec{u}(t, x)$ is an N-vector valued function which may be allowed to take complex values. We consider the Cauchy problem for (3.1) in a neighborhood of $(t, x)=(0, 0)$ with the initial data on the hyperplane $t=0$. Our main interest here is of the uniqueness of the solution which is analytic in a neighborhood of $(t, x)=(0, 0)$. We shall be concerned only with the analytic solution, the term "solution" means always the solution which is analytic in a neighborhood of $(t, x)=(0, 0)$ hereafter.

According to Theorem 2.1, we obtain immediately

Theorem 3.1. *It is necessary that $F(\tau)=\det(\tau A-B)$ does not vanish identically as a polynomial in τ for the solution of the Cauchy problem for* (3.1) *to be unique.*

However, we can see easily that $F(\tau)\not\equiv 0$ does not give the sufficient condition for the uniqness. For example, let $n=1$, and A, $B=B_1$ and C be as follows;

$$A=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

then we can see easily that the Cauchy problem for (3.1) has a null solution.

For general equations, it seems difficult to state the condition under which the solution is unique, so we restrict oursleves to rather special equation, that is, of type (p, p).

The following lemma is easy.

Lemma 3.1. *Let the equation* (3.1) *is of type* (p, p), *then* (3.1) *is reduced to*

$$\begin{pmatrix} E_p & 0 \\ 0 & 0 \end{pmatrix} \frac{\partial \vec{u}}{\partial t} = \begin{pmatrix} P(D) & K(D) \\ L(D) & Q(D) \end{pmatrix} \vec{u} + \begin{pmatrix} S & 0 \\ 0 & E_{N-p} \end{pmatrix} \vec{u}, \tag{3.2}$$

where D *stands for* $(\partial/\partial x_1, .., \partial/\partial x_n)$.

We state the principal result in this section.

Theorem 3.2. *A necessary and sufficient condition in order that the solution of the Cauchy problem for* (3.2) *is unique is that the matrix* $Q(\xi)$ *is nilpotent for any unit vector* $\xi=(\xi_1,..., \xi_n)$.

The proof of sufficiency is very easy, so we may omit it. For the proof of necessity, we make some preparative considerations. Our aim is to show that there exists a null solution of (3.2) if $Q(\xi)$ is not nilpotent for some unit vector ξ. In this case we may assume, after a suitable exchange of independent variable if necessary, that the coefficient matrix of $\partial/\partial x_1$ in $Q(D)$ is not nilpotent, and we write it Q also. We seek a null solution which depends only on t and x_1, and we remove the suffix 1 of x_1. For $P(D)$, $K(D)$ and $L(D)$, we denote by P, K and L the coefficients of $\partial/\partial x_1$ respectively. Hence we seek a null solution of

$$\begin{pmatrix} E_p & 0 \\ 0 & 0 \end{pmatrix} \frac{\partial \vec{u}}{\partial t} = \begin{pmatrix} P & K \\ L & Q \end{pmatrix} \frac{\partial \vec{u}}{\partial x} + \begin{pmatrix} S & 0 \\ 0 & E_{N-p} \end{pmatrix} \vec{u}. \tag{3.3}$$

Let $\sum \vec{u}_n(x)t^n$ be the formal solution of (3.3) with $\vec{u}_0(x)=0$, then we have

$$(n+1)v_n(x) = Pv'_{n-1}(x) + Kw'_{n-1}(x) + Sv_{n-1}(x) \tag{3.4}$$

and

$$0 = Lv'_n(x) + Qw'_n(x) - w'_n(x), \tag{3.5}$$

for $n=1, 2,...$, where $\vec{u}_n(x) = {}^t(v_n(x), w_n(x))$ and $'$ means the differentiation by x.

First, let us assume that Q is regular, and denoting its inverse matrix by Q also and rewriting $Q^{-1}L$ and $-L$ also, we obtain from (3.5)

$$w_n'(x) = Qw_n(x) + Lv_n'(x). \tag{3.6}$$

Here we introduce the notations $|c|$ and $\|A\|$ to denote the length of a vector c and the operator norm of a matrix A respectively.

Since $v_0(x) = v_1(x) = 0$ and $w_1(x) = 0$, we have by (3.6)

$$w_1(x) = e^{xQ}c \tag{3.7}$$

for some constant $(N-p)$-vector c, and we assume $|c| = 1$. Let us define $w_n(x)$ for $n \geqq 2$ by

$$w_n(x) = \int_0^x e^{(x-y)Q} L v_n'(y) dy, \tag{3.8}$$

then we have two sequences $\{v_n(x)\}$ and $\{w_n(x)\}$ by using (3.4), satisfying (3.4) and (3.6).

