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Asymptotic behavior of sample functions of Gaussian random fields

By

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1. Introduction.

Let $\{X(s); s \in S\}$ be a path continuous real valued centered Gaussian random field with a parameter space S, where S is a non compact locally compact topological space. Denote by $S \cup \{\infty\}$ a one point compactification of S. Then, after P. Lévy, we can formulate the asymptotic behavior at the infinity point $\{\infty\}$ of sample functions of the Gaussian random field as follows:

Definition 1. A positive continuous function $\varphi(s)$ defined on a neighborhood of $\{\infty\}$ is called a function belonging to *the upper class* $U(X)$ if there exists a neighborhood U of $\{\infty\}$, with probability 1, such that $X(s) < \gamma_0(s)\varphi(s)$ holds for any $s \in U \cap S$, where $\gamma_0(s) = (E[X(s)^2])^{1/2}$.

Definition 2. A positive continuous function $\varphi(s)$ defined on a neighborhood of $\{\infty\}$ is called a function belonging to *the lower class* $L(X)$ if there exists no such neighborhood U of $\{\infty\}$, with probability 1, that $X(s) < \gamma_0(s)\varphi(s)$ holds for any $s \in U \cap S$.

In case of Brownian motion, I. Petrovsky [6] and K. L. Chung-P. Erdös-T. Sirao [1] first proved the criterion which determines whether $\varphi(s)$ is of *the upper class* or *the lower class.* T. Sirao [7] also proved in case of Lévy's Brownian motion with multidimensional parameter. Recently many authors have investigated the asymptotic behavior of sample functions of Gaussian random fields in our sense,

[2], [3], [5], [8], [9], [10], [11], [12], [13].

In this paper, we investigate a path continuous real valued centered Gaussian random field $\{X(t)\,;\,t\in R^N\}$ such that

$$E[(X(s)-X(t))^2]=\gamma^2(s-t)=\sum_{i=1}^{d}\sigma_i^2(|s_i-t_i|), \tag{1}$$

where $s_i, t_i\in R^{N_i}$, $N_1+\cdots+N_d=N$, $s=(s_1,\cdots,s_d)$, $t=(t_1,\cdots,t_d)$ and $|s_i-t_i|$ is the usual N_i-dimensional Euclidean metric.

If each $\sigma_i^2(|s_i-t_i|)$ is conditionally positive definite, then so is $\gamma^2(s-t)=\sum_{i=1}^{d}\sigma_i^2(|s_i-t_i|)$. Therefore we can easily give many examples satisfying our assumption (1).

We investigate the following five cases but the methods of the proofs of the theorems corresponding to each case are each other almost the same.

Case [I]. *The uniform upper class* $U_u(X;D)$ and *the uniform lower class* $L_u(X;D)$. Let D be compact set of R^N which contains an N-dimensional ball. Set

$$S=D\times D-\{(t,t)\,;\,t\in D\}. \tag{2}$$

Then S is naturally a locally compact set and after our definition 1 and 2, we call a function $\varphi(x)$ defined on the real line belonging to *the uniform upper class* $U_u(X;D)$ or *the uniform lower class* $L_u(X;D)$ of the random field $\{X(t)\,;\,t\in D\}$ when the function $\varphi(\gamma(s-t))$ belongs to *the upper class* or *the lower class* of the random field $\{Y(s,t)=X(s)-X(t)\,;\,(s,t)\in S\}$ in the sense of Definition 1 and 2 respectively.

Case [II]. *The local upper class* $U_l(X;t_0)$ and *the local lower class* $L_l(X;t_0)$. Set

$$S=D-\{t_0\}, \tag{3}$$

where D is the closed ball of R^N which has the radius 1 and the center t_0. Then S is naturally a locally compact set. We call a function $\varphi(x)$ belonging to *the local upper class* $U_l(X;t_0)$ or *the local lower class* $L_l(X;t_0)$ when the function $\varphi(\gamma(s-t_0))$ belongs to *the upper class* or *the lower class* of the random field $\{Y(s)=X(s)$

$-X(t_0)\,;s\in S\}$ in the sense of Definition 1 and 2 respectively.

Case [III]. *The upper class* $U_{ut}(X;t_0)$ and *the lower class* $L_{ut}(X;t_0)$. Set

$$D^+=\{(x_1,\cdots,x_N)\in R^N;t_j{}^0\leq x_j\leq t_j{}^0+1,1\leq j\leq N\},$$

$$D^-=\{(x_1,\cdots,x_N)\in R^N;t_j{}^0-1\leq x_j\leq t_j{}^0,1\leq j\leq N\},$$

and

$$S=D^+\times D^- -\{(t_0,t_0)\}, \tag{4}$$

where $t_0=(t_1{}^0,\cdots,t_N{}^0)$. Then S is naturally a locally compact set. We call a function $\varphi(x)$ belonging to *the upper class* $U_{ut}(X;t_0)$ or *the lower class* $L_{ut}(X;t_0)$ when the function $\varphi(\gamma(s-t))$ belongs to *the upper class or the lower class* of the random field $\{Y(s,t)=X(s)-X(t)\,;\,(s,t)\in S\}$ in the sense of Definition 1 and 2 respectively.

Case [IV]. *The upper class* $U_l^\infty(X)$ and *the lower class* $L_l^\infty(X)$. In this case, in addition to (1) we assume that $\gamma(t)$ satisfies

$$\lim_{|t|\to+\infty}\gamma(t)=+\infty. \tag{5}$$

Then we call a function $\varphi(x)$ belonging to *the upper class* $U_l^\infty(X)$ or *the lower class* $L_l^\infty(X)$ when the function $\varphi(\gamma(t))$ belongs to *the upper class* or *the lower class* of the random field $\{X(t)-X(0)\,;t\in R^N\}$ in the sense of Definition 1 and 2 respectively.

Case [V]. *The upper class* $U_b^\infty(X)$ and *the lower class* $L_b^\infty(X)$. In this case, in addition to (1) we assume that

$$E[X(t)^2]=1$$

holds for all $t\in R^N$. Then we call a function $\varphi(x)$ belonging to *the upper class* $U_b^\infty(X)$ or *the lower class* $L_b^\infty(X)$ when the function $\varphi(|t|)$ belongs to *the upper class* or *the lower class* of the random field $\{X(t)\,;t\in R^N\}$ in the sense of Definition 1 and 2 respectively.

In § 2, we give the integral tests which determine whether $\varphi(x)$ belongs to *the upper class* or *the lower class* corresponding to each case under some regularity conditions. In § 3 we prove Theorem

1-5 when the integral tests converge and in § 4 we prove Theorem 6-10 when the integral tests diverge. In § 5 we discuss the invariance of *the upper classes* or *the lower classes* between two random fields such that $E[(X(s)-X(t))^2]=\sum_{i=1}^d |s_i-t_i|^{2\alpha_i}$, $0<\alpha_i<1$, $s_i, t_i \in R^{N_i}$, $s, t \in R^N$, and $N_1+\cdots+N_d=N$.

Finally, we are concerned only with, real valued, path continuous, centered Gaussian random fields, which we will generally refer to simply as a Gaussian random field.

I would like to express my hearty thanks to Professor H. Watanabe for his valuable suggestions and discussions.

2. Integral tests.

First we define a nearly regular varying function with exponent $\alpha(>0)$ (*n.r.v.f.* (α)) *at* $x=+\infty$ (or *at* $x=0$).

Definition 3. A real valued function $f(x)$ defined on the half line is called *n.r.v.f.* (α) *at* $x=+\infty$ (*at* $x=0$) if and only if there exists a regular varying function $r(x)$ with exponent $\alpha(>0)$ such that

$$f(x) \asymp r(x)^{*)},\ x \uparrow +\infty \ (x \downarrow 0). \tag{6}$$

Remark 1. A locally bounded measurable function $r(x)$ defined on $(0, +\infty)$ is called a regular varying function with exponent $\alpha(>0)$ at $x=+\infty$ (at $x=0$) if and only if

$$\lim_{x\to+\infty(+0)} r(tx)/r(x)=t^\alpha$$

holds for any $t>0$. Especially, if a continuous function $f(x)$ is a *n.r.v.f.* (α) *at* $x=+\infty$ (*at* $x=0$) then there exists a non-decreasing continuous regular varying function $r(x)$ with exponent α such that $f(x) \asymp r(x)$, $x\uparrow +\infty$ $(x \uparrow 0)$, and the inverse function $f^{-1}(x)=\inf\{y; f(y)=x\}$ of a non-decreasing continuous *n.r.v.f.* (α) *at* $x=+\infty$ (*at* $x=0$) is also a *n.r.v.f.* $(1/\alpha)$ *at* $x=+\infty$ (*at* $x=0$).

*) We describe by $f(t) \asymp g(t)$, $|t|\uparrow+\infty$ $(|t|\downarrow 0)$ when there exist two positive constants c and C such that

$$0<c\le \varliminf_{|t|\to+\infty(+0)} f(t)/g(t) \le \varlimsup_{|t|\to+\infty(+0)} f(t)/g(t) \le C<+\infty.$$

To describe our theorems, set

$$\Phi(x) = \int_x^\infty e^{-u^2/2} du/\sqrt{2\pi},$$

$$K(a\overline{N}; \gamma) = \prod_{i=1}^{d} [\sigma_i^{-i}(x)]^{aN_i},$$

$$K(a\overline{N}/b\overline{N}; \gamma, \varphi)(x) = K(a\overline{N}; \gamma)(x)/K(b\overline{N}; \gamma)(x/\varphi(x)),$$

$$\sigma_i^{-1}(x) = \inf\{y; \sigma_i(y) = x\}.$$

Now we have the theorem about *the upper class* corresponding to each case under the Assumption U.

Assumption U. Each $\sigma_i(x)$ is a non-decresing continuous *n.r.v.f.* (α_i) *at* $x=0$. In case of Theorem 4, in addition to this it is a *n.r.v.f.* (α_i') *at* $x=+\infty$.

Theorem 1. *A non-increasing positive continuous function* $\varphi(x)$ *is a fuuction belonging to the uniform upper class* $U_u(X; D)$ *if*

$$I_u(\gamma; \varphi) \equiv \int_{+0} x^{-1}\Phi(\varphi(x)) K(\overline{N}/2\overline{N}; \gamma, \varphi)(x) dx < +\infty.$$

Theorem 2. *A non-increasing positive continuous function* $\varphi(x)$ *is a function belonging to the local upper class* $U_l(X; t_0)$ *if*

$$I_l(\gamma; \varphi) \equiv \int_{+0} x^{-1}\Phi(\varphi(x)) K(\overline{N}/\overline{N}; \gamma, \varphi)(x) dx < +\infty.$$

Remark 2. The integral tests of Theorem 1 and 2 are essentially equivalent to those of Theorem 3 and 7 in [3] when $d=1$ under the condition $x\sigma_i'(x) \asymp \sigma_i(x)$ ($\sigma_i'(x)$ is the derivative of $\sigma_i(x)$). This condition is satisfied if $\sigma_i^2(x)$ is a concave *n.r.v.f.* *at* $x=0$.

Theorem 3. *A non-increasing positive continuous function* $\varphi(x)$ *is a function belonging to the upper class* $U_{ul}(X)$ *if*

$$I_{ul}(\gamma; \varphi) \equiv \int_{+0} x^{-1}\Phi(\varphi(x)) K(2\overline{N}/2\overline{N}; \gamma, \varphi)(x) dx < +\infty.$$

Theorem 4. *A non-decreasing positive continuous function* $\varphi(x)$ *is a function belonging to the upper class* $U_l^\infty(X)$ *if*

$$I_l^\infty(\gamma;\varphi)\equiv\int^{+\infty} x^{-1}\Phi(\varphi(x))K(\overline{N}/\overline{N};\gamma,\varphi)(x)\,dx<+\infty.$$

Theorem 5. *A non-decreasing positive continuous function* $\varphi(x)$ *is a function belonging to the upper class* $U_b^\infty(X)$ *if*

$$I_b^\infty(\gamma;\varphi)\equiv\int^{+\infty} x^{N-1}\Phi(\varphi(x))/K(\overline{N};\gamma)(1/\varphi(x))\,dx<+\infty.$$

Remark 3. Just the same result of Theorem 5 is obteined for the asymptotic behavior at $\{\infty\}$ of the random field $\{X(t)-X(0)\,;\ t\in R^N\}$ under the condition $\gamma^2(t)\asymp C$ instead of the assumption of Case [V].

Remark 4. Theorem 1-5 are still valid under the weaker condition $E[(X(s)-X(t))^2]\asymp\sum_{i=1}^d\sigma_i^2(|s_i-t_i|)$.

Next we have Theorem 6-9 about *the lower class* corresponding to the case [I]-[IV] respectively under the Assumption L which is stronger than the Assumption U. In case of [V] we have Theorem 10 about *the lower class* under the Assumption U with an additional condition.

Assumption L. Each $\sigma_i(x)$ is a non-decreasing twice continuously differentiable *n.r.v.f.* (α_i) *at* $x=0$ which satisfies the relations

$$x|\sigma_i'(x)|\leq d_0\sigma_i(x), \tag{7·a}$$

$$x^2|\sigma_i''(x)|\leq d_1\sigma_i(x), \tag{7·b}$$

in the mentioned domain and

$$0<\alpha_i<1 \quad \text{for each } i. \tag{7·c}$$

In case of Theorem 9, in addition to this it is a *n.r.v.f.* (α_i') *at* $x=+\infty$ such that $0<\alpha_i'<1$, where d_0 and d_1 are constants independent of x and i.

