

A necessary condition for the well posed hyperbolic mixed problem with variable coefficients

By

Kunihiko KAJITANI

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§1. Introduction

In the present article we shall derive a necessary condition (say, Lopatinski's condition) for the well posed mixed problem of the first order hyperbolic systems with variable coefficients. Here we say that the mixed problem is well posed, if this problem has a local solution for any sufficiently smooth data and a finite propagation speed.

In the case of constant coefficients, R. Hersh in [5] and T. Kasahara in [10] proved the global existence of the solution for the mixed problem of the first order hyperbolic systems under Lopatinski's condition, and Hersh in [6] and [7] researched into the existence and the finite propagation speed of the solution for the higher order hyperbolic systems. Moreover T. Shirota in [14] studied precisely the finite propagation speed of the solution for the single higher order hyperbolic equations. Recently R. Sakamoto in [13] characterized completely the necessary and sufficient condition for the existence and the finite propagation speed of the solution for the single higher order hyperbolic equations.

In the case of variable coefficients, M. Ikawa proved in [3] that the mixed problem for the wave equation with an oblique derivative boundary condition is well posed. J. Chazarain in [2] and R. Beals in [1] proved the existence of the solution in Gevrey-class for the

mixed problem for hyperbolic systems (not necessary strictly hyperbolic) by use of semigroup theory and derived a finite propagation speed. In [7] the author showed a necessary condition for the well posed mixed problem for 2×2 hyperbolic systems with analytic coefficients. In this article we shall extend the result obtained in [7] to the general systems. Namely, we shall prove that Lopatinski's condition for the principal part of the hyperbolic systems considered and for the boundary condition is necessary under some assumptions, if the mixed problem for the systems is well posed. It is seemed that this problem is corresponding to those which P. Lax in [11] and S. Mizohata in [12] studied for the Cauchy problem.

We consider the following mixed problem in Ω a neighbourhood of the origin in $R^k \times (0, \infty)$

$$L[u] = \frac{\partial}{\partial t} u - \sum_{j=1}^k A_j(x, t) \frac{\partial}{\partial x_j} u - B(x, t)u = f(x, t), \quad (x, t) \text{ in } \Omega$$

$$(1.1) \quad u(x, 0) = g(x), \quad x \text{ in } D = \Omega \cap \{t=0\},$$

$$P(x', t)u(x', 0, t) = h(x', t), \quad (x', t) = (x_1, \dots, x_{k-1}, t)$$

$$\text{in } G = \Omega \cap \{x_k=0\},$$

where $A_j(x, t)$ and $B(x, t)$ are $m \times m$ matrices of infinitely differentiable elements, $u(x, t)$, $f(x, t)$ defined in Ω and $g(x)$ defined in D are vector valued functions of k components, $P(x', t)$ is a $l \times m$ matrix of infinitely differentiable elements and $h(x, t)$ is a vector valued function defined in G of l components.

We pose the following assumptions,

[A. I] L is hyperbolic at origin, that is the characteristic matrix $\sum_{j=1}^k A_j(0, 0)\eta_j$ has only real eigen values for any (η_1, \dots, η_k) in $R^k - \{0\}$. $A_k(0, 0)$ is non singular.

[A. II] the rank of $P(0, 0)$ is equal to l and l is the number of negative eigen values of $A_k(0, 0)$.

We put $M(\lambda, i\eta) = A_k^{-1}(0, 0)(\lambda - \sum_{j=1}^{k-1} A_j(0, 0)\eta_j)$ then it follows from hyperbolicity of L that the real part of eigen values of $M(\lambda, i\eta)$ is

not zero for $\text{Re } \lambda > 0, \eta$ in R^{k-1} and that the number of these eigen values with negative real part is equal to l . We define by $E^-(\lambda, \eta)$ (resp. $E^+(\lambda, \eta)$) the generalized eigen space corresponding to eigen values with negative (resp. positive) real part of $M(\lambda, \eta)$. We define

$$(1.2) \quad Q^\pm(\lambda, \eta) = \frac{1}{2\pi i} \oint_{\Gamma_\pm} (\zeta - M)^{-1} d\zeta,$$

where Γ_+ (resp. Γ_-) is a Jordan's curve containing only eigen values with positive (resp. negative) real part. We define $A(\lambda, \eta)$,

$$A(\lambda, \eta) = \frac{1}{2\pi i} \oint_{\Gamma_+} (\zeta - M)^{-1} A_k^{-1}(0, 0) (\zeta - M)^{-1} Q^- d\zeta.$$

[A. III] (i) $E^-(\lambda, \eta)$ consists of only eigen vectors of $M(\lambda, \eta)$ (has not generalized eigen vectors) for $\text{Re } \lambda > 0, \eta$ in R^{k-1} .