Now take $\varepsilon > 0$ so that $\varepsilon\|Q\| \leqq 1$, then there exist $a(n, k)$ and $b(n, k)$ which may depend on ε with satisfying

$$|v_n^{(k)}(x)| \leqq a(n, k)\|Q\|^{n+k-1} \tag{3.9}$$

and

$$|w_n^{(k)}(x)| \leqq b(n, k)\|Q\|^{n+k-1}, \tag{3.10}$$

where n and k run over all natural numbers and $|x| \leqq \varepsilon$, and we can take $a(1, k) = 0$, $b(1, k) = \sup_{|x| \leqq \varepsilon} \|e^{xQ}\|$ and $a(2, k) = \|K\|/2$ for all k. It should be noted that (3.9) and (3.10) are lead by induction, if we notice (3.4) and (3.6). Since we may assume $\|Q\|$ is larger than 1, we may assume $S = 0$ by (3.9). Thus, we obtain

$$(n+1)a(n+1, k) \leqq \|P\|a(n, k+1) + \|K\|b(n, k+1) \tag{3.11}$$

and

$$b(n, k) \leqq b(n, k-1) + \|L\|a(n, k+1), \tag{3.12}$$

with $b(n, 0) \leqq M\|L\| a(n, 1)$, where $M = M(\varepsilon) = \sup_{\substack{|y| \leqq |x| \\ |x| \leqq \varepsilon}} \|e^{(x-y)Q}\|$. Hence we have

$$(3.13) \qquad (n+1)a(n+1, k) \leqq \|P\| a(n, k) + mMa(n, 1) + m \sum_{s=1}^{k+1} a(n, s)$$

with $m = \|K\|\|L\|$.

Now let $\sigma = 2\max.\{\|P\|, mM, m\}$, and set $A(n, k) = n!\,a(n, k)$, then we have

Lemma 3.2. *It follows*

$$(3.14) \qquad A(n, k) \leqq \sigma^{n-2} \|K\| n^k n!,$$

for $n \geqq 2$ *and* $k \geqq 0$.

Proof. We shall prove (3.14) by induction in n. For $n=2$, it is easily seen that (3.14) is valid for all $k \geqq 0$, and assume (3.14) for some n and all $k \geqq 0$. From (3.13) it follows $A(n+1, k) \leqq \|K\| \sigma^{n-2} n! (\|P\| n^{k+1} + mnM + m \sum_{s=1}^{k+1} n^s)$, and hence $A(n+1, k) \leqq \sigma^{n-1} \|K\| (n+1)! \sum_{s=1}^{k+1} n^s (n+1)^{-1}$. Thus we obtain (3.14) for $n+1$, since $\sum_{s=1}^{k+1} n^s \leqq (n+1)^{k+1}$, and this completes the proof. Q.E.D.

By virtue of Lemma 3.2, we have

$$(3.15) \qquad |v_n(x)| \leqq \sigma^{n-2} \|K\| \|Q\|^{n-1}$$

and

$$(3.16) \qquad |w_n(x)| \leqq n\sigma^{n-2} M \|K\| \|L\| Q\|^{n-1},$$

for $n \geqq 2$, if we notice (3.9) and (3.10), and this leads the convergency of our formal solution when $\sigma\|Q\||t| < 1$. Thus we can obtain a null solution which is analytic in a neighborhood of $(t, x) = (0, 0)$.

Now consider the case when Q is singular. In this case we may assume Q is given by

$$\begin{pmatrix} Q_0^{-1} & * \\ 0 & R \end{pmatrix},$$

where Q_0 is a regular matrix of degree $(N-p-r)$ and R a nilpotent of degree r, and r is a positive integer less than $N-p$. Then, for formal solution $\Sigma \vec{u}_n(x)t^n$, with $\vec{u}_n(x)={}^t(v_n(x), w_n(x))={}^t(v_n(x), \lambda_n(x), \mu_n(x))$, we obtain for $n \geqq 0$,

$$(n+1)v_{n+1}(x)=Pv_n'(x)+Kw_n'(x)+Sv_n(x) \tag{3.17}$$

$$\lambda_n'(x)=Q_0\lambda_n(x)+\sum_{s=1}^{r+1} J_s v_n^{(s)}(x) \tag{3.18}$$

and

$$\mu_n(x)=\sum_{s=1}^{r} L_s v_n^{(s)}(x), \tag{3.19}$$

with some suitable matrices J_s $(1 \leqq s \leqq r+1)$ and L_s $(1 \leqq s \leqq r)$.