Strictly speaking, it does not need the assumption (7·b) for

Theorem 7 and 9.

Theorem 6. *A non-increasing positive continuous function* $\varphi(x)$ *is a function belonging to the uniform lower class* $L_u(X; D)$ *if*

$$I_u(\gamma; \varphi) = +\infty.$$

Theorem 7. *A non-increasing positive continuous function* $\varphi(x)$ *is a function belonging to the local lower class* $L_l(X; t_0)$ *if*

$$I_l(\gamma; \varphi) = +\infty.$$

Remark 5. Theorem 6 and 7 are still valid under the condition that $\sigma_i^2(x)$ is a concave *n.r.v.f.* $(2\alpha_i)$ *at* $x=0$ such that $0<\alpha_i<1/2$ instead of the Assumption L. (c.f. [3]).

Theorem 8. *A non-increasing positive continuous function* $\varphi(x)$ *is a function belonging to the lower class* $L_{ul}(X; t_0)$ *if*

$$L_{ul}(\gamma; \varphi) = +\infty.$$

Theorem 9. *A non-decreasing positive continuous function* $\varphi(x)$ *is a function belonging to the lower class* $L_l^\infty(X)$ *if*

$$I_l^\infty(\gamma; \varphi) = +\infty.$$

Theorem 10. *A non-decreasing positive continuous function* $\varphi(x)$ *is a function belonging to the lower class* $L_b^\infty(X)$ *if*

$$I_b^\infty(\gamma; \varphi) = +\infty,$$

under an additional condition

$$|E[X(t)X(s)]| \leq 0(|\log|t-s||^{-\beta}), \quad (|t-s| \uparrow +\infty),$$

$$\beta > N/\alpha' + 4,$$

where $\alpha' = \min(\alpha_1, \cdots, \alpha_d, 1-\alpha)$, $\alpha = \max(\alpha_1, \cdots, \alpha_d)$.

Remark 6. The additional condition of Theorem 10 is weaker than that of [12].

Remark 7. Theorem 10 is still valid under the condition $E[(X(t)-X(s))^2]\geq\sum_{1=i}^{d}\sigma_i^2(|t_i-s_i|)$.

Remark 8. It is remarkable that the integral test for the Case [IV] is not dependent on the asymptotic bahavior at the origin of $\sigma_i(x)$ but on the asymptotic behavior at the infinity of $\sigma_i(x)$, whereas that of the Case [V] is dependent only on the asymptotic behavior at the origin of $\sigma_i(x)$. This phenomena is, as far as I know, first pointed out by M. B. Marcus [4].

3. Upperbounds.

First we show that it is sufficient to prove our theorems under the restricted condition respectively, described in Lemma 1 bellow.

Lemma 1. (i) *It is sufficient to prove Theorem 1 and 6 for $\varphi(x)$ such that*

$$2\log 1/K(\bar{N};\gamma)(x)-2\log_{(2)}1/x\leq\varphi^2(x)\leq 3\log 1/K(\bar{N};\gamma)(x) \tag{8}$$

holds near the origin.

(ii) *It is sufficient to prove Theorem 2, 3, 7 and 8 for $\varphi(x)$ such that*

$$2\log_{(2)}1/x-2\log_{(3)}1/x\leq\varphi^2(x)\leq 3\log_{(2)}1/x \tag{9}$$

holds near the origin.

(iii) *It is sufficient to prove Theorem 4 and 9 for $\varphi(x)$ such that*

$$2\log_{(2)}x-2\log_{(3)}x\leq\varphi^2(x)\leq 3\log_{(2)}x \tag{10}$$

holds near the infinity.

(iv) *It is sufficient to prove Theorem 5 and 10 for $\varphi(x)$ such that*

$$2N\log x-2\log_{(2)}x\leq\varphi^2(x)\leq 3N\log x \tag{11}$$

holds near the infinity.

Proof of (i). Set

$$\varphi_1^2(x)=2\log 1/K(\bar{N};\gamma)(x)-2\log_{(2)}1/x,$$

$$\varphi_2^2(x)=3\log 1/K(\bar{N};\gamma)(x).$$

We assume that there exists a sequence $x_n\downarrow 0$ such that $\varphi(x_n)\leq\varphi_1(x_n)$. Denote by n_0 the maximal number of m such that $x_n\leq 2^{-m}$. Since $\sigma_i^{-1}(x)$ is a nearly regular varying function and $\varphi(x)$ is a non-increasing function, we have

$$\begin{aligned}I_u(\gamma;\varphi)&\geq\int_{2^{-n_0}}^{2^{-n_0+1}}x^{-1}\Phi(\varphi(x))K(\bar{N}/2\bar{N};\gamma,\varphi)(x)dx\\&\geq\Phi(\varphi(x_n))\int_{2^{-n_0}}^{2^{-n_0+1}}x^{-1}\prod_{i=1}^{d}[\sigma_i^{-1}(x)]^{-N_i}K(2\bar{N}/2\bar{N};\gamma,\varphi)(x)dx\\&\geq c_1\Phi(\varphi_1(x_n))\prod_{i=1}^{d}[\sigma_i^{-1}(2^{-n_0+1})]^{-N_i}\\&\geq c_2(\log 1/x_n)^{1/2}\uparrow+\infty\quad\text{as}\quad x_n\downarrow 0.\end{aligned}\tag{12}$$

This implies that if $I_u(\gamma;\varphi)<+\infty$, then $\varphi_1(x)<\varphi(x)$ holds near the origin. Moreover by the trivial relations

$$I_u(\gamma;(\varphi_1\vee\varphi)\wedge\varphi_2)\leq I_u(\gamma;\varphi_1\vee\varphi)+I_u(\gamma;\varphi_2),$$

$$I_u(\gamma;\varphi_2)<+\infty,\tag{13}$$

where $\varphi_1\vee\varphi(x)=\max\{\varphi_1(x),\varphi(x)\}$, $(\varphi_1\vee\varphi)\wedge\varphi_2(x)=\min\{\varphi_1\vee\varphi(x),\varphi_2(x)\}$, we have $I_u(\gamma;(\varphi_1\vee\varphi)\wedge\varphi_2)<+\infty$ from $I_u(\gamma;\varphi)<+\infty$. Therefore if we can prove Theorem 1 under the condition (8), then φ is of $U_u(X;D)$ when $I_u(\gamma;\varphi)<+\infty$ because of $(\varphi_1\vee\varphi)\wedge\varphi_2\leq\varphi$ and $(\varphi_1\vee\varphi)\wedge\varphi_2\in U_u(X;D)$.

To prove $I_u(\gamma;(\varphi_1\vee\varphi)\wedge\varphi_2)=+\infty$ under the condition $I_u(\gamma;\varphi)=+\infty$, set

$$\Delta^+=\{x;\varphi_1\vee\varphi(x)\leq\varphi_2(x)\},$$

$$\Delta^-=\{x;\varphi_1\vee\varphi(x)>\varphi_2(x)\},$$

$$I_u(\gamma;(\varphi_1\vee\varphi)\wedge\varphi_2)=\int_{\Delta^+}+\int_{\Delta^-}\equiv I_1+I_2,$$

$$I_u(\gamma;\varphi_1\vee\varphi)=\int_{\Delta^+}+\int_{\Delta^-}\equiv I_1+I_3.$$

Since $\sigma_i^{-1}(x)$ is a nearly regular varying function, we have

$$I_3 \leq c_3 \int_{\Delta^-} \Phi(\varphi_1 \vee \varphi(x))\,(xK(\bar{N};\gamma)(x))^{-1}(\varphi_1 \vee \varphi(x))^{3N(\bar{\alpha})}dx$$

$$\leq c_4 \int_{\Delta^-} \Phi(\varphi_2(x))\,(xK(\bar{N};\gamma)(x))^{-1}(\varphi_2(x))^{3N(\bar{\alpha})}dx$$

$$\leq c_4 \int_{+0} \Phi(\varphi_2(x))\,(xK(\bar{N};\gamma)(x))^{-1}(\varphi_2(x))^{3N(\bar{\alpha})}dx$$

$$< +\infty, \tag{14}$$

where $N(\bar{\alpha}) = N_1/\alpha_1 + \cdots + N_d/\alpha_d$. On the other hand, from (12) obviously it follows that $I_u(\gamma;\varphi) = +\infty$ yields $I_1 + I_3 = +\infty$. Therefore combining this with (14) we have $I_1 + I_2 = +\infty$. Furthermore if we can prove $(\varphi_1 \vee \varphi) \wedge \varphi_2 \in L_u(X;D)$, then $\varphi_1 \vee \varphi$ is also of $L_u(X;D)$ because of $\varphi_2 \in U_u(X;D)$ by Theorem 1 and if $\varphi_1 \vee \varphi$ is of $L_u(X;D)$ then φ is also of $L_u(X;D)$ by definition. This completes the proof of (i).

Proof of (ii). Set

$$\varphi_1^2(x) = 2\log_{(2)} 1/x - 2\log_{(3)} 1/x,$$

$$\varphi_2^2 = 3\log_{(2)} 1/x.$$

We assume that there exists a sequence $x_n \downarrow 0$ such that $\varphi(x_n) \leq \varphi_1(x_n)$. Since $\varphi(x)$ is a non-increasing function, we have

$$I_l(\gamma;\varphi) \geq \int_{x_n} x^{-1}\Phi(\varphi(x))K(\bar{N}/\bar{N};\gamma,\varphi)(x)dx$$

$$\geq c_5\Phi(\varphi(x_n)) \int_{x_n} x^{-1}dx$$

$$\geq c_5\Phi(\varphi_1(x_n)) \int_{n_n} \varphi^{-1}dx$$

$$\geq c_6(\log_{(2)} 1/x_n)^{1/2} \uparrow +\infty \quad \text{as} \quad x_n \downarrow 0.$$

By virtue of the same argument of (12), (13) and (14) it is sufficient to prove Theorem 2 and 7 under the condition (9). In case of $I_{ul}(\gamma;\varphi)$, the proof is just the same as that of $I_l(\gamma;\varphi)$. This completes the proof of (ii). The proof of (iii) is also just the same as that of (ii), so we omit it.

Proof of (iv). Set

$$\varphi_1^2(x)=2N\log x-2\log_{(2)}x,$$

$$\varphi_2^2(x)=3N\log x.$$

We assume that there exists a sequence $x_n\uparrow+\infty$ such that $\varphi(x_n)\leq\varphi_1(x_n)$. Since $\varphi(x)$ is a non-decreasing function, we have

$$I_b^\infty(\gamma;\varphi)\geq\int^{x_n}x^{N-1}\Phi(\varphi(x))/K(\bar{N};\gamma)(1/\varphi(x))dx$$

$$\geq c_7\int^{x_n}x^{N-1}\Phi(\varphi(x))dx$$

$$\geq c_7\Phi(\varphi_1(x_n))\int^{x_n}x^{N-1}dx$$

$$\geq c_8(\log x_n)^{1/2}\uparrow+\infty\quad\text{as}\quad x_n\uparrow+\infty.$$

By virtue of the same argument of (12), (13) and (14), it is sufficient to prove Theorem 5 and 10 under the condition (11). This completes the proof of Lemma 1.

Next, to describe our fundamental lemma in the general form, we introduce a metric space S satisfying condition A with a dimension N or a condition B with a dimension M, respectively.

Definition 4. We call that a complete metric space S satisfies *the condition A with a dimension N* if there exist a positive constant d_2 and a positive integer N such that for any $\varepsilon>0$ and for any compact subset K of S,

$$N(\varepsilon;K)\leq d_2(d(K)/\varepsilon)^N$$

holds, where $N(\varepsilon;K)$ is the minimal number of ε-net on K and $d(K)$ is the diameter of K.

Definition 5. We call that a complete metric space S satisfies *the condition B with a dimension M* if there exist two positive constants d_3, d_4 and a positive integer M such that for any $\varepsilon>0$ and any compact subset K of S,

$$d_3(d(K)/\varepsilon)^M\leq M(\varepsilon;K)\leq d_4(d(K)/\varepsilon)^M$$

holds, where $M(\varepsilon; K)$ is the maximal number of ε-distinguishable set on K.

Now under the following situation we have Lemma 2, essentially due to T. Sirao [7]. Let $\{X(s); s=(s_1, \cdots, s_d) \in S=S_1\times\cdots\times S_d\}$ be a Gaussian random field, where each S_i is a compact metric space with a metric ρ_i satisfying *the condition A with a dimension* N_i. We assume that $E[X(s)^2]=1$ and that there exist non-decreasing continuous functions $\sigma_i(x)$, $i=1, \cdots, d$ such that

$$(E[(X(s)-X((s)_i'))^2])^{1/2}\leq d_5\sigma_i(\rho_i(s_i, (s)_i')),$$

where $s=(s_1, \cdots, s_d)$, $(s)_i'=(s_1, \cdots, s_{i-1}, s_i', s_{i+1}, \cdots s_d)\in S$.