(ii) $\{E^-(\lambda, \eta) \cup (\text{Ker } P(0, 0) \cap E^+(\lambda, \eta))\} \cap \{A(\lambda, \eta)(E^-(\lambda, \eta) \cap \text{Ker } P(0, 0))\} = \{0\}$. and $\text{rank } A(\lambda, \eta) \geq \dim(E^-(\lambda, \eta) \cap \text{Ker } P(0, 0))$, for $\text{Re } \lambda > 0, \eta$ in R^{k-1} .

Remark 1. We put $E_0^\pm(\lambda, \eta) = \text{Ker } F(0, 0) \cap E^\pm(\lambda, \eta)$ and $E_\mp^\pm(\lambda, \eta) = \{h \in E^\pm(\lambda, \eta) \text{ and } h \notin E_0^\pm(\lambda, \eta)\}$. Then (ii) of [A. III] is equivalent (ii)' A is an one to one and onto map from $E_0^-(\lambda, \eta)$ to $E_\mp^+(\lambda, \eta)$ for $\text{Re } \lambda > 0, \eta$ in R^{k-1} .

Remark 2. In the case of $m=2$ (cf. [7]), (ii) or (ii)' is valid, if and only if it holds that for $\text{Re } \lambda > 0$,

$$(1.3) \quad E^-(\lambda, 0) \cap \text{Ker } P(0, 0) = \{0\}.$$

We say that (1.1) has a finite propagation speed, if there exist a positive number δ_0 and (x_0, t_0) such that (1.1) has the unique solution in $\Omega(x_0, t_0) = \{(x, t); |x - x_0| < \delta_0(t_0 - t), x_k > 0, t > 0\}$ for any $(x_0, t_0), x_{0k} > 0, t > 0$, in a neighbourhood of the origin.

Definition 1. We say that the mixed problem (1.1) is well posed in a neighbourhood of the origin, if for any data $\{f(x, t), g(x), h(x', t)\}$ infinitely differentiable in a neighbourhood of the origin, with the

compatibility condition, there exists a neighbourhood Ω of the origin such that (1.1) has a solution $u(x, t)$ in $C^\infty(\bar{\Omega})$ and has a finite propagation speed.

Definition 2. We denote by L_0 the principal part of L , that is, $L_0 = \frac{\partial}{\partial t} - \sum_{j=1}^k A_j(x, t) \frac{\partial}{\partial x_j}$. We say that (L_0, P) satisfies Lopatinski's condition at the origin, if it holds

$$(1.4) \quad E^-(\lambda, \eta) \cap \text{Ker } P(0, 0) = \{0\}, \quad \text{for } \text{Re } \lambda > 0, \quad \eta \text{ in } R^{k-1}.$$

Now we state our result,

Theorem. Suppose that [A. I], [A. II] and [A. III] are valid. Then (L_0, P) satisfies Lopatinski's condition at the origin, if the mixed problem (1.1) is well posed in a neighbourhood of the origin.

Remark 3. If we do not assume (ii) of [A. III], our theorem does not generally hold. For example ($m=2, k=2$ and $l=1$), we put

$$L = \frac{\partial}{\partial t} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial x_1} - \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial x_2} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$P = (0, 1).$$

We can see easily that this example does not satisfy (1.3) and (1.4). But the mixed problem for (L, B) is well posed (cf. [13]). In the case of constant coefficients, (1.1) is well posed and (1.3) is valid, if and only if (L_0, P) satisfies (1.4) (cf. [6], [13] and [14]).

To prove our theorem we shall apply to the mixed problem the method of the asymptotic solution which P. D. Lax in [11] used for the Cauchy problem.