Set

$$\lambda_1(x)=e^{xQ_0}c \tag{3.20}$$

for some $(N-p-r)$-vector c with $|c|=1$, and for $n \geqq 2$

$$\lambda_n(x)=e^{xQ_0}\int_0^x e^{-yQ_0}\Big(\sum_{s=1}^{r+1} J_s v_n^{(s)}(y)\Big)dy\,, \tag{3.21}$$

then we obtain three sequences $\{v_n(x)\}$, $\{\lambda_n(x)\}$ and $\{\mu_n(x)\}$ satisfying (3.17), (3.18) and (3.19).

Similarly to the previous case, for a small $\varepsilon>0$ such as $\varepsilon\|Q_0\| \leqq 1$, we have $a(n, k)$, $b(n, k)$ and $c(n, k)$ such that

$$|v_n^{(k)}(x)| \leqq a(n, k)\|Q_0\|,^{(n-2)(r+1)+k+1} \tag{3.22}$$

$$|\lambda_n^{(k)}(x)| \leqq b(n, k)\|Q_0\|^{(n-1)(r+1)+k} \tag{3.23}$$

and

$$|\mu_n^{(k)}(x)| \leqq c(n, k)\|Q_0\|,^{(n-1)(r+1)+k} \tag{3.24}$$

where $n \geqq 2$ and $k \geqq 0$, and $|x| \leqq \varepsilon$. Moreover, we may set $a(1, k)=c(1, k)=0$ and $b(1, k)=\sup_{|x| \leqq \varepsilon}\|e^{xQ_0}\|$. And repeating the arguments as in the proof of Lemma 3.2, we can obtain

$$a(n, k) \leqq \alpha \sup_{|x| \leqq \varepsilon} \|e^{xQ_0}\| \omega^{n-2} n^{k+r} \|Q_0\|^{(n-2)(r+1)+k+1}, \tag{3.25}$$

where α is a positive constant and ω is a positive number which depends on ε and r, and (3.25) yields the convergency of formal solution if $\omega\|Q_0\|^{r+1}|t|<1$. Summing up the above arguments, we can complete the proof of Theorem 3.2.

4. The case of variable coefficients I (necessary condition)

Let us consider the case of variable coefficients. Here, we treat only the equation of type $(N-1, N-1)$, so after some transform of dependent variables, if necessary, we may start with the following equation;

$$\begin{pmatrix} E_{N-1} & 0 \\ 0 & 0 \end{pmatrix} \frac{\partial \vec{u}}{\partial t} = \sum_{j=1}^{n} \begin{pmatrix} B_j(t, x) & K_j(t, x) \\ L_j(t, x) & \psi^j(t, x) \end{pmatrix} \frac{\partial \vec{u}}{\partial x_j} + \begin{pmatrix} C(t, x) & P(t, x) \\ 0 & -1 \end{pmatrix} \vec{u}, \tag{4.1}$$

where $B_j(t, x)$ $(j=1,\dots, n)$ and $C(t, x)$ are $(N-1)\times(N-1)$ matrices whose entries are all analytic functions in a neighborhood of $(t, x)=(0, 0)$, and so are all other coefficients also.

The principal aim of this present section is to establish the following theorem.

Theorem 4.1. *It is necessary*

$$\psi^j(0, 0)=0 \qquad (j=1,\dots, n) \tag{4.2}$$

in order that the solution of the Cauchy problem for (4.1) *is unique.*

To prove Theorem 4.1, we must show if $\psi^j(0, 0)\neq 0$ for some j, then there exists a null solution of (4.1).