Lemma 2. *If there exist constants* $d_6>0$ *and* $\beta_i>0$, $i=1, \cdots, d$ *such that*

$$\sigma_i(tu)/\sigma_i(t)\leq d_6 u^{\beta_i}$$

holds for all $0<t\leq(\sum_{i=1}^d d(S_i)^2)^{1/2}$ *and* $0\leq u\leq 1$, *then there exists a constant* d_7 *independent of* d, d_5, x *and* S *such that*

$$P[\sup_{s\in S} X(s)\geq x]\leq d_7\Phi(x)\prod_{i=1}^d N(\varepsilon_i(x); S_i)$$

holds for any $x\geq 1$, *where* $\varepsilon_i(x)=\sigma_i^{-1}(1/(d\, d_5 x))/2$.

Proof. Take any compact subset K_i of S_i such that

$$d_5\sum_{i=1}^d \sigma_i(d(K_i))x\leq 1,$$

and set

$$K=K_1\times\cdots\times K_d,$$

$$F_{\sigma_i}(x)=d_5\int_0^\infty \sigma_i(x\,\mathrm{e}^{-u^2})du,$$

$$A=\{\omega; \sup_{s\in K} X(s)\geq x+c_9\sum_{i=1}^d F_{\sigma_i}(d(K_i))\}.$$

Then by the slight modification of Lemma 9 in [3], choosing a sufficiently large constant c_9, we have

$$P(A)\leq c_{10}\Phi(x), \tag{15}$$

where c_9 and c_{10} are constants independent of d, d_5, x and K_i. Now let $\{t_{i,j}; 1 \leq j \leq N(\varepsilon_i(x); S_i)\}$ be a $\varepsilon_i(x)$-net of S_i and set

$$B_{i,j} = \{s \in S_i; \rho_i(t_{i,j}, s) \leq \varepsilon_i(x)\},$$

$$A_{j_1,\dots,j_d} = \{\omega; \sup_{s \in B_{1,j_1} \times \cdots \times B_{d,j_d}} X(s) \geq x\},$$

$$E = \{\omega; \sup_{s \in S} X(s) \geq x\}.$$

Since it follows that

$$d_5 x \sum_{i=1}^{d} \sigma_i(d(B_{i,j})) \leq d_5 x \sum_{i=1}^{d} \sigma_i(2\varepsilon_i(x)) \leq 1,$$

and

$$\sum_{i=1}^{d} F_{\sigma_i}(d(B_{i,j})) \leq \sum_{i=1}^{d} F_{\sigma_i}(2\varepsilon_i(x))$$

$$\leq \sum_{i=1}^{d} \int_0^\infty d_6 \, e^{-\beta_i u^2} du/x,$$

applying (15) to $A_{j_1,\dots,j_d}$, we have

$$P(A_{j_1,\dots,j_d}) \leq d_7 \Phi(x),$$

where d_7 is a constant independent of d, d_5, x and $(j_1, \cdots, j_d)$. Therefore it follows that

$$P(E) = P(\bigcup_{(j_1,\dots,j_d)} A_{j_1,\dots,j_d})$$

$$\leq d_7 \Phi(x) \prod_{i=1}^{d} N(\varepsilon_i(x); S_i).$$

This completes the proof of Lemma 2.

Now we begin to prove Theorem 1 under the condition (8) by virtue of Lemma 1.

Choose a closed ball D_i of R^{N_i} such that $D \subset D_1 \times \cdots \times D_d$ and set

$$K_n = \{(s, t) \in D \times D; 2^{-n-1} \leq \gamma(s-t) \leq 2^{-n}\},$$

$$S_i = D_i \times D_i,$$

$$L_{n,i,1} = \{(s_i, s_i') \in S_i; \sigma_i(|s_i - s_i'|) \leq 2^{-n-2}/\sqrt{d}\},$$

$$L_{n,i,2} = \{(s_i, s_i') \in S_i; 2^{-n-2}/\sqrt{d} \leq \sigma_i(|s_i - s_i'|) \leq 2^{-n}\},$$

here S_i is naturally a metric space with a $2N_i$-dimensional Euclidean metric. Then easily we have

$$E\Big[\Big(\frac{X(s)-X(t)}{\gamma(s-t)}-\frac{X((s)_i')-X((t)_i')}{\gamma((s)_i'-(t)_i')}\Big)^2\Big]$$

$$=\{E[(X(s)-X((s)_i')+X((t)_i')-X(t))^2]-(\gamma(s-t)$$

$$-\gamma((s)_i'-(t)_i'))^2\}\{\gamma(s-t)\gamma((s)_i'-(t)_i')\}^{-1}$$

$$\leq c_{11}\sigma_i^2((|s_i-s_i'|^2+|t_i-t_i'|^2)^{1/2})2^{2n} \quad \text{on} \quad L_{n,1,i_1}\times\cdots\times L_{n,d,i_d}$$

for $(i_1,\cdots,i_d)\neq(1,\cdots,1)$,

$$N(\varepsilon;L_{n,i,\theta})\leq c_{12}(\sigma_i^{-1}(2^{-n})/\varepsilon)^{N_i}(d(D_i)/\varepsilon)^{N_i},\ \theta=1,2,$$

and

$$K_n\subset\bigcup_{(i_1,\cdots,i_d)\neq(1,\cdots,1)}L_{n,1,i_1}\times\cdots\times L_{n,d,i_d}.$$

Since each $\sigma_i(x)$ is a nearly regular varying function, the condition of Lemma 2 is satisfied, so setting

$$A_n=\{\omega;\sup_{(s,t)\in K_n}(X(s)-X(t))/\gamma(s-t)\geq\varphi(2^{-n})\},$$

from Lemma 2 we have

$$P(A_n)\leq c_{13}\Phi(\varphi(2^{-n}))\prod_{i=1}^{d}N(\varepsilon_i(\varphi(2^{-n}));L_{n,i,2})$$

$$\leq c_{14}\Phi(\varphi(2^{-n}))K(\bar{N}/2\bar{N};\gamma,\varphi)(2^{-n}),$$

where c_{13} and c_{14} are independent of n. Finally by virtue of nearly regular varyingness of $\sigma_i^{-1}(x)$ and the condition (8), we have

$$\sum_n P(A_n)\leq c_{15}I_u(\gamma;\varphi)<+\infty.$$

Therefore by Borel-Cantelli lemma, there exists an $n_0(\omega)$, with probability 1, such that

$$\sup_{(s,t)\in K_n}(X(s)-X(t))/\gamma(s-t)<\varphi(2^{-n})$$

holds for any $n\geq n_0(\omega)$. By non-increasingness of $\varphi(x)$, it follows that

$$X(s)-X(t)<\gamma(s-t)\varphi(\gamma(s-t))$$

holds for any $\gamma(s-t)<2^{-n_0(\omega)}$. This completes the proof of Theorem 1.

Proof of Theorem 2. First we notice that it is sufficient to prove the theorem under the condition (9) of Lemma 1. Set

$$K_n=\{s\in D;\, 2^{-n-1}\leq\gamma(s-t_0)\leq 2^{-n}\},$$

$$L_{n,i,1}=\{s_i\in R^{N_i};\, \sigma_i(|s_i-t_i^0|)\leq 2^{-n-2}/\sqrt{d}\},$$

$$L_{n,i,2}=\{s_i\in R^{N_i};\, 2^{-n-2}/\sqrt{d}\leq\sigma_i(|s_i-t_i^0|)\leq 2^{-n}\},$$

$$A_n=\{\omega;\, \sup_{s\in K_n}(X(s)-X(t_0))/\gamma(s-t_0)\geq\varphi(2^{-n})\},$$

where $t_0=(t_1^0,\cdots,t_d^0)$, $t_i^0\in R^{N_i}$. Then we have

$$E\left[\left(\frac{X(s)-X(t_0)}{\gamma(s-t_0)}-\frac{X((s)_i')-X(t_0)}{\gamma((s)_i'-t_0)}\right)^2\right]\leq c_{16}\sigma_i^2(|s_i-s_i'|)\,2^{2n}$$

on $L_{n,1,i_1}\times\cdots\times L_{n,d,i_d}$ for $(i_1,\cdots,i_d)\neq(1,\cdots,1)$,

$$N(\varepsilon;\, L_{n,i,\theta})\leq c_{17}(\sigma_i^{-1}(2^{-n})/\varepsilon)^{N_i},\quad \theta=1,2,$$

and

$$K_n\subset\bigcup_{(i_1,\cdots,i_d)\neq(1,\cdots,1)}L_{n,1,i_1}\times\cdots\times L_{n,k,i_d}.$$

Therefore by just the same argument of the proof of Theorem 1, we have

$$\begin{aligned}\sum_n P(A_n)&\leq c_{18}\sum_n\Phi(\varphi(2^{-n}))\prod_{i=1}^{d}N(\varepsilon_i(\varphi(2^{-n}));\, L_{n,i,2})\\ &\leq c_{19}\sum_n\Phi(\varphi(2^{-n}))K(\bar{N}/\bar{N};\,\gamma,\varphi)\,(2^{-n})\\ &\leq c_{20}I_t(\gamma;\,\varphi)<+\infty,\end{aligned}$$

and we get Theorem 2.

Proof of Theorem 3. First we notice that it is sufficient to prove the theorem under the condition (9) of Lemma 1. Set

$$D_i^+=\{(x_1,\cdots,x_{N_i})\in R^{N_i};\, t_{i,j}^0\leq x_j\leq t_{i,j}^0+1,\ 1\leq j\leq N_i\},$$

$$D_i^-=\{(x_1,\cdots,x_{N_i})\in R^{N_i};\, t_{i,j}^0-1\leq x_j\leq t_{i,j}^0,\ 1\leq j\leq N_i\},$$

where $t^0=(t_1^0, \cdots, t_d^0)$, $t_i^0=(t_{i,1}^0, \cdots, t_{i,N_i}^0)$,

$$K_n=\{(s, t)\in D^+\times D^-; 2^{-n-1}\leq\gamma(s-t)\leq 2^{-n}\},$$

$$L_{n,i,1}=\{(s_i, s_i')\in D_i^+\times D_i^-; \sigma_i(|s_i-s_i'|)\leq 2^{-n-2}/\sqrt{d}\},$$

$$L_{n,i,2}=\{(s_i, s_i')\in D_i^+\times D_i^-; 2^{-n-2}/\sqrt{d}\leq\sigma_i(|s_i-s_i'|)\leq 2^{-n}\},$$

and

$$A_n=\{\omega; \sup_{(s,t)\in K_n}(X(s)-X(t))/\gamma(s-t)\geq\varphi(2^{-n})\}.$$

Then we have

$$E\left[\left(\frac{X(s)-X(t)}{\gamma(s-t)}-\frac{X((s)_i')-X((t)_i')}{\gamma((s)_i'-(t)_i')}\right)^2\right]$$

$$\leq c_{21}\sigma_i^2((|s_i-s_i'|^2+|t_i-t_i'|^2)^{1/2})2^{2n}$$

$$\text{on}\quad L_{n,1,i_1}\times\cdots\times L_{n,d,i_d}$$

for $(i_1, \cdots, i_d)\neq(1, \cdots, 1)$,

$$N(\varepsilon; L_{n,i,\theta})\leq c_{22}(\sigma_i^{-1}(2^{-n})/\varepsilon)^{2N_i},\quad \theta=1, 2,$$

and

$$K_n\subset\bigcup_{(i_1,\cdots,i_d)\neq(1,\cdots,1)}L_{n,1,i_1}\times\cdots\times L_{n,d,i_d}.$$

Therefore just by the same argument of the proof of Theorem 1, we have

$$\sum_n P(A_n)\leq c_{23}\sum_n\Phi(\varphi(2^{-n}))\prod_{i=1}^d N(\varepsilon_i(\varphi(2^{-n})); L_{n,i,2})$$

$$\leq c_{24}\sum_n\Phi(\varphi(2^{-n}))K(2\overline{N}/2\overline{N}; \gamma, \varphi)(2^{-n})$$

$$\leq c_{25}I_{u\,t}(\gamma; \varphi)<+\infty,$$

and we get Theorem 3.

Proof of Theorem 4. First we notice that it is sufficient to prove the theorem under the condition (10) of Lemma 1. Set

$$K_n=\{s\in R^N; 2^n\leq\gamma(s)\leq 2^{n+1}\},$$

$$L_{n,i,1}=\{s_i\in R^{N_i}; \sigma_i(|s_i|)\leq 2^{n-1}/\sqrt{d}\},$$

$$L_{n,i,2}=\{s_i\in R^{N_i};\,2^{n-1}/\sqrt{d}\leq\sigma_i(|s_i|)\leq 2^{n+1}\},$$

$$A_n=\{\omega;\,\sup_{s\in K_n}(X(s)-X(0))/\gamma(s)\geq\varphi(2^n)\}.$$

Then we have

$$E\Big[\Big(\frac{X(s)-X(0)}{\gamma(s)}-\frac{X((s)_i')-X(0)}{\gamma((s)'_i)}\Big)^2\Big]\leq\gamma^2(s-(s)_i')\,(\gamma(s)\gamma((s)_i'))^{-1}$$

$$\leq c_{26}\sigma_i^2(|s_i-s_i'|)\,2^{-2n}\quad\text{on}\quad L_{n,1,i_1}\times\cdots\times L_{n,d,i_d}$$

for $(i_1,\cdots,i_d)\neq(1,\cdots,1)$,

$$N(\varepsilon;L_{n,i,\theta})\leq c_{27}(\sigma_i^{-1}(2^n)/\varepsilon)^{N_i},\ \theta=1,2,$$

and

$$K_n\subset\bigcup_{(i_1,\cdots,i_d)\neq(1,\cdots,1)}L_{n,1,i_1}\times\cdots\times L_{n,d,i_d}.$$

Therefore just by the same argument of the proof of Theorem 1 we have

$$\sum_n P(A_n)\leq c_{28}\sum_n\Phi(\varphi(2^n))\prod_{i=1}^{d}N(\varepsilon_i(\varphi(2^n));L_{n,i,2})$$

$$\leq c_{29}\sum_n\Phi(\varphi(2^n))K(\overline{N}/\overline{N};\gamma,\varphi)(2^n)$$

$$\leq c_{30}I_i^\infty(\gamma;\varphi)<+\infty,$$

and we get Theorem 4.