§2. Proof of Theorem

We shall prove our theorem by contradiction. To do so, we need an inequality derived by the closed graph theorem. We denote $\Omega(x_0, t_0) = \{(x, t); |x - x_0| \leq \delta(t_0 - t); 0 \leq t \leq t_0, x \in R^k\}$, where x_0 in R^k and $t_0 > 0$ are fixed, and δ is a positive constant. Then, if the mixed problem is well posed in a neighbourhood of the origin, it follows

from the closed graph theorem that for any (x_0, t_0) in a neighbourhood of the origin and for any $u(x, t)$ in $C^\infty(\Omega(x_0, t_0))$ there exist a positive constant $C(x_0, t_0)$ and a positive integer s_0 such that

$$(2.1) \quad |u|_{0, \Omega(x_0, t_0)} \leq C(x_0, t_0) \{ |Lu|_{s_0, \Omega(x_0, t_0)} + |u|_{s_0, D(x_0, t_0)} + |Pu|_{s_0, G(x_0, t_0)} \},$$

where $D(x_0, t_0) = \Omega(x_0, t_0) \cap \{t=0\}$ and $G(x_0, t_0) = \Omega(x_0, t_0) \cap \{x_k=0\}$. We note that $C(x_0, t_0)$ may be generally dependent of (x_0, t_0) . But we can see a behaviour of $C(x_0 n^{-1}, t_0 n^{-1})$ for $n \rightarrow \infty$. We write $\Omega(n) = \Omega(x_0 n^{-1}, t_0 n^{-1})$, $D(n) = D(x_0 n^{-1}, t_0 n^{-1})$ and $G(n) = G(x_0 n^{-1}, t_0 n^{-1})$. Then we have,

Lemma 2.1. *Suppose that the mixed problem (1.1) is well posed in a neighbourhood of the origin. Then for any positive integer n there exist a positive constant C_0 and positive integers s_0 and s_1 (independent of n) such that for any $u(x, t)$ in $C^\infty(\Omega(x_0, t_0))$,*

$$(2.2) \quad |u|_{0, \Omega(n)} \leq C_0 n^{s_1} \{ |Lu|_{s_0, \Omega(n)} + |u|_{s_0, D(n)} + |Pu|_{s_0, G(n)} \}$$

We can derive this lemma from (2.1) by the same method as Lemma 2.1 in [9].

We put

$$M(n) = A_k^{-1}(n^{-1}x, n^{-1}t) \left\{ n \frac{\partial}{\partial t} - n \sum_{j=1}^k A_j(n^{-1}x, n^{-1}t) \frac{\partial}{\partial x_j} - B(n^{-1}x, n^{-1}t) \right\}$$

and

$$P(n) = P(n^{-1}x', n^{-1}t).$$

Then we have,

Lemma 2.2. *Suppose that the mixed problem (1.1) is well posed in a neighbourhood of the origin. Then there exist a positive constant C_1 and positive integers s_0 and s_1 such that for any positive integer n and for any $u(x, t)$ in $C^\infty(\Omega(x_0, t_0))$,*

$$(2.3) \quad |u|_{0, \Omega(x, t)} \leq C_1 n^{s_1 + s_0} \{ |M(n)u|_{s_0, \Omega(x_0, t)} + |u|_{s_0, D(x_0, t_0)} \\ + |P(n)u|_{s_0, G(x_0, t_0)} \}$$

Proof. This lemma was suggested by V. J. Ivrii [4]. We apply the inequality (2.2) to $u_n(x, t) = u(nx, nt)$. We obtain easily (2.3) under the change of variables, $y = nx$ and $s = nt$. q.e.d.

Proof of Theorem. Suppose that (L_0, P) does not satisfy the Lopatinski's condition, that is, there exist linearly independent vectors z_1, \dots, z_l in C^m and $\lambda_0, \operatorname{Re} \lambda_0 > 0$, η_0 in R^{k-1} such that

$$(2.4) \quad \{z_1, \dots, z_l\} \subset E^-(\lambda_0, \eta_0) \cap \operatorname{Ker} P(0, 0)$$

We denote

$$M^0 = A_k^{-1}(0, 0)(\lambda_0 - i \Sigma A_j(0, 0)\eta_j)$$

Let $\xi_j^-, j = 1, \dots, l$ be the eigen values with a negative real part of M^0 and h_j (resp. ω_j) the right (resp. left) eigen vectors corresponding to ξ_j^- . For simplicity, we assume that $\xi_j^-, j = 1, \dots, l$, are simple eigen values.