Set $\vec{u}={}^t(v, w)$, where v denotes the $(N-1)$-vector given by the first $(N-1)$ compoents of $\vec{u}$, then (4.1) is written as follows;

$$\begin{aligned} \partial v/\partial t = \sum_{j=1}^{n} B_j(t, x)\partial v/\partial x_j + \sum_{j=1}^{n} K_j(t, x)\partial w/\partial x_j \\ + C(t, x)v + P(t, x)w, \end{aligned} \tag{4.3}$$

and

(4.4) $$0=\sum_{j=1}^{n} L_j(t, x)\partial v/\partial x_j+\sum_{j=1}^{n}\psi^j(t, x)\partial w/\partial x_j-w\,.$$

Now let $\psi^1(0, 0)\neq 0$ without loss of generality, then after deviding the both sides by $\psi^1(t, x)$ and denoting x_1 and $(x_2,\dots, x_n)$ by y and x' respectively, it suffices to consider

(4.5) $$\partial w/\partial y=\sum_{j=2}^{n}\phi^j(t, y, x')\partial w/\partial x_j+\alpha(t, y, x')w$$
$$+\sum_{j=1}^{n}\sum_{k=1}^{N-1} a_j^k(t, y, x')\partial v_k/\partial x_j,$$

in place of (4.4), where $\alpha(t, y, x')$ and $a_j^k(t, y, x')$ $(1\leqq j\leqq n,\ 1\leqq k\leqq N-1)$ are all analytic in a neighborhood of the origin.

We have to show there exists a non-trivial solution ${}^t(v, w)$ satisfying (4.3) and (4.5) such that $v(0, y, x')=0$ and $w(0, y, x')=0$. In doing so, our main tool will be the method of the majorant series.

Let us consider the formal solution of (4.3) and (4.5) and denote it by

(4.6) $$v(t, y, x')\sim\Sigma v_{pqr}t^p y^q x'^r$$

and

(4.7) $$w(t, y, x')\sim\Sigma w_{pqr}t^p y^q x'^r,$$

where r is given by multi-index $(r_2,\dots, r_n)$. By the initial condition we have to set $v_{0qr}=0$ and $w_{0qr}=0$, and after expanding all the coefficients into power series and using (4.3) and (4.5), we can determine v_{pqr} and w_{pqr} successively if we give w_{p0r} for $p\neq 0$ and for all r. Then take $w_{100}=1$ and $w_{p0r}=0$ when $(p, r)\neq(1, 0)$.

Consider now the equations which all the coefficients are replaced by their majorant series, we call it the majorant equation, in (4.3) and (4.5), and denoting the formal solution of the majorant equation by

(4.8) $$V(t, y, x')\sim\Sigma V_{pqr}t^p y^q x'^r$$

and

(4.9) $$W(t, y, x') \sim \Sigma W_{pqr} t^p y^q x'^r,$$

we can see easily

(4.10) $$|v^j_{pqr}| \leqq V^j_{pqr} \qquad (j=1,\dots, N-1)$$

and

(4.11) $$|w_{pqr}| \leqq W_{pqr}$$

where v^j_{pqr} and V^j_{pqr} denote the j-th component of v_{pqr} and V_{pqr} respectively, if we choose V^j_{0qr} $(j=1,\dots, N-1)$, W_{0qr} and W_{p0s} $((p, r) \neq (1, 0))$ non-negative and $W_{100} \geqq 1$.

Now put $z=\sigma y+x'+\rho t=\sigma y+x_2+\cdots+x_n+\rho t$, where σ and ρ are positive constants which should be determined later, then for all the coefficients of (4.3) and (4.5), take a common majorant series of the form; $M\{1-z/\gamma\}^{-1}$, where M and γ are positive constants which can be determined only from the coefficients. Thus we consider the majorant equation of (4.3) and (4.5) of the following form;

(4.12) $$\partial V^k/\partial t = M(1-z/\gamma)^{-1}\{\sum_{m=1}^{N-1}\sum_{j=1}^{n} \partial V^m/\partial x_j + \sum_{j=1}^{n} \partial W/\partial x_j + \sum_{m=1}^{N-1} V^m + W\}, \qquad (k=1,\dots, N-1)$$

and

(4.13) $$\partial W/\partial y = M(1-z/)\gamma^{-1}\{\sum_{m=1}^{N-1}\sum_{j=1}^{n} \partial V^m/\partial x_j + \sum_{j=2}^{n} \partial W/\partial x_j + W\}.$$

We seek the solution of (4.12) and (4.13) which depend only on z, and moreover we assume all the components of V are same, which we denote by V also. Thus denoting the differentiation with respect to z by $'$, we obtain from (4.12) and (4.13)

(4.14) $$\rho V'(z) = \gamma M(\gamma-z)^{-1}\{(N-1)(\sigma+n-1)V'(z)+(\sigma+n-1)W'(z) + \gamma(N-1)V(z)+\gamma W(z)\},$$

and

$$\sigma W'(z)=\gamma M(\gamma-z)^{-1}\{(N-1)(\sigma+n-1)V'(z)+(n-1)W'(z)+\gamma W(z)\}. \tag{4.15}$$