Proof of Theorem 5. First we notice that it is sufficient to prove the theorem under the condition (11) of Lemma 1. Set

$$C_n=\{s\in R^N;\ \text{absolute value of each coordinate of } s\leq n\},$$

$$K_n=C_{n+1}-C_n^{\,i}=\bigcup_{j=1}^{p_n}S_{n,j},\ \ (C_n^{\,i}\text{ is the interior of }C_n)$$

$$A_{n,j}=\{\omega;\,\sup_{s\in S_{n,j}}X(s)\geq\varphi(n)\},$$

where $S_{n,j}$ is a N-dimensional cube whose coordinate of vertexes are integer and $p_n\leq 2^N(n+1)^{N-1}$. Then we have

$$E[(X(s)-X((s)_i'))^2]=\gamma^2(s-(s)_i')$$

$$=\sigma_i^2(|s_i-s_i'|).$$

Therefore applying Lemma 2 to $A_{n,j}$ we have

$$P(A_{n,j}) \leq c_{31} \Phi(\varphi(n)) / K(\bar{N}; \gamma)(1/\varphi(n))$$

and

$$\sum_n \sum_{j=1}^{p_n} P(A_{n,j}) \leq c_{32} \sum_n \Phi(\varphi(n)) n^{N-1} / K(\bar{N}; \gamma)(1/\varphi(n))$$

$$\leq c_{33} I_b^{\infty}(\gamma; \varphi) < +\infty .$$

This proves Theorem 5.

4. Lowerbounds.

To prove our theorems about *the lower classes*, we have to prepare some lemmas. The following Lemma 3 is essentially obtained by rewriting Lemma 2.1 of [11].

Let $\{X(s); s=(s_1, \cdots, s_d) \in S=S_1 \times \cdots \times S_d\}$ be a Gaussian random field with a parameter space S and let each S_i be a compact metric space with a metric ρ_i satisfying *the condition B with a dimension* N_i. Naturally S is a compact metric space. We assume that $E[X(s)^2] = 1$ and that there exist non-decreasing continuous functions $\sigma_i(x)$, $i=1, \cdots, d$ such that

$$E[(X(s)-X(t))^2] \geq d_8 \sum_{i=1}^{d} \sigma_i^2(\rho_i(s_i, t_i)),$$

where $s=(s_1, \cdots, s_d)$, $t=(t_1, \cdots, t_d)$, $s_i, t_i \in S_i$. Set

$$A(\varepsilon_1, \cdots, \varepsilon_d; S_1 \times \cdots \times S_d; x)$$

$$= \{\omega; \sup_{j_1, \cdots, j_d} X(t_{j_1}^{(1)}, \cdots, t_{j_d}^{(d)}) \geq x\},$$

where $\{t_{j_i}^{(i)}; 1 \leq j_i \leq M(\varepsilon_i; S_i)\}$ is an ε_i-distinguishable set of S_i. Then we have

Lemma 3. *If there exist constants d_9 and $\delta_i > 0$, $i=1, \cdots, d$ such that*

$$\sigma_i(tu)/\sigma_i(t) \geq d_9 u^{\delta_i}$$

holds for all $0 < t \leq (\sum_{i=1}^{d} d(S_i)^2)^{1/2}$ and $u \geq 1$, then there exists a positive constant d_{10} independent of d_8, x and S such that

$$P(A(\varepsilon_1(x),\cdots,\varepsilon_d(x);S_1\times\cdots\times S_d;x))$$
$$\geq \tfrac{1}{2}\Phi(x)\prod_{i=1}^{d} M(\varepsilon_i(x);S_i)$$

holds for all $x\geq d_{10}/\{\sqrt{d_8}\min\{\sigma_i(d(S_i))\}\}$, *where* $\varepsilon_i(x)=\sigma_i^{-1}(d_{10}/(\sqrt{d_8}x))$.

Proof. Denote by $A_{j_1,\cdots,j_d}$ the event $\{\omega;X(t_{j_1}^{(1)},\cdots,t_{j_d}^{(d)})\geq x\}$, then we have

$$P(A(\varepsilon_1(x),\cdots,\varepsilon_d(x);S_1\times\cdots\times S_d,x))$$
$$\geq\sum_{(j_1,\cdots,j_d)}P(A_{j_1,\cdots,j_d})-\sum_{\substack{(j_1,\cdots,j_d)\\ \neq(k_1,\cdots,k_d)}}P(A_{j_1,\cdots,j_d}\cap A_{k_1,\cdots,k_d}).\tag{16}$$

In order to estimate the second term of the right hand side of (16), set

$$B(j_i,n_i)=\{s\in S_i;n_i\varepsilon_i(x)\leq\rho_i(t_{j_i}^{(i)},s)\leq(n_i+1)\varepsilon_i(x)\},$$
$$n_i=1,2,\cdots.$$

By virtue of *the condition B with a dimension* N_i, the cardinal number of the set $\{t_{k_i}^{(i)};t_{k_i}^{(i)}\in B(j_i,n_i)\}$ is not larger than $d_4(n_i+1)^{N_i}$. Since it follows that

$$P(A_{j_1,\cdots,j_d}\cap A_{k_1,\cdots,k_d})\leq c_{34}\Phi(x)\exp\left\{-\frac{1-r}{4}x^2\right\},$$

where

$$1-r\equiv1-E[X(t_{j_1}^{(1)},\cdots,t_{j_d}^{(d)})X(t_{k_1}^{(1)},\cdots t_{k_d}^{(d)})]$$
$$=E[(X(t_{j_1}^{(1)},\cdots,t_{j_d}^{(d)})-X(t_{k_1}^{(1)},\cdots t_{k_d}^{(d)}))^2]/2$$
$$\geq d_8/2\sum_{i=1}^{d}\sigma_i^2(\rho_i(t_{j_i}^{(i)},t_{k_i}^{(i)}))$$
$$\geq d_8/2\sum_{i=1}^{d}\sigma_i^2(n_i\varepsilon_i(x))\quad\text{for}\quad t_{k_i}^{(i)}\in B(j_i,n_i),$$

and c_{34} is a constant independent of x and r, we have

$$\sum_{\substack{(j_1,\cdots,j_d)\\ \neq(k_1,\cdots,k_d)}}P(A_{j_1,\cdots,j_d}\cap A_{k_1,\cdots,k_d})$$
$$\leq2c_{34}\Phi(x)\sum_{(j_1,\cdots,j_d)}\sum_{\substack{1\leq i\leq d\\ n_i\geq1}}d_4(n_i+1)^{N_i}\exp\{-d_8x^2\sum_{i=1}^{d}\sigma_i^2(n_i\varepsilon_i(x))/8\}$$

$$\leq 2c_{34}d_4\Phi(x)\prod_{i=1}^{d} M(\varepsilon_i(x)\,;S_i)\sum_{\substack{1\leq i\leq d\\ n_i\geq 1}}(n_i+1)^{N_i}\exp\{-d_{10}^2d_9^2\sum_{i=1}^{d} n_i^{2\delta_i}/8\}.$$

Therefore we can choose a sufficiently large d_{10} such that

$$2c_{34}d_4\sum_{\substack{1\leq i\leq d\\ n_i\geq 1}}(n_i+1)^{N_i}\exp\{-d_9^2d_{10}^2\sum_{i=1}^{d} n_i^{2\delta_i}/8\}\leq 1/2. \tag{17}$$

Combining (16) and (17) we get the proof of Lemma 3.

Lemma 4. *Let $\sigma_i(x)$ be a function which satisfies the assumption L without the additional condition for Theorem 9 and let s_p, $1\leq p\leq 4$ be four points of R^{N_i}. Set $|s_p-s_q|=r_{pq}$ and*

$$R_i\equiv R_i(s_1, s_2\,; s_3, s_4)=\sigma_i^2(r_{14})+\sigma_i^2(r_{23})-\sigma_i^2(r_{13})-\sigma_i^2(r_{24}).$$

Then there exists a constant d_{11} independent of r_{ij} and r_0 such that

$$|R_i|\leq d_{11}(r_{12}r_{34}/r_0^2)^{1-\alpha}\sigma_i(r_{12})\sigma_i(r_{34}) \tag{18}$$

if $+\infty>d_{12}>r_{13}, r_{24}, r_{14}, r_{23}\geq r_0\geq r_{12}/4\geq r_{34}/4,$

$$|R_i|\leq d_{11}(r_{34}/r_{12})^{\alpha'}\sigma_i(r_{12})\sigma_i(r_{34}) \tag{19}$$

if $+\infty>d_{12}>r_{12}/4\geq r_{23}\wedge r_{24}\geq r_{34},$

and

$$|R_i|\leq d_{11}(r_{34}/r_{12})^{1-\alpha}\sigma_i(r_{12})\sigma_i(r_{34})+d_{11}\sigma_i^2(r_{34}) \tag{20}$$

if $+\infty>d_{12}>r_{12}/4\geq r_{34}\geq r_{23}\wedge r_{24},$

where $1>\alpha>\max(\alpha_1, \cdots, \alpha_d)$, $\min(\alpha_1, \cdots, \alpha_d, 1-\alpha))>\alpha'>0$. Moreover if each $\sigma_i(x)$ satisfies the additional condition for Theorem 9, we can drop d_{12} by setting $1>\alpha>\max(\alpha_1, \cdots, \alpha_d, \alpha_1', \cdots, \alpha_d')$, $\min(\alpha_1, \alpha_1', \cdots, \alpha_d', 1-\alpha)>\alpha'>0$.

Proof. By twice continuous differentiability of $\sigma_i^2(x)$, it follows that

$$R_i(s_1, s_2\,; s_3, s_4)=(r_{14}+r_{23}-r_{24}-r_{13})\frac{d\sigma_i^2(r_{24})}{dx}$$
$$+\int_{r_{24}}^{r_{23}}(r_{23}-x)\frac{d^2\sigma_i^2(x)}{dx^2}dx+\int_{r_{13}}^{r_{14}}(r_{13}-x)\frac{d\sigma_i^2(x)}{dx^2}dx$$

$$+(r_{14}-r_{13})\int_{r_{24}}^{r_{14}}\frac{d^2\sigma_i(x)}{dx^2}dx. \tag{21}$$

By virtue of the four point propetry (c.f. Lemma 7 of [3]) and the Assumption L, we have

$$|r_{14}+r_{23}-r_{24}-r_{13}|\leq 4r_{12}r_{34}/r_{24},$$

$$4r_{12}r_{34}\frac{d\sigma_i^2(r_{24})}{dx}/r_{24}\leq c_{35}r_{12}r_{34}\sigma_i^2(r_0)/r_0^2$$

$$=c_{35}(r_{12}r_{34}/r_0^2)(\sigma_i(r_0)/\sigma_i(r_{34}))(\sigma_i(r_0)/\sigma_i(r_{12}))\sigma_i(r_{12})\sigma_i(r_{34})$$

$$\leq c_{36}(r_{12}r_{34}/r_0^2)^{1-\alpha}\sigma_i(r_{12})\sigma_i(r_{34}),$$

and

$$\left|\int_{r_{24}}^{r_{23}}(r_{23}-x)\frac{d^2\sigma_i^2(x)}{dx^2}dx\right|\leq r_{34}\int_{r_{24}}^{r_{23}}\left|\frac{d^2\sigma_i^2(x)}{dx^2}\right|dx$$

$$\leq c_{37}r_{34}^2\sigma_i^2(r_0)/r_0^2$$

$$\leq c_{38}(r_{12}r_{34}/r_0^2)^{1-\alpha}\sigma_i(r_{12})\sigma_i(r_{34}),$$

where c_{36} and c_{38} are constants depending on α and d_{12}. Since we obtain the same estimates as above with respect to the third and the fourth term of (21) respectively, we get the first argument of Lemma 4.