We shall construct the asymptotic solution of (1.1) having the following form,

$$(2.5) \quad u(x, t) = \sum_{j \geq 0} n^{-j} \sum_{p=1}^l \exp \{n(\xi_p^- x_k + \lambda_0 t - i x' \eta_0)\} u_j^{(p)}(x, t),$$

here, $u_j^{(p)}$ are functions to be determined.

Applying $M(n)$ to (2.5), we have,

$$(2.6) \quad M(n)[u] = \sum_{p=1}^l \left[n^2 (\xi_p^- - M^0) u_0^{(p)} + n \left\{ (\xi_p^- - M^0) u_1^{(p)} \right. \right. \\ \left. \left. - \left(\frac{\partial}{\partial x_k} - M_0(D_t, D_x) - M_1(x, t; \lambda_0, \eta_0) \right) u_0^{(p)} \right\} + \dots \right. \\ \left. + n^{-j+2} \left\{ (\xi_p^- - M^0) u_j^{(p)} + \left(\frac{\partial}{\partial x_k} - M_0(D_t, D_x) \right. \right. \right. \\ \left. \left. \left. - M_1(x, t; \lambda_0, \eta_0) \right) u_{j-1}^{(p)} \right\} \right]$$

$$\begin{aligned} & - \sum_{\substack{i+s=j \\ s < j-1}} M_i(x, t; \lambda_0, i\eta_0) u_s^{(p)} - \sum_{\substack{i+s=j-1 \\ s < j-1}} M_i(x, t, D_t, D_{x'}) u_s^{(p)} \Big\} \\ & + \dots \Big] \exp \{n(\xi_p^- x_k + \lambda_0 t - ix' \cdot \eta_0)\}, \end{aligned}$$

where,

$$M_0(D_t, D_{x'}) = A_k^{-1}(0, 0) \left(\frac{\partial}{\partial t} - \sum_{j=1}^{k-1} A_j(0, 0) \frac{\partial}{\partial x_j} \right),$$

$$M_j(x, t, D_t, D_{x'}) = \sum_{|v|+i=j} x'^v t^i M_{i,v} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x'} \right) + \sum_{|v|+i=j-1} x'^v t^i B_{v,i},$$

and operating $P(n)$ to u , we get

$$\begin{aligned} (2.7) \quad & P(n)[u] |_{x_k=0} = \sum_{p=1}^l \exp n(\lambda_0 t - ix' \cdot \eta_0) [P_0 u_0^{(p)}] |_{x_k=0} \\ & + n^{-1} \{P_0 u_1^{(p)} + P_1(x, t) u_0^{(p)}\} |_{x_k=0} + \dots \\ & + n^{-j} \{P_0 u_j^{(p)} + \sum_{i+s=j-1} P_i(x', t) u_s^{(p)}\} |_{x_k=0} + \dots \end{aligned}$$

where $P_0 = P(0, 0)$,

$$P_j(x', t) = \sum_{|v|+i=j} x'^v t^i (v! i!)^{-1} \left(\left(\frac{\partial}{\partial x'} \right)^v \left(\frac{\partial}{\partial t} \right)^i P \right) (0, 0).$$

Thus if we work out formally the coefficients of n^{-j} in (2.6) and (2.7) and set each of them equal to zero, we obtain the sequence of the recursion formulas,

$$(2.8) \quad (\xi_p^- - M^0) u_0^{(p)} = 0, \quad p = 1, \dots, l$$

$$(2.9) \quad (\xi_p^- - M^0) u_1^{(p)} + \left(\frac{\partial}{\partial x_k} - M_0(D_t, D_{x'}) - M_1(x, t; \lambda_0, i\eta_0) \right) u_0^{(p)} = 0,$$

$$(2.10) \quad (\xi_p^- - M^0) u_{j+1}^{(p)} + \left(\frac{\partial}{\partial x_k} - M_0(D_t, D_{x'}) - M_1(x, t; \lambda_0, i\eta_0) \right) u_j^{(p)} = f_j^{(p)} \quad \begin{matrix} p=1; 2; \dots; \\ j=1; 2; \dots; \end{matrix}$$