It should be noted that the system (4.14) and (4.15) has a unique analytic solution in a neighbornood of $z=0$ for any initial data, and hence we expand it into power series as follows; $V(z)=\sum_{\nu=0}^{\infty} V_\nu z^\nu$ and $W(z)=\sum_{\nu=0}^{\infty} W_\nu z^\nu$. And then, introducing new parameters $\gamma M\rho$ and $\gamma M\sigma$ instead of ρ and σ, and denoting them by ρ and σ also, we obtain from (4.14) and (4.15) the following equation

$$\gamma\rho\nu V_\nu-(N-1)(\sigma+n-1)\nu V_\nu-(\sigma+n-1)\nu W_\nu=\rho(\nu-1)V_{\nu-1}+\gamma(N-1)V_{\nu-1}+\gamma W_{\nu-1}, \tag{4.16}$$

and

$$\gamma\sigma\nu W_\nu-(n-1)\nu W_\nu-(N-1)(\sigma+n-1)\nu V_\nu=\gamma W_{\nu-1}. \tag{4.17}$$

Thus if we take ρ and σ sufficiently large so that

$$\{\gamma\rho-(N-1)(\gamma\sigma-n+1)\}(\gamma\sigma-n+1)-(N-1)(\sigma+n-1)^2>0, \tag{4.18}$$

we can see easily $V_\nu\geqq 0$ and $W_\nu\geqq 0$ for $\nu\geqq 1$, if we choose V_0 and W_0 non-negative. Moreover, since W_1 is given by

$$W_1=\gamma[\{\gamma\rho-(N-1)(\gamma\sigma-n+1)\}(\gamma\sigma-n+1)-(N-1)(\sigma+n-1)]^{-1}\times\{(N-1)^2(\sigma+n-1)V_0+(N\sigma+(N-2)(n-1))W_0\}, \tag{4.19}$$

we can take V_0 and W_0 which may be dependent of ρ and σ, of course, with kept the non-negativity of V_0 and W_0, so that $W_1\geqq 1$. This yields the convergency of the formal solutions given by (4.6) and (4.7) and they give a null solution of (4.1). This completes the proof of Theorem 4.1.

As an immediate consequence, we have

Corollary 4.1. *Let us consider the following equation of type* $(N-p, N-p)$;

$$(4.20)\quad \begin{pmatrix} E_{N-p} & 0 \\ 0 & 0 \end{pmatrix} \frac{\partial \vec{u}}{\partial t} = \sum_{j=1}^{n} \begin{pmatrix} B_j(t, x) & K_j(t, x) \\ L_j(t, x) & M_j(t, x) \end{pmatrix} \frac{\partial \vec{u}}{\partial x_j} + \begin{pmatrix} C(t, x) & P(t, x) \\ 0 & -E_p \end{pmatrix} \vec{u}.$$

A necessary condition in order that the solution of the Cauchy problem for (4.20) *is unique is that the matrix* $\sum_{j=1}^{n} \xi_j M_j(0, 0)$ *is singular for any unit vector* $\xi = (\xi_1, \dots, \xi_n)$.

5. The case of variable coefficients II (sufficient condition)

In this section we give a sufficient condition under which the solution of the Cauchy problem for (4.1) is unique.

Our result can be stated as

Theorem 5.1. *Let* $\lambda_1, \dots, \lambda_n$ *be eigenvalues of the matirx* $(\partial \psi^j(0, 0)/\partial x_k)$ $(j, k = 1, \dots, n)$. If, for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ with components of non-genative integer, it holds

$$(5.1)\quad \sum_{j=1}^{n} \alpha_j \lambda_j \neq 1,$$

then the solution of the Cauchy problem for (4.1) *is unique.*

To prove Theorem 5.1, we prepare two lemmas.

Lemma 5.1. *Consider a first order differential operator* Γ;

$$(5.2)\quad \Gamma = \sum_{j=1}^{n} \gamma^j(x) \partial / \partial x_j,$$

where $\gamma^j(x) = \gamma^j(x_1, \dots, x_n)$ $(j = 1, \dots, n)$ *are all analytic functions in a neighborhood of the origin (which may be valued in complex number) with satisfying* $\gamma^j(0) = 0$ *for all* j *and* $\partial \gamma^j(0)/\partial x_k = 0$ *for* $k > j$. *If, for any multi-index* $\alpha = (\alpha_1, \dots, \alpha_n)$ *with components of non-negative integer, it holds*

$$(5.3)\quad \sum_{j=1}^{n} \alpha_j \partial \gamma^j(0)/\partial x_j \neq 1,$$

then the equation

$$(5.4)\quad f(x) = \Gamma f(x)$$

has no solution which is analytic in a neighborhood of origin except $f(x)\equiv 0$.