Next let us prove the second part of Lemma 4. In this case, by differentiability, nearly regular varyingness of $\sigma_i(x)$ and the relation r_{13}, $r_{14}\geq r_{12}/2$, we have

$$|\sigma_i^2(r_{14})-\sigma_i^2(r_{13})|=\left|\int_{r_{13}}^{r_{14}}\frac{d\sigma_i^2(x)}{dx}dx\right|$$

$$\leq c_{39}r_{34}\sigma_i^2(r_{12})/r_{12}$$

$$=c_{39}(r_{34}/r_{12})(\sigma_i(r_{12})/\sigma_i(r_{34}))\sigma_i(r_{12})\sigma_i(r_{34})$$

$$\leq c_{40}(r_{34}/r_{12})^{1-\alpha}\sigma_i(r_{34})\sigma(r_{12}), \tag{22}$$

and

$$|\sigma_i^2(r_{23})-\sigma_i^2(r_{24})|=\left|\int_{r_{24}}^{r_{23}}\frac{d\sigma_i^2(x)}{dx}dx\right|$$

$$\leq c_{41}r_{34}\sigma_i^2(r_{23}\wedge r_{24})/(r_{23}\wedge r_{24})$$

$$\leq c_{42}(r_{34}/r_{23}\wedge r_{24})(\sigma_i^2(r_{23}\wedge r_{24})/\sigma_i(r_{34}))$$
$$\times(\sigma_i(r_{23}\wedge r_{24})/\sigma_i(r_{12}))\sigma_i(r_{12})\sigma_i(r_{34})$$
$$\leq c_{43}(r_{34}/r_{23}\wedge r_{24})^{1-\alpha}(r_{23}\wedge r_{24}/r_{12})^{\alpha'}\sigma_i(r_{12})\sigma_i(r_{34})$$
$$\leq c_{43}(r_{34}/r_{12})^{\alpha'}\sigma_i(r_{12})\sigma_i(r_{34}). \tag{23}$$

Combining (22) and (23) we get (19). In the third case, (22) is still valid and from the relation $\sigma_i^2(r_{23})+\sigma_i^2(r_{24})\leq 2\sigma_i^2(r_{23}\wedge r_{24}+r_{34})\leq c_{44}\sigma_i^2(r_{34})$, we get (20). This completes the proof of Lemma 4.

Lemma 5. [10] *Let $\{X_{n,k}; n=1,2,\cdots, k=1,2,\cdots,k(n)\}$ be a Gaussian system such that each random variable obeys a standard normal distribution and set*

$$r_{(n,k)}^{(m,j)}=E[X_{n,k}X_{m,j}],$$
$$A_{n,k}=\{\omega; X_{n,k}\geq x_n\},$$
$$B_n=\bigcap_{k=1}^{k(n)}A_{n,k}^c.$$

Then it follows that

$$|P(\bigcap_{n=1}^{\infty}B_n)-\prod_{n=1}^{\infty}P(B_n)|$$
$$\leq 1/2\sum_{n\neq m}\sum_{k=1}^{k(n)}\sum_{j=1}^{j(m)}|r_{(n,k)}^{(m,j)}|\int_0^1 g(\lambda r_{(n,k)}^{(m,j)}; x_n, x_m)d\lambda, \tag{24}$$

where $g(\lambda r,x,y)=(2\pi\sqrt{1-\lambda^2r^2})^{-1}\exp\{-(x^2+y^2-2\lambda rxy)/2(1-\lambda^2r^2)\}$. Therefore we notice that if $\sum_{n=1}^{\infty}P(B_n^c)=+\infty$ and the right hand side of (24) converges then we have $P(\overline{\lim}\,B_n^c)=1$.

Now we begin to prove Theorem 6. First we notice that it is sufficient to prove the theorem under the condition (8) of Lemma 1. Choose a closed ball D_i of R^{N_i} such that $D\supset D_1\times\cdots\times D_d$ and let $\{t_{n,j}^{(i)}; 1\leq j\leq M(\varepsilon_n^{(i)}; D_i)\}$ be an $\varepsilon_n^{(i)}$-distinguishable set of D_i, where $\varepsilon_n^{(i)}=\sigma_i^{-1}(\delta^n)$ and a constant δ $(1>\delta>0)$ is chosen suitably later. Set

$$V_{n,j}^{(i)}=\{s\in D_i; |s-t_{n,j}^{(i)}|\leq\sigma_i^{-1}(a\delta^n)\}, \tag{25}$$

then we can make each ball contain two closed balls $B_{n,j,1}^{(i)}$ and $B_{n,j,2}^{(i)}$

such that $d(B_{n,j,1}^{(i)})=d(B_{n,j,2}^{(i)})=\sigma_i^{-1}(b\delta^n)$ and $\inf\{|s-t|;s\in B_{n,j,1}^{(i)},t\in B_{n,j,2}^{(i)}\}=\sigma_i^{-1}(c\delta^n)$, where a,b and c $(c>b)$ are constants independent of n chosen suitably later. To apply Lemma 3, let us estimate the following:

$$v(s,t;s',t')\equiv E\left[\left(\frac{X(s)-X(t)}{\gamma(s-t)}-\frac{X(s')-X(t')}{\gamma(s'-t')}\right)^2\right]$$

$$=\{\gamma^2(s-s')+\gamma^2(t-t')-(\gamma(s-t)-\gamma(s'-t'))^2$$

$$+\sum_{i=1}^d R_i\}(\gamma(s-t)\gamma(s'-t'))^{-1},$$

where

$R_i=R_i(s_i,s_i';t_i',t_i)$, (defined in Lemma 4),

$s=(s_1,\cdots,s_d)$, $s'=(s_1',\cdots,s_d')$,

$t=(t_1,\cdots,t_d)$, $t'=(t_1',\cdots,t_d')$,

$s_i,s_i'\in B_{n,j,1}^{(i)}$, $t_i,t_i'\in B_{n,j,2}^{(i)}$.

Then by Lemma 4 we have

$$|R_i|\le c_{45}(b/c)^{2(1-\alpha)/\alpha_i}\sigma_i^2((|s_i-s_i'|^2+|t_i-t_i'|)^{1/2}).$$

On the other hand by virtue of differentiability of $\sigma_i(x)$, we have

$$(\gamma(s-t)-\gamma(s'-t'))^2=(\gamma^2(s-t)-\gamma^2(s'-t'))^2/(\gamma(s-t)+\gamma(s'-t'))^2$$

$$\le c_{46}\sum_{i=1}^d(b/c)^{(2-2\alpha)/\alpha_i}\sigma_i^2((|s_i-s_i'|^2+|t_i-t_i'|^2)^{1/2}).$$

Since we can choose constants b and c such that b/c is sufficiently small, we have

$$v(s,t;s',t')\ge c_{47}\delta^{-2n}\sum_{i=1}^d\sigma_i^2((|s_i-s_i'|^2+|t_i-t_i'|^2)^{1/2})$$

if $s=(s_1,\cdots,s_d)$, $s'=(s_1',\cdots,s_d')\in B_{n,j_1,1}^{(1)}\times\cdots\times B_{n,j_d,1}^{(d)}$

$t=(t_1,\cdots,t_d)$, $t'=(t_1',\cdots,t_d')\in B_{n,j_1,2}^{(1)}\times\cdots\times B_{n,j_d,2}^{(d)}$,

where c_{47} is a constant independent of n and δ.

Now set

$$A_{(\bar\mu)}^{(\nu)}(n;j_1,\cdots,j_d)=\left\{\omega;\frac{X(s(\bar\mu))-X(s'(\bar\nu))}{\gamma(s(\bar\mu)-s'(\bar\nu))}\ge\varphi(c\delta^n)\right\},$$

$$A^{(n)}_{(j_1,\cdots,j_d)}=\bigcup_{(\bar\mu),(\bar\nu)} A^{(\bar\nu)}_{(\bar\mu)}\quad (n;j_1,\cdots,j_d),$$

$$s(\bar\mu)=(s^{(1)}_{\mu_1},\cdots,s^{(d)}_{\mu_d}),\ s'(\bar\nu)=(s'^{(1)}_{\nu_1},\cdots,s'^{(d)}_{\nu_d}),$$

$$(\bar\mu)=(\mu_1,\cdots,\mu_d),\ (\bar\nu)=(\nu_1,\cdots,\nu_d),$$

$$1\le\mu_i\le M(\varepsilon^{(i)}_n;B^{(i)}_{n,j_i,1}),\ 1\le\nu_i\le M(\varepsilon^{(i)}_n;B^{(i)}_{n,j_i,2}),$$

where $\{s^{(i)}_{\mu_i}\}$, $\{s'^{(i)}_{\nu_i}\}$ are $\varepsilon^{(i)}_n$-distinguishable set of $B^{(i)}_{n,j_i,1}$ and $B^{(i)}_{n,j_i,2}$ respectively. Then setting

$$\varepsilon^{(i)}_n=c_{48}\sigma_i^{-1}(\delta^n/\varphi(c\delta^n))$$

in Lemma 3, we have

$$P(A^{(n)}_{(j_1,\cdots,j_d)})\ge 1/2\Phi(\varphi(c\delta^n))\prod_{i=1}^{d}M(\varepsilon^{(i)}_n;B^{(i)}_{n,j_i,1}\times B^{(i)}_{n,j_i,2})$$

$$\ge c_{49}\Phi(\varphi(c\delta^n))K(2\bar N/2\bar N;\gamma,\varphi)(c\delta^n).$$

Since the cardinal number of $V^{(1)}_{n,j_1}\times\cdots\times V^{(d)}_{n,j_d}$ for fixed n is larger than $c_{50}/K(\bar N;\gamma)(c\delta^n)$, we have

$$\sum_n\sum_{(j_1,\cdots,j_d)}P(A^{(n)}_{(j_1,\cdots,j_d)})\ge c_{51}\sum_n\Phi(\varphi(c\delta^n))K(\bar N/2\bar N;\gamma,\varphi)(c\delta^n)$$

$$\ge c_{52}I_u(\gamma;\varphi)=+\infty. \tag{26}$$

Next with the help of Lemma 5, we have to show

$$P(\bigcap_{(n,j_1,\cdots,j_d)}A^{(n)c}_{(j_1,\cdots,j_d)})=0.$$

Set

$$\Lambda_1(n,\bar j)=\{(m,k_1,\cdots k_d);\ m\ge n+c_{53}\log n\},$$

$$\Lambda_2(n,\bar j)=\{(m,k_1,\cdots,k_d);\ n\le m\le n+c_{53}\log n,$$

$$|t^{(i)}_{n,j_i}-t^{(i)}_{m,k_i}|\ge n^{c_{54}}\sigma_i^{-1}(\delta^n),\ i=1,\cdots,d\},$$

$$\Lambda_{3,i}(n,\bar j)=\{(m,k_1,\cdots,k_d)\ne(n,j_1,\cdots j_d);\ n\le m\le n+c_{53}\log n,$$

$$|t^{(i)}_{n,j_i}-t^{(i)}_{m,k_i}|\le n^{c_{54}}\sigma_i^{-1}(\delta^n)\},$$

where $c_{53}>\alpha(2N/\alpha'+3)/(\alpha'\log 1/\delta)$ and $c_{54}>\alpha(2N/\alpha'+3)/\{2(1-\alpha)\}$. Then by Lemma 5, we have

$$|P(\bigcap_{(n,j_1,\cdots,j_d)}A^{(n)c}_{(j_1,\cdots,j_d)})-\prod_{(n,j_1,\cdots,j_d)}P(A^{(n)c}_{(j_1,\cdots,j_d)})$$

$$\leq \sum_{(n,j_1,\cdots,j_d)} \{ \sum_{A_1(n,\bar{j})} + \sum_{A_2(n,\bar{j})} + \sum_{i=1}^{d} \sum_{A_{3,i}} \} \sum_{(\bar{\mu}),(\bar{\nu})} \sum_{(\bar{\mu}'),(\bar{\nu}')}$$

$$|r^{(m,\bar{k},\bar{\mu}',\bar{\nu}')}_{(n,\bar{j},\bar{\mu},\bar{\nu})}| \left| \int_0^1 g(\lambda r^{(m,\bar{k},\bar{\mu}',\bar{\nu}')}_{(n,\bar{j},\bar{\mu},\bar{\nu})}; \varphi(c\delta^n), \varphi(c\delta^m)) d\lambda \right.$$

$$\equiv I_1 + I_2 + I_3,$$

where

$$r^{(m,\bar{k},\bar{\mu}',\bar{\nu}')}_{(n,\bar{j},\bar{\mu},\bar{\nu})} = E\left[\frac{X(s(\bar{\mu})) - X(s'(\bar{\nu}))}{\gamma(s(\bar{\mu}) - s'(\bar{\nu}))} \cdot \frac{X(s(\bar{\mu}')) - X(s'(\bar{\nu}'))}{\gamma(s(\bar{\mu}') - s'(\bar{\nu}))} \right]$$

$$= \{\gamma(s(\bar{\mu}) - s'(\bar{\nu}))\gamma(s(\bar{\mu}') - s'(\bar{\nu}'))\}^{-1} \sum_{i=1}^{d} R_i/2,$$

$$R_i = R_i(s^{(i)}_{\mu_i}, s'^{(i)}_{\nu_i}; s^{(i)}_{\mu_i'}, s'^{(i)}_{\nu_i'}),$$

$$s(\bar{\mu}) = (s^{(1)}_{\mu_1}, \cdots, s^{(d)}_{\mu_d}) \in B^{(1)}_{n,j_1,1} \times \cdots \times B^{(d)}_{n,j_d,1},$$

$$s'(\bar{\nu}) = (s'^{(1)}_{\nu_1}, \cdots, s'^{(d)}_{\nu_d}) \in B^{(1)}_{n,j_1,2} \times \cdots \times B^{(d)}_{n,j_d,2},$$

$$s(\bar{\mu}') = (s^{(1)}_{\mu_1'}, \cdots, s^{(d)}_{\mu_d'}) \in B^{(1)}_{m,k_1,1} \times \cdots \times B^{(d)}_{m,k_d,1},$$

$$s'(\bar{\nu}') = (s'^{(1)}_{\nu_1'}, \cdots, s'^{(d)}_{\nu_d'}) \in B^{(1)}_{m,k_1,2} \times \cdots \times B^{(d)}_{m,k_d,2}.$$