and

$$(2.11) \quad \sum_{p=1}^l P_0 u_0^{(p)} |_{x_k=0} = 0$$

$$(2.12) \quad \sum_{p=1}^l (P_0 u_1^{(p)} + P_1(x', t) u_0^{(p)})|_{x_k=0} = 0$$

$$(2.13) \quad \sum_{p=1}^l (P_0 u_j^{(p)} + P_1(x', t) u_{j-1}^{(p)})|_{x_k=0} = g_j$$

where

$$f_j^{(p)} = \sum_{\substack{i+s=j+1 \\ s < j}} M_i(x, t; \lambda_0, i\eta_0) u_s^{(p)} + \sum_{\substack{i+s=j \\ s < j}} M_i(x, t, D_t, D_{x'}) u_s^{(p)}$$

$p=1, 2, \dots, l,$

and

$$g_j = - \sum_{\substack{i+s=j-1 \\ i < 2}} \sum_{p=1}^l P_i(x', t) u_s^{(p)}|_{x_k=0}, \quad j=1, 2, \dots$$

From (2.8) and (2.11), we have

$$(2.14) \quad u_0^{(p)} = \sigma_0^{(p)}(x, t) h_p, \quad p=1, 2, \dots, l, \quad \text{and}$$

$$(2.15) \quad \sum_{p=1}^l u_0^{(p)}|_{x_k=0} = \sum_{p=1}^{l'} \tilde{\sigma}_0^{(p)}(x', t) z_p,$$

where $\sigma_0^{(p)}$ and $\tilde{\sigma}_0^{(p)}$ are scalar functions. Next, we seek $u_1^{(p)}$ satisfied with (2.9) and (2.12). We can find the particular solution $\hat{u}_1^{(p)}$ of (2.9), if $u_0^{(p)}$ are satisfied with

$$\omega_p \cdot \left(\frac{\partial}{\partial x_k} - M_0(D_t, D_{x'}) - M_1(x, t; \lambda_0, i\eta_0) \right) u_0^{(p)} = 0,$$

that is,

$$(2.16) \quad \left(\frac{\partial}{\partial x_k} - a_p(D_t, D_{x'}) - b_p(x, t) \right) \sigma_0^{(p)} = 0, \quad p=1, \dots, l,$$

where

$$a_p(D_t, D_{x'}) = \omega_p \cdot M_0 h_p \quad \text{and} \quad b_p = \omega_p \cdot M_1 h_p$$

The Cauchy Kowalweski's theorem guarantees the existence of the analytic solution $\sigma_0^{(p)}$ of (2.16) with the intial conditions (2.15). Here we note that $\tilde{\sigma}_0^{(p)}(x', t)$, $p=1, 2, \dots, l'$, remain undetermined.

From above reasoning we obtain the general solution of (2.9) as follows

$$(2.17) \quad u_1^{(p)} = \sigma_1^{(p)}(x, t)h_p + \hat{u}_1^{(p)}, \quad p = 1, \dots, l,$$

where $\hat{u}_1^{(p)}$ is the particular solution of (2.9). Next, we determine $\tilde{\sigma}_0^{(p)}(x', t)$, $p = 1, \dots, l'$, so that $u_1^{(p)}|_{x_k=0}$ verify (1.12). We put $H = (h_1, h_2, \dots, h_p)$, $\sigma_j = (\sigma_j^{(1)}, \dots, \sigma_j^{(l)})$, $\tilde{\sigma}_j = (\tilde{\sigma}_j^{(1)}, \dots, \tilde{\sigma}_j^{(l')})$ and $Z = (z_1, z_2, \dots, z_{l'})$. Inserting (2.17) into (2.12), we obtain

$$(2.18) \quad P_0 H \sigma_1(x', 0, t) + \sum_{p=1}^l (P_0 \hat{u}_1^{(p)} + P_1 u_0^{(p)})|_{x_k=0} = 0.$$