Proof. Let $f(x)$ be a solution of (5.4). We expand it into power series and show $D^\alpha f(0)=0$ for any multi-index α. We shall carry out the proof by the induction in $|\alpha|$. Clearly $f(0)=0$, and now consider the case of $|\alpha|=1$. Differentiating both sides of (5.4) in each x_j and observing the equation which the n-vector, ordered the first order derivatives of $f(x)$ at $x=0$ lexicographically, satisfies, we can see easily $D^\alpha f(0)=0$ ($|\alpha|=1$ from (5.3).

Now let $D^\alpha f(0)=0$ when $|\alpha|\leqq m$, then applying $D^\alpha=D_1^{\alpha_1}\cdots\cdot D_n^{\alpha_n}$ ($|\alpha|=m+1$) to both sides of (5.4) and using the formula of Leibniz, we have

$$D^\alpha f(0)=\sum_{j=1}^{n}\sum_{k\leqq j}\alpha_k\partial\gamma^j(0)/\partial x_k D_1^{\alpha_1}\cdots D_k^{\alpha_k-1}\cdots D_n^{\alpha_n}D_jf(0)\,. \tag{5.5}$$

Hence if we order all the $(m+1)$-th derivatives of $f(x)$ at origin lexicographically, we can see that any $D^\alpha f(0)$ is given as a linear combination of those terms which should be ordered after it. Thus (5.3) implies $D^\alpha f(0)=0$ when $|\alpha|=m+1$, and this completes the proof.

Q.E.D.

Lemma 5.2. *Let* $\Gamma=\sum_{j=1}^{n}\gamma^j(x)\partial/\partial x_j$, *where* $\gamma^j(x)$ $(j=1,\dots,n)$ *are analytic in a neighborhood of the origin. There exists a constant unitary matrix* U *such that if we introduce the new coordinate variable* y *by* $y=U^{-1}x$, *then it follows*

$$\partial\tilde{\gamma}^j(0)/\partial y_k=0 \qquad (k>j), \tag{5.6}$$

where $\tilde{\Gamma}=\sum_{j=1}^{n}\tilde{\gamma}^j(y)\partial/\partial y_k$ *is the expression of* Γ *with respect to the new variable* y.

Proof. As is well known, we can find a unitary matrix U which makes $U^{-1}(\partial\gamma^j(0)/\partial x_k)U$ triangular type, and this U is our demanded one.

Q.E.D.

Now we are in a position to prove Theorem 5.1. Set $u={}^t(v, w)$ and $\Gamma=\sum_{j=1}^{n}\psi^j(t, x)\partial/\partial x_j$, and expand (4.1) into power series in t, then we obtain for $n=0, 1, 2,\dots,$

$$(5.7)\qquad (n+1)v_{n+1}=\sum_{k=0}^{n}B_{n-k}(x, \partial/\partial x)v_k+\sum_{k=0}^{n}K_{n-k}(x, \partial/\partial x)w_k$$

$$+\sum_{k=0}^{n}C_{n-k}(x)v_k+\sum_{k=0}^{n}P_{n-k}(x)w_k,$$

and

$$(5.8)\qquad 0=\sum_{k=0}^{n}L_{n-k}(x, \partial/\partial x)v_k+\sum_{k=0}^{n}\Gamma_{n-k}(x, \partial/\partial x)w_k-w_n,$$

comparing the coefficients of t^n for each n, where $B_k(x, \partial/\partial x)$ and the rest denote the coefficients of the expansion of $\sum_{j=1}^{n}B_j(t, x)\partial/\partial x_j$ and the rest respectively.

At first, we have

$$(5.9)\qquad w_1=\Gamma_0(x, \partial/\partial x)w_1,$$

since $v_0=0$, $w_0=0$, and hence $v_1=0$ by (5.7). Observing Lemma 5.2, and noticing (5.1), we can apply Lemma 5.1 to $\Gamma_0(x, \partial/\partial x)$ and obtain $w_1=0$. Repeating this procedure, we can obtain $\vec{u}_n=0$ successively. This completes the proof.

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