Choosing a of (25) and δ sufficiently small, then by Lemma 4 we have

$$|r^{(m,\bar{k},\bar{\mu}',\bar{\nu}')}_{(n,\bar{j},\bar{\mu},\bar{\nu})}| \leq c_{55}\delta^{(m-n)\alpha'/\alpha} \quad \text{and} \quad |r^{(m,\bar{k},\bar{\mu}',\bar{\nu}')}_{(n,\bar{j},\bar{\mu},\bar{\nu})}\varphi(\delta^n)\varphi(\delta^m)| \leq c_{56}$$

if $(m, k_1, \cdots, k_d) \in A_1(n, \bar{j})$,

$$|r^{(m,\bar{k},\bar{\mu}',\bar{\nu}')}_{(n,\bar{j},\bar{\mu},\bar{\nu})}| \leq c_{57} n^{-c_{54}2(1-\alpha)/\alpha} \quad \text{and} \quad |r^{(m,\bar{k},\bar{\mu}',\bar{\nu}')}_{(n,\bar{j},\bar{\mu},\bar{\nu})}\varphi(c\delta^n)\varphi(c\delta^m)| \leq c'_{57}$$

if $(m, k_1, \cdots, k_d) \in A_2(n, \bar{j})$, and

$$|r^{(m,\bar{k},\bar{\mu}',\bar{\nu}')}_{(n,\bar{j},\bar{\mu},\bar{\nu})}| \leq \varepsilon < \min(N_1, \cdots, N_d)/N(\bar{\alpha})$$

if $(m, k_1, \cdots, k_d) \in A_{3,i}(n, \bar{j})$, $i = 1, \cdots, d$, $N(\bar{\alpha}) = \sum_{i=1}^{d} N_i/\alpha_i$, where $c_{55} \sim c'_{57}$ are independent of $(n, \bar{j}, \bar{\mu}, \bar{\nu})$ and $(m, \bar{k}, \bar{\mu}', \bar{\nu}')$. Since the cardinal number of $(\mu_1, \cdots, \mu_d)$ and $(\nu_1, \cdots, \nu_d)$ are less than $c_{58} n^{N/\alpha'}$ for each $(n, j_1, \cdots, j_d)$, and the cardinal number of $(n, j_1, \cdots, j_d)$ for fixed n is less than $c_{59}(K(\bar{N}; \gamma)(c\delta^n))^{-1}$, we have

$$I_1 \leq c_{60} \sum_{(n,j_1,\cdots,j_d)} \sum_{A_1(n,\bar{j})} \delta^{(m-n)\alpha'/\alpha} n^{N/\alpha'} m^{N/\alpha'} \exp\{-(\varphi^2(c\delta^n) + \varphi^2(c\delta^m))/2\}$$

$$\leq c_{61} \sum_{n} \sum_{m \geq n + c_{53}\log n} \delta^{(m-n)\alpha'/\alpha} n^{N/\alpha'+1} m^{N/\alpha'+1}$$

$$\leq c_{62}\sum_{n} n^{2N/\alpha'+2-c_{53}(\log 1/\delta)\alpha'/\alpha} < +\infty ,$$

$$I_2 \leq c_{63} \sum_{(n,j_1,\cdots,j_d)} \sum_{A_2(n,j)} n^{N/\alpha'} m^{N/\alpha'} n^{-c_{54}2(1-\alpha)/\alpha} \exp\{-(\varphi(c\delta^n)+\varphi(c\delta^m))/2\}$$

$$\leq c_{64}\sum_{n} (\log n) n^{2N/\alpha'+2-c_{54}2(1-\alpha)/\alpha} < +\infty .$$

Since the cardinal number of $(m, k_1, \cdots, k_d) \in A_{3,i}$ for fixed m and i is less than $c_{65}n^{c_{54}N_i}[\sigma_i^{-1}(c\delta^n)]^{N_i}/K(\overline{N};\gamma)(c\delta^n)$, we have

$$I_3 \leq c_{66}\sum_{i=1}^{d} \sum_{(n,j_1,\cdots,j_d)} \sum_{A_{3,i}} n^{N/\alpha'} m^{N/\alpha'} \exp\{-(\varphi^2(c\delta^n)$$

$$+\varphi^2(c\delta^m)-2\varepsilon\varphi(c\delta^n)\varphi(c\delta^m))/2\}$$

$$\leq c_{67}\sum_{i=1}^{d}\sum_{n} n^{2N/\alpha'+c_{54}N_i+2}[\sigma_i^{-1}(c\delta^n)]^{N_i}[K(\overline{N};\gamma)(c\delta^n)]^{-\varepsilon} < +\infty.$$

These yield $I_1+I_2+I_3 < +\infty$. Therefore with the help of (26) we have

$$P(\cap A^{(n)c}_{(j_1,\cdots,j_d)}) = 0. \tag{27}$$

This completes the proof of Theorem 6.

Proof of Theorem 7. First we notice that it is sufficient to prove the theorem under the condition (9) of Lemma 1. Choose a sequence of points $(t_1^{(n)}, \cdots, t_d^{(n)}) \in D$ such that

$$|t_i^{(n)}-t_i^0| = (1+c)\sigma_i^{-1}(\delta^n),\ 1>\delta>0,\ 1>c>0,$$

$i=1, \cdots, d$, and set

$$V_n^{(i)} = \{s_i \in D_i;\ |t_i^{(n)}-s_i| \leq c\sigma_i^{-1}(\delta^n)\},$$

where δ and c are constants indepent of n chosen suitably later. To apply Lemma 3, let us estimate the following:

$$v(s,t) \equiv E\Big[\Big(\frac{X(s)-X(t_0)}{\gamma(s-t_0)}-\frac{X(t)-X(t_0)}{\gamma(t-t_0)}\Big)^2\Big]$$

$$=(\gamma(s-t_0)\gamma(t-t_0))^{-1}\{\gamma^2(s-t)-(\gamma(s-t_0)-\gamma(t-t_0))^2\}.$$

By the Assumption L for $\sigma_i(x)$, we have

$$(\gamma(s-t_0)-\gamma(t-t_0))^2 = (\gamma^2(s-t_0)-\gamma^2(t-t_0))^2/(\gamma(s-t_0)+\gamma(t-t_0))^2$$

$$\leq \sum_{i=1}^{d} c_{68}|s_i - t_i|^2 \sigma_i^2(r_i)/r_i^2,$$

where $r_i = |s_i - t_i^0| \wedge |t_i - t_i^0|$. Therefore choosing sufficiently small $c>0$, we have

$$v(s, t) \geq c_{69}\delta^{-2n}\gamma^2(s-t),$$

for $s, t \in V_n^{(1)} \times \cdots \times V_n^{(d)}$, where c_{69} is a constant independent of n. Now set

$$A_{(n, j_1, \cdots, j_d)} = \{\omega;\ (X(s(\bar{j})) - X(t_0))/\gamma(s(\bar{j}) - t_0) \geq \varphi(\delta^n)\},$$

$$A_n = \bigcup_{(j_1, \cdots, j_d)} A_{(n, j_1, \cdots, j_d)},$$

$$s(\bar{j}) = (s_{j_1}^{(1)}, \cdots, s_{j_d}^{(d)}),$$

where $\{s_{j_i}^{(i)}\}$ is a $\varepsilon_n^{(i)}$-distinguishable set of $V_n^{(i)}$. Then setting

$$\varepsilon_n^{(i)} = c_{70}\sigma_i^{-1}(\delta^n/\varphi(\delta^n))$$

in Lemma 3, we have

$$P(A_n) \geq 1/2\Phi(\varphi(\delta^n)) \prod_{i=1}^{d} M(\varepsilon_n^{(i)}; V_n^{(i)})$$

$$\geq c_{71}\Phi(\varphi(\delta^n)) K(\bar{N}/\bar{N}; \gamma, \varphi)(\delta^n).$$

Hence we have

$$\sum_n P(A_n) \geq c_{72} I_t(\gamma; \varphi) = +\infty.$$

Next with the help of Lemma 5, we have to show $P(\bigcap_n A_n^c) = 0$. By Lemma 5, we have

$$|P(\bigcap_n A_n^c) - \prod_n P(A_n^c)|$$

$$\leq \sum_n \Big(\sum_{m \geq m(n)} + \sum_{n < m < m(n)} \Big) \sum_{(j_1, \cdots, j_d)} \sum_{(k_1, \cdots, k_d)}$$

$$|r_{(n, \bar{j})}^{(m, \bar{k})}| \int_0^1 g(\lambda r_{(n, \bar{j})}^{(m, \bar{k})}; \varphi(\delta^n), \varphi(\delta^m))\, d\lambda$$

$$\equiv I_1 + I_2,$$

where

$$r_{(n, \bar{j})}^{(m, \bar{k})} = E\Big[\frac{X(s(\bar{j})) - X(t_0)}{\gamma(s(\bar{j}) - t_0)} \cdot \frac{X(t(\bar{k})) - X(t_0)}{\gamma(t(\bar{k}) - t_0)} \Big]$$

$$= (\gamma(s(\bar{j}) - t_0)\gamma(t(\bar{k}) - t_0))^{-1} \sum_{i=1}^{d} R_i/2,$$

$$R_i = R_i(s_{j_i}^{(i)}, t_i^0; t_{k_i}^{(i)}, t_i^0),$$

$$s(\bar{j}) = (s_{j_1}^{(1)}, \cdots, s_{j_d}^{(d)}) \in V_n^{(1)} \times \cdots \times V_n^{(d)},$$

$$t(\bar{k}) = (t_{k_1}^{(1)}, \cdots, t_{k_d}^{(d)}) \in V_m^{(1)} \times \cdots \times V_m^{(d)},$$

$$m(n) = n + c_{73} \log_{(2)} n, \quad c_{73} > \alpha/(\alpha' \log 1/\delta).$$

Choose δ sufficiently small, then by Lemma 4 we have

$$|r_{(n,\bar{j})}^{(m,\bar{k})}| \leq c_{74}\delta^{(m-n)\alpha'/\alpha} \quad \text{and} \quad |r_{(n,\bar{j})}^{(m,\bar{k})}\varphi(\delta^n)\varphi(\delta^m)| \leq c_{75}$$

if $m \geq m(n)$, and

$$|r_{(n,\bar{j})}^{(m,\bar{k})}| \leq \varepsilon < 1/2$$

if $n < m \leq m(n)$. Since the cardinal number of $(n, j_1, \cdots, j_d)$ for fixed n is less than $c_{76}(\log n)^{N/(2\alpha')}$, we have

$$I_1 \leq c_{77} \sum_n \sum_{m \geq m(n)} \delta^{(m-n)\alpha'/\alpha} (\log n \log m)^{N/(2\alpha')}$$
$$\times \exp\{-(\varphi^2(\delta^n) + \varphi^2(\delta^m))/2\}$$
$$\leq c_{78} \sum_n \sum_{m \geq m(n)} \delta^{(m-n)\alpha'/\alpha} (\log n \log m)^{N/(2\alpha')+1} (nm)^{-1}$$
$$\leq c_{79} \sum n^{-2} (\log n)^{c_{80}} < +\infty,$$

and

$$I_2 \leq c_{81} \sum_n \sum_{n < m < m(n)} (\log n \log m)^{N/(2\alpha')}$$
$$\times \exp\{-(\varphi^2(\delta^n) + \varphi^2(\delta^m) - 2\varepsilon\varphi(\delta^n)\varphi(\delta^m))/2\}$$
$$\leq c_{82} \sum_n n^{-2+2\varepsilon} (\log n)^{N/\alpha'+2} \log_{(2)} n < +\infty.$$

This yields the proof of Theorem 7 by just the same argument of (27).