From the assumption, the rank of $P_0 H$ is $l - l'$. Hence there are left nullvectors $r_1, \dots, r_{l'}$ of $P_0 H$. We write $R = (r_1, \dots, r_{l'})$, $l' \times m$ matrix. If $\hat{u}_1^{(p)}$ and $u_0^{(p)}$ verifies

$$(2.19) \quad R \cdot \sum_{p=1}^l (P_0 \hat{u}_1^{(p)} + P_1(x', t)u_0^{(p)})|_{x_k=0} = 0,$$

there exists a solution $\sigma_1(x', 0, t)$ of (2.18). We express (2.19) as an equation of $\tilde{\sigma}_0$. To do so, we must solve $\hat{u}_1^{(p)}$ explicitly from (2.9). We use operators Q^\pm defined in (1.2). We put $M^\pm = Q^\pm M$. We note that Q^\pm is an orthogonal projection into $E_0^\pm(\lambda_0, i\eta_0)$ along $E_0^\mp(\lambda_0, i\eta_0)$. Since $Q^- \hat{u}_1^{(p)}$ is in E_0^- , $R P_0 Q^- \hat{u}_1^{(p)} = 0$. Hence by virtue of (2.9), we obtain

$$(2.20) \quad Q^+ \hat{u}_1^{(p)} = (\xi_p^- - M^+)^{-1} Q^+ (M_0(D_r, D_{x'}) + M_1(x, t, \lambda_0, i\eta_0))u_0^{(p)}$$

Noting that

$$(\xi_p^- - M^+)^{-1} Q^+ = \frac{1}{2\pi i} \oint_{\Gamma_+} (\xi_p^- - \xi)^{-1} (\xi - M^0)^{-1} d\xi,$$

and

$$(\xi_p^- - \xi)^{-1} h_p = (M^- - \xi)^{-1} h_p \quad \text{for } \xi \in \Gamma_+, p = 1, \dots, l,$$

we have by virtue of (2.14) and (2.20),

$$\sum_{p=1}^l Q^+ u_1^{(p)}|_{x_k=0} = \frac{1}{2\pi i} \oint_{\Gamma_+} (\xi - M_0)^{-1} \{M_0(D_r, D_{x'}) +$$

$$\begin{aligned}
& + M_1(x', 0, t; \lambda_0, i\eta_0) \} \\
& \quad \times (M^- - \xi)^{-1} d\xi \sum_{p=1}^l h_p \sigma_0^{(p)}(x', 0, t) \\
& = \frac{1}{2\pi i} \oint_{\Gamma_+} (\xi - M^0)^{-1} \left\{ A_k^{-1}(0, 0) \frac{\partial}{\partial t} + A_k^{-1} \sum_{j=1}^{k-1} A_j(0, 0) \frac{\partial}{\partial x_j} + M_1 \right\} \\
& \quad \cdot (\xi - M^-)^{-1} d\xi \times Z \cdot \tilde{\sigma}_0(x', t),
\end{aligned}$$

which implies with (2.19),

$$(2.21) \quad T_0 \frac{\partial}{\partial t} \tilde{\sigma}_0 + \sum_{j=1}^{k-1} T_j \frac{\partial}{\partial x_j} \tilde{\sigma}_0 + S(x', t) \tilde{\sigma}_0 = 0,$$

$$\text{where, } T_0 = RP_0 \Lambda Z, \Lambda = \frac{1}{2\pi i} \oint_{\Gamma_+} (\xi - M^0)^{-1} A_k^{-1}(0, 0) (M^- - \xi) d\xi,$$

and

$$\begin{aligned}
T_j = \frac{1}{2\pi i} \oint_{\Gamma_+} RP(\xi - M^0)^{-1} A_k^{-1}(0, 0) A_j(0, 0) (M^- - \xi)^{-1} Z d\xi, \\
j = 1, \dots, k-1,
\end{aligned}$$

are $l' \times l'$ constant matrices, and $S(x, t)$ is a $l' \times l'$ matrix with elements analytic in (x', t) . From the assumption (ii) of [A.III] it follows that the $l' \times l'$ matrix T_0 is non singular. Hence there exists a non trivial solution $\tilde{\sigma}_0(x', t)$ of (2.21) and therefore we have a solution $\sigma_1(x', 0, t)$ of (2.18) such that

$$(2.22) \quad H\sigma_1(x', 0, t) = Z\tilde{\sigma}_1(x', t) + H\hat{\sigma}_1(x', t)$$

where $\hat{\sigma}_1(x', t)$ is a particular solution of (2.18).