Proof of Theorem 8. First we notice that it is sufficient to prove the theorem under the condition (9) of Lemma 1. Choose two sequences of points $(s_1^{(n)}, \cdots, s_d^{(n)}) \in D^+$, $(t_1^{(n)}, \cdots, t_d^{(n)}) \in D^-$ such that

$$|s_i^{(n)} - t_i^0| = (1+c)\sigma_i^{-1}(\delta^n)/2,$$

$$|t_i^{(n)}-t_i^0|=(1+c)\sigma_i^{-1}(\delta^n)/2,$$

$$|s_i^{(n)}-t_i^{(n)}|=(1+c)\sigma_i^{-1}(\delta^n),\ i=1,\cdots,d,\ 1>\delta>0,\ 1>c>0,$$

and set

$$V_{n,1}^{(i)}=\{s_i\in D_i^+;\ |s_i-s_i^{(n)}|\leq c\sigma_i^{-1}(\delta^n)/2\},$$

$$V_{n,2}^{(i)}=\{s_i\in D_i^-;\ |s_i-t_i^{(n)}|\leq c\sigma_i^{-1}(\delta^n)/2\},$$

where δ and c are constants independent of n chosen suitably later. To apply Lemma 3, let us estimate the following:

$$v(s,t;s',t')\equiv E\Big[\Big(\frac{X(s)-X(t)}{\gamma(s-t)}-\frac{X(s')-X(t')}{\gamma(s'-t')^2}\Big)^2\Big].$$

By just the same argument as in case of Theorem 6, choosing c sufficienly small, we have

$$v(s,t;s',t')\geq c_{82}'\delta^{-2n}\sum_{i=1}^{d}\sigma_i^2((|s_i-s_i'|^2+|t_i-t_i'|^2)^{1/2})$$

$$\text{if}\quad s=(s_1,\cdots,s_d),\ s'=(s_1',\cdots,s_d')\in V_{n,1}^{(1)}\times\cdots\times V_{n,1}^{(d)},$$

$$t=(t_1,\cdots,t_d),\ t'=(t_1',\cdots,t_d')\in V_{n,2}^{(1)}\times\cdots\times V_{n,2}^{(d)},$$

where c_{82}' is a constant independent of n. Now set

$$A_{(j_1,\cdots,j_d,k_1,\cdots,k_d)}^{(n)}=\Big\{\omega;\ \frac{X(s(\bar{j}))-X(t(\bar{k}))}{\gamma(s(\bar{j})-t(\bar{k}))}\geq\varphi(\delta^n)\Big\},$$

$$A_n=\bigcup_{\substack{(j_1,\cdots,j_d)\\(k_1,\cdots,k_d)}}A_{(j_1,\cdots,j_d,k_1,\cdots,j_d)}^{(n)},$$

$$s(\bar{j})=(s_{j_1}^{(1)},\cdots,s_{j_d}^{(d)}),$$

$$t(\bar{k})=(t_{k_1}^{(1)},\cdots,t_{k_d}^{(d)}),$$

where $\{s_{j_i}^{(i)}\}$ and $\{t_{k_i}^{(i)}\}$ are $\varepsilon_n^{(i)}$-distinguishable set of $V_{n,1}^{(i)}$ and $V_{n,2}^{(i)}$ respectively. Then setting

$$\varepsilon_n^{(i)}=c_{83}\sigma_i^{-1}(\delta^n/\varphi(\delta^n))$$

in Lemma 3, we have

$$P(A_n)\geq 1/2\Phi(\varphi(\delta^n))\prod_{i=1}^{d}M(\varepsilon_n^{(i)};\ V_{n,1}^{(i)}\times V_{n,2}^{(i)})$$

$$\geq c_{84}\Phi(\varphi(\delta^n))K(2\bar{N}/2\bar{N};\gamma,\varphi)(\delta^n).$$

Hence we have

$$\sum_n P(A_n) \geq c_{85} I_{u\,t}(\gamma\,;\varphi) = +\infty\,.$$

Next with the help of Lemma 5, we have to show $P(\bigcap_n A_n{}^c) = 0$. By Lemma 5 we have

$$\begin{aligned}
&|P(\cap A_n{}^c) - \prod_n P(A_n{}^c)| \\
&\leq \sum_n \Big(\sum_{m \geq m(n)} + \sum_{n < m \leq m(n)} \Big) \sum_{\substack{(j_1, \cdots, j_d) \\ (k_1, \cdots, k_d)}} \sum_{\substack{(j'_1, \cdots, j'_d) \\ (k'_1, \cdots, k'_d)}} \\
&\quad |r_{(n,\bar{j},\bar{k})}^{(m,\bar{j}',\bar{k}')}| \int_0^1 g(\lambda r_{(n,\bar{j},\bar{k})}^{(m,\bar{j}',\bar{k}')}\,;\varphi(\delta^n), \varphi(\delta^m))\,d\lambda \\
&\equiv I_1 + I_2\,,
\end{aligned}$$

where

$$\begin{aligned}
r_{(n,\bar{j},\bar{k})}^{(m,\bar{j}',\bar{k}')} &= E\Big[\frac{X(s(\bar{j})) - X(t(\bar{k}))}{\gamma(s(\bar{j}) - t(\bar{k}))} \cdot \frac{X(s(\bar{j}')) - X(t(\bar{k}'))}{\gamma(s(\bar{j}') - t(\bar{k}'))}\Big] \\
&= (\gamma(s(\bar{j}) - t(\bar{k}))\gamma(s(\bar{j}') - t(\bar{k}')))^{-1} \sum_{i=1}^{d} R_i/2\,, \\
&\quad R_i = R_i(s_{j_i}^{(i)}, t_{k_i}^{(i)}\,; s_{j'_i}^{(i)}, t_{k'_i}^{(i)})\,, \\
&\quad s(\bar{j}) = (s_{j_1}^{(1)}, \cdots, s_{j_d}^{(d)}) \in V_{n,1}^{(1)} \times \cdots \times V_{n,1}^{(d)}, \\
&\quad t(\bar{k}) = (t_{k_1}^{(1)}, \cdots, t_{k_d}^{(d)}) \in V_{n,2}^{(1)} \times \cdots \times V_{n,2}^{(d)}, \\
&\quad s(\bar{j}') = (s_{j'_1}^{(1)}, \cdots, s_{j'_d}^{(d)}) \in V_{m,1}^{(1)} \times \cdots \times V_{m,1}^{(d)}, \\
&\quad t(\bar{k}') = (t_{k'_1}^{(1)}, \cdots, t_{k'_d}^{(d)}) \in V_{m,2}^{(1)} \times \cdots \times V_{m,2}^{(d)}, \\
&\quad m(n) = n + c_{86} \log_{(2)} n,\ \ c_{86} > \alpha/(\alpha' \log 1/\delta)\,.
\end{aligned}$$

Choose δ sufficiently small, then by Lemma 4 we have

$$|r_{(n,\bar{j},\bar{k})}^{(m,\bar{j}',\bar{k}')}| \leq c_{87} \delta^{(m-n)\alpha'/\alpha} \quad \text{and} \quad |r_{(n,\bar{j},\bar{k})}^{(m,\bar{j}',\bar{k}')} \varphi(\delta^n)\varphi^m)| \leq c_{88}$$

if $m \geq m(n)$, and

$$|r_{(n,\bar{j},\bar{k})}^{(m,\bar{j}',\bar{k}')}| \leq \varepsilon < 1/2$$

if $n < m \leq m(n)$. Since the cardinal number of $(n, \bar{j}, \bar{k})$ for fixed n is less than $c_{89}(\log n)^{N/\alpha'}$, we have

$$I_1 \leq c_{90} \sum_n \sum_{m \geq n(n)} (\log n \log m)^{N/\alpha'} \delta^{(m-n)\alpha'/\alpha}$$

$$\times \exp\{-(\varphi^2(\delta^n) + \varphi^2(\delta^m))/2\}$$

$$\leq c_{91} \sum_n \sum_{m \geq m(n)} (\log n \log m)^{N/\alpha'+1} \delta^{(m-n)\alpha'/\alpha} (nm)^{-1}$$

$$\leq c_{92} \sum_n n^{-2} (\log n)^{c_{93}} < +\infty,$$

and

$$I_2 \leq c'_{93} \sum_n \sum_{n < m \leq m(n)} (\log n \log m)^{N/\alpha'}$$

$$\times \exp\{-(\varphi^2(\delta^n) + \varphi^2(\delta^m) - 2\varepsilon\varphi(\delta^n)\varphi(\delta^m))/2\}$$

$$\leq c_{94} \sum n^{-2+2\varepsilon} (\log n)^{2N/\alpha'+2} \log_{(2)} n < \infty.$$

This yields the proof of Theorem 8 by just the same argument of (27).

Proof of Theorem 9. First we notice that it is sufficient to to prove the theorem under the condition (10) of Lemma 1. Choose a sequence of points $(t_1^{(n)}, \cdots, t_d^{(n)}) \in R^N$ such that

$$|t_i^{(n)}| = (1-c)\sigma_i^{-1}(\delta^n/d), \ \delta > 1, 1 > c > 0, \ i = 1, \cdots, d,$$

and set

$$V_n^{(i)} = \{s_i \in R^{N_i}; |t_i^{(n)} - s_i| \leq c\sigma_i^{-1}(\delta^n/d)\},$$

where δ and c are constants independent of n chosen suitably later. To apply Lemma 3, let us estimate the following:

$$v(s, t) \equiv E\left[\left(\frac{X(s) - X(0)}{\gamma(s)} - \frac{X(t) - X(0)}{\gamma(t)}\right)^2\right]$$

$$= \{\gamma^2(s-t) - (\gamma(s) - \gamma(t))^2\}/(\gamma(s)\gamma(t)).$$

By the Assumption L for $\sigma_i^2(x)$, we have

$$(\gamma(s) - \gamma(t))^2 = (\gamma^2(s) - \gamma^2(t))^2/(\gamma(s) + \gamma(t))^2$$

$$\leq c_{95} \sum_{i=1}^{d} |s_i - t_i|^2 \sigma_i^2(r_i)/r_i^2,$$

where $r_i = |s_i| \wedge |t_i|$. Therefore choosing sufficiently small c, we have

$$v(s,t)\geq c_{96}\delta^{-2n}\gamma^2(s-t)$$

for $s,t\in V_n^{(1)}\times\cdots\times V_n^{(d)}$, where c_{96} is a constant independent of n. Now set

$$A^{(n)}_{(j_1,\cdots,j_d)}=\{\omega;\ (X(s(\bar{j}))-X(0))/\gamma(s(\bar{j}))\geq\varphi(\delta^n)\},$$

$$A_n=\bigcup_{(j_1,\cdots,j_d)}A^{(n)}_{(j_1,\cdots,j_d)},$$

$$s(\bar{j})=(s^{(1)}_{j_1},\cdots,s^{(d)}_{j_d}),$$

where $\{s^{(i)}_{j_i}\}$ is an $\varepsilon_n{}^{(i)}$-distinguishable set of $V_n{}^{(i)}$. Then setting

$$\varepsilon_n{}^{(i)}=c_{97}\sigma_i{}^{-1}(\delta^n/\varphi(\delta^n))$$

in Lemma 3, we have

$$\begin{aligned}P(A_n)&\geq 1/2\Phi(\varphi(\delta^n))\prod_{i=1}^{d}M(\varepsilon_n{}^{(i)};V_n{}^{(i)})\\&\geq c_{98}\Phi(\varphi(\delta^n))K(\bar{N}/\bar{N};\gamma,\varphi)(\delta^n).\end{aligned}$$

Hence we have

$$\sum_n P(A_n)\geq c_{99}I^{\infty}_{\ell}(\gamma;\varphi)=+\infty.$$

Next with the help of Lemma 5, we have to show $P(\bigcap A_n{}^c)=0$. By Lemma 5 we have

$$\begin{aligned}&|P(\bigcap A_n{}^c)-\prod_n P(A_n{}^c)|\\&\leq\sum_n\Big(\sum_{n\geq m(n)}+\sum_{n<m\leq m(n)}\Big)\sum_{(k_1,\cdots,k_d)}\sum_{(j_1,\cdots,j_d)}\\&\qquad|r^{(m,\bar{k})}_{(n,\bar{j})}|\int_0^1 g(\lambda r^{(m,\bar{k})}_{(n,\bar{j})};\varphi(\delta^n),\varphi(\delta^m))d\lambda\\&\equiv I_1+I_2,\end{aligned}$$

where

$$\begin{aligned}r^{(m,\bar{k})}_{(n,\bar{j})}&=E\Big[\frac{X(s(\bar{j}))-X(0)}{\gamma(s(\bar{j}))}\cdot\frac{X(t(\bar{k}))-X(0)}{\gamma(t(\bar{k}))}\Big]\\&=\{\gamma(s(\bar{j}))\gamma(t(\bar{k}))\}^{-1}\sum_{i=1}^{d}R_i/2,\end{aligned}$$

$$R_i=R_i(s^{(i)}_{j_i},0;t^{(i)}_{k_i},0),$$

$$s(\bar{j})=(s_{j_1}^{(1)},\cdots,s_{j_d}^{(1)})\in V_n^{(1)}\times\cdots\times V_n^{(d)},$$

$$t(\bar{k})=(t_{k_1}^{(1)},\cdots,t_{k_d}^{(d)})\in V_m^{(1)}\times\cdots\times V_m^{(d)},$$

$$m(n)=n+c_{100}\log_{(2)}n,\ c_{100}>\alpha/(\alpha'\log 1/\delta).$$

Choosing δ sufficiently large, by Lemma 4 we have

$$|r_{(n,\bar{j})}^{(m,\bar{k})}|\leq c_{101}\delta^{-(m-n)\alpha'/\alpha}\quad\text{and}\quad|r_{(n,\bar{j})}^{(m,\bar{k})}\varphi(\delta^n)\varphi(\delta^m)|\leq c_{102}$$

if $m\geq m(n)$, and

$$|r_{(n,\bar{j})}^{(m,\bar{k})}|\leq\varepsilon<1/2$$

if $n<m\leq m(n)$. Since the cardinal number of $(j_1,\cdots,j_d)$ for fixed n is less than $c_{103}(\log n)^{N/(2\alpha')}$, we have

$$I_1\leq c_{104}\sum_n\sum_{m\geq m(n)}(\log n\log m)^{N/(2\alpha')}\delta^{-(m-n)\alpha'/\alpha}$$

$$\times\exp\{-(\varphi^2(\delta^n)+\varphi^2(\delta^m))/2\}$$

$$\leq c_{105}\sum_n\sum_{m\geq m(n)}(\log n\log m)^{N/(2\alpha')+1}\delta^{-(m-n)\alpha'/\alpha}(nm)^{-1}$$

$$\leq c_{106}\sum_n n^{-2}(\log n)^{c_{107}}<+\infty,$$

and

$$I_2\leq c_{108}\sum_n\sum_{n<m\leq m(n)}(\log n\log m)^{N/(2\alpha')}$$

$$\times\exp\{-(\varphi^2(\delta^n)+\varphi^2(\delta^m)-2\varepsilon\varphi(\delta^n)\varphi(\delta^m))/2\}$$

$$\leq c_{109}\sum(\log n)^{N/\alpha'+2}n^{-2+2\varepsilon}\log_{(2)}n<+\infty.$$

This yields the proof of Tneorem 9 by just the same argument of (27).