Summarising the above result, we find $u_0^{(p)}$ as follows,

$$u_0^{(p)}(x, t) = \sigma_0^{(p)}(x, t) h_p, \quad p = 1, 2, \dots, l$$

$$H\sigma_0(x', 0, t) = Z\tilde{\sigma}_0(x', t),$$

here $\sigma_0(x, t) = (\sigma_0^{(1)}(x, t), \dots, \sigma_0^{(l)}(x, t))$ is a solution of (2.16) and its initial value $\tilde{\sigma}_0(x', t)$ is a solution of (2.21). Next, we have $u_1^{(p)}(x, t)$ as follows,

$$u_1^{(p)}(x, t) = \sigma_1^{(p)}(x, t)h_p + \hat{u}_1^{(p)}(x, t), \quad p=1, 2, \dots, l,$$

$$H\sigma_1(x, 0, t) = Z\tilde{\sigma}_1(x', t) + H\hat{\sigma}_1(x', t),$$

where $\hat{u}_1^{(p)}$ is a particular solution of (2.9) and $\hat{\sigma}_1(x', t)$ is a particular solution of (2.18), and $\sigma_1(x, t) = (\sigma_1^{(1)}, \dots, \sigma_1^{(l)})$ and $\tilde{\sigma}_1(x', t) = (\sigma_1^{(1)}, \dots, \sigma_1^{(l)})$ are arbitrary functions in order to construct $u_2^{(p)}(x, t)$, $p=1, \dots, l$.

Generally, we put $u_s^{(p)}(x, t)$ such that

$$u_s^{(p)}(x, t) = \sigma_s^{(p)}(x, t)h_p + \hat{u}_s^{(p)}(x, t), \quad p=1, \dots, l,$$

and

$$H\sigma_s(x', 0, t) = Z\tilde{\sigma}_s(x, t) + H\hat{\sigma}_s(x', t).$$

where $\hat{u}_s^{(p)}$ is a particular solution of (2.10) with $j=s-1$ and $H\hat{\sigma}_s(x', t)$ is a particular solution of (2.13) with $j=s$. Then if we choose $\sigma_s^{(p)}(x, t)$ and $\tilde{\sigma}_s(x', t)$ such that,

$$\left(\frac{\partial}{\partial x_k} - a_p(D_t, D_{x'}) - b_p(x, t) \right) \sigma_s^{(p)} = \tilde{f}_s^{(p)}(x, t), \quad p=1, \dots, l,$$

and

$$\left(T_0 \frac{\partial}{\partial t} + \sum_{j=1}^{k-1} T_j \frac{\partial}{\partial x_j} + S(x', t) \right) \tilde{\sigma}_s = \tilde{g}_s(x', t),$$

where $\tilde{f}_s^{(p)} = \omega_p \cdot f_s^{(p)} - \omega_p \cdot \left(\frac{\partial}{\partial x_k} - M_0(D_t, D_x) - M_1(x, t; \lambda_0, i\eta) \right) u_s^{(p)}(x, t)$ and $\tilde{g}_s = Rg_{s+1} - \sum_{p=1}^l RP_1(x', t) \hat{u}_s^{(p)}(x', 0, t) - RP_1(x, t) H\hat{\sigma}_s(x', t)$, we can construct successively $u_{s+1}^{(p)}(x, t)$ satisfying (2.10) with $j=s$ and (2.13) with $j=s+1$.

We put

$$u(x, t) = \sum_{j=0}^N n^{-j} \sum_{p=1}^l \exp \{ n(\xi_p^- \cdot x_k + \lambda_0 t - ix' \eta_0) \} u_j^{(p)}(x, t).$$

Then if N is sufficiently large, it is obvious that $u(x, t)$ violates (2.3). Thus the proof of our theorem is complete.

Remark. In above statement, we assumed that the matrix M_0 has simple eigen values. But this is not essential. In fact, it follows from the assumption (i) of [A. III] that, if ξ_p^- is an eigen value of multiplicity v_p , there exist the number v_p of linearly independent eigen vectors corresponding to ξ_p^- . We can repeat the above reasoning by use of those eigen vectors.

DEPARTMENT OF APPLIED MATHEMATICS
AND PHYSICS, KYOTO UNIVERSITY

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