Proof of Theorem 10. First we notice that it is sufficient to prove the theorem under the condition (11) of Lemma 1. Let $\{x_{n,j};\ 1\leq j\leq M(h;B_n)\}$ be an h-distinguishable set of $B_n=\{x=(x_1,\cdots,x_N)\in R^N,\ x_i>0,\ i=1,\cdots,N,\ (2n-1)h\leq|x|\leq 2nh\}$ and set

$$V_{n,j}^{(i)}=\{s_i\in R^{N_i};|x_{n,j}^{(i)}-s_i|\leq 1\},\ 1\leq j\leq M(h,B_n),$$

$$x_{n,j}=(x_{n,j}^{(1)},\cdots,x_{n,j}^{(d)}),\ x_{n,j}^{(i)}\in R^{N_i},$$

where h is constant chosen suitably later. Set

$$A^{(n)}_{(j,\mu_1,\cdots,\mu_d)}=\{\omega;X(s(\bar{\mu}))\geq\varphi(2nh)\},$$

$$A_j{}^{(n)}=\bigcup_{(\mu_1,\cdots,\mu_d)}A^{(n)}_{(j,\mu_1,\cdots,\mu_d)},$$

$$s(\bar{\mu})=(s^{(1)}_{\mu_1},\cdots,s^{(d)}_{\mu_d}),$$

where $\{s^{(i)}_{\mu_i}\}$ is $\varepsilon^{(i)}_n$-distinguishable set of $V^{(i)}_{n,j}$. Then setting

$$\varepsilon_n{}^{(i)}=c_{110}\sigma_i{}^{-1}(1/\varphi(2nh))$$

in Lemma 3, we have

$$P(A_j{}^{(n)})\geq 1/2\Phi(\varphi(2nh))\prod_{i=1}^{d}M(\varepsilon_n{}^{(i)};V^{(i)}_{n,j})$$

$$\geq c_{111}\Phi(\varphi(2nh))/K(\bar{N};\gamma)(1/\varphi(2nh)).$$

Hence we have

$$\sum_n\sum_j P(A_j{}^{(n)})\geq c_{112}\sum_n\Phi(\varphi(2nh))n^{N-1}/K(\bar{N};\gamma)(1/\varphi(2nh))$$

$$\geq c_{113}I_b^{\infty}(\gamma;\varphi)=+\infty.$$

Next with the help of Lemmr 5, we have to show $P(\bigcap_{(n,j)}A_j{}^{(n)c})=0$. Set.

$$\Lambda_1(n,j)=\{(m,k);m\geq n^{1+a}\},\ (a>0),$$

$$\Lambda_2(n,j)=\{(m,k);n\leq m\leq n^{1+a},$$

$$|x^{(i)}_{n,j}-x^{(i)}_{m,k}|\geq n^{1-b}h,\ i=1,\cdots,d\},\ (1>b>0),$$

$$\Lambda_{3,i}(n,j)=\{(m,k)\neq(n,j);n\leq m\leq n^{1+a},\ |x^{(i)}_{n,j}-x^{(i)}_{m,k}|\leq n^{1-b}h\}.$$

Then by Lemma 5 we have

$$|P(\bigcap_{(n,j)}A_j{}^{(n)c})-\prod_{(n,j)}P(A_j{}^{(n)c})|$$

$$\leq\sum_{(n,j)}\Big(\sum_{\Lambda_1(n,j)}+\sum_{\Lambda_2(n,j)}+\sum_{i=1}^{d}\sum_{\Lambda_{3,i}(n,j)}\Big)\sum_{(\mu_1,\cdots\mu_d)}\sum_{(\nu_1,\cdots\nu_d)}$$

$$|r^{(m,k,\nu)}_{(n,j,\mu)}|\int_0^1 g(\lambda r^{(m,k,\nu)}_{(n,j,\mu)};\varphi(2nh),\varphi(2mh))d\lambda$$

$$\equiv I_1+I_2+I_3,$$

where

$$r^{(m,k,\bar\nu)}_{(n,j,\bar\mu)}=E[X(s(\bar\mu))X(t(\bar\nu))],$$

$$s(\bar\mu)=(s^{(1)}_{\mu_1},\cdots,s^{(d)}_{\mu_d})\in V^{(1)}_{n,j}\times\cdots\times V^{(d)}_{n,j},$$

$$t(\bar\nu)=t(t^{(1)}_{\nu_1},\cdots,t^{(d)}_{\nu_d})\in V^{(1)}_{m,k}\times\cdots\times V^{(d)}_{m,k}.$$

By the additional condition of Theorem 10, we have

$$|r^{(m,k,\bar\nu)}_{(n,j,\bar\mu)}|\leq c_{114}(\log(m-n))^{-\beta}\quad\text{and}$$

$$|r^{(m,k,\bar\nu)}_{(n,j,\bar\mu)}\varphi(2nh)\varphi(2mn)|\leq c_{115}$$

if $(m,k)\in\Lambda_1(n,j)$,

$$|r^{(m,k,\bar\nu)}_{(n,j,\bar\mu)}|\leq c_{115}(\log n)^{-\beta}\quad\text{and}$$

$$|r^{(m,k,\bar\nu)}_{(n,j,\bar\mu)}\varphi(2nh)\varphi(2mh)|\leq c_{117}$$

if $(m,k)\in\Lambda_2(n,j)$, and

$$|r^{(m,k,\bar\nu)}_{(n,j,\bar\mu)}|\leq\varepsilon<bN_0/(2N)$$

if $(m,k)\in\Lambda_{3,i}(n,j)$ for sufficiently large h, where $N_0=\min(N_1,\cdots,N_d)$. Since the cardinal number of $(\mu_1,\cdots,\mu_d)$ for fixed (n,j) is less than $c_{118}(\log n)^{N/(2\alpha')}$, we have

$$I_1\leq c_{119}\sum_n\sum_{m\geq n^{1+a}}(\log n\log m)^{N/(2\alpha')}n^{N-1}m^{N-1}(\log(m-n))^{-\beta}$$
$$\times\exp\{-(\varphi^2(2nh)+\varphi^2(2mh))/2\}$$
$$\leq c_{120}\sum_n\sum_{m\geq n^{1+a}}(\log n\log m)^{N/(2\alpha')+1}(nm)^{-1}(\log(m-n))^{-\beta}$$
$$\leq c_{121}\sum_n(\log n)^{N/\alpha'+2-\beta}n^{-1}<+\infty,$$

$$I_2\leq c_{122}\sum_n\sum_{n\leq m\leq n^{1+a}}(\log n\log m)^{N/(2\alpha')}n^{N-1}m^{N-1}(\log n)^{-\beta}$$
$$\times\exp\{-(\varphi^2(2nh)+\varphi^2(2mh))/2\}$$
$$\leq c_{123}\sum_n\sum_{n\leq m\leq n^{1+a}}(\log n)^{N/\alpha'+2-\beta}(nm)^{-1}$$
$$\leq c_{124}\sum_n(\log n)^{N/\alpha'+3-\beta}n^{-1}<+\infty,$$

and

$$I_3\leq c_{125}\sum_n\sum_{n\leq m\leq n^{1+a}}(\log n\log m)^{N/(2\alpha')}n^{N-1}m^{N-1-bN_0}$$
$$\times\exp\{-(\varphi^2(2nh)+\varphi^2(2mh)-2\varepsilon\varphi(2nh)\varphi(2mh))/2\}$$

$$\leq c_{126} \sum_{n} \sum_{n \leq m \leq n^{1+a}} (\log n)^{N/\alpha'+2} n^{-1} m^{-1-bN_0+2\varepsilon N}$$

$$\leq c_{127} \sum_{n} (\log n)^{N/\alpha'+2} n^{-1-bN_0+2\varepsilon N} < +\infty .$$

This yields the proof of Theorem 10 by just the same argument of (27).

5. Examples.

In this section we consider the following examples. Let $\{X_{\alpha_i}(t) ; t \in R^{N_i}\}$, $i=1, \cdots, d$ be independent Gaussian random fields such that

$$E[(X_{\alpha_i}(s) - X_{\alpha_i}(t))^2] = |s-t|^{2\alpha_i}, \ (0 < \alpha_i < 1),$$

and set

$$X_{\bar{\alpha}}(t) = \sum_{i=1}^{d} X_{\alpha_i}(t_i), \ t = (t_1, \cdots, t_d) ; t_i \in R^{N_i},$$

$$N_1 + \cdots + N_d = N, \ N(\bar{\alpha}) = \sum_{i=1}^{d} N_i/\alpha_i .$$

Then we can apply our Theorem 1-4 and Theorem 6-9 for the Gaussian random field $\{X_{\bar{\alpha}}(t) ; t \in R^N\}$, and the examples of *the upper classes* or *the lower classes* are following:

$$\varphi^2(x) = 2N(\bar{\alpha}) \log 1/x + (2N(\bar{\alpha}) + 1) \log_{(2)} 1/x + (2+\varepsilon) \log_{(3)} 1/x$$
$$\in U_u(X_{\bar{\alpha}} ; D) \quad \text{if} \quad \varepsilon > 0,$$
$$\in L_u(X_{\bar{\alpha}} ; D) \quad \text{if} \quad \varepsilon \leq 0,$$

$$\varphi^2(x) = 2 \log_{(2)} 1/x + (N(\bar{\alpha}) + 1) \log_{(3)} 1/x + (2+\varepsilon) \log_{(4)} 1/x$$
$$\in U_l(X_{\bar{\alpha}} ; t_0) \quad \text{if} \quad \varepsilon > 0,$$
$$\in L_l(X_{\bar{\alpha}} ; t_0) \quad \text{if} \quad \varepsilon \leq 0,$$

$$\varphi^2(x) = 2 \log_{(2)} 1/x + (2N(\bar{\alpha}) + 1) \log_{(3)} 1/x + (2+\varepsilon) \log_{(4)} 1/x$$
$$\in U_{ul}(X_{\bar{\alpha}} ; t_0) \quad \text{if} \quad \varepsilon > 0,$$
$$\in L_{ul}(X_{\bar{\alpha}} ; t_0) \quad \text{if} \quad \varepsilon \leq 0,$$

$$\varphi^2(x) = 2 \log_{(2)} x + (N(\bar{\alpha}) + 1) \log_{(3)} x + (2+\varepsilon) \log_{(4)} x$$
$$\in U_l^{\infty}(X_{\bar{\alpha}}) \quad \text{if} \quad \varepsilon > 0,$$
$$\in L_l^{\infty}(X_{\bar{\alpha}}) \quad \text{if} \quad \varepsilon \leq 0.$$

Next let us consider two Gaussian random fields $\{X_{\bar{\alpha}}(t)\,;t\in R^N\}$ and $\{X_{\bar{\alpha}'}(t')\,;t'\in R^{N'}\}$, $(N_1+\cdots+N_d=N,\ N_1'+\cdots+N_{d'}'=N')$. Then as the easy corollary of our theorems, we have the following theorem concerning the invariance of *the upper classes* or *the lower classes* between two random fields $\{X_{\bar{\alpha}}(t)\}$ and $\{X_{\bar{\alpha}'}(t')\}$.

Theorem 11. *The upper classes* $U_u(X_{\bar{\alpha}}; D)$, $U_l(X_{\bar{\alpha}}; t_0)$, $U_{ul}(X_{\bar{\alpha}}; t_0)$ *and* $U_l^\infty(X_{\bar{\alpha}})$ *or the lower classes* $L_u(X_{\bar{\alpha}}; D)$, $L_l(X_{\bar{\alpha}}; t_0)$, $L_{ul}(X_{\bar{\alpha}}; t_0)$ *and* $L_l^\infty(X_{\bar{\alpha}})$ *coincide with the upper classes or the lower classes corresponding to the random field* $\{X_{\bar{\alpha}'}(t')\}$ *respectively if and only if*

$$N_1/\alpha_1+\cdots+N_d/\alpha_d=N_1'/\alpha_1'+\cdots+N_d'/\alpha_d'.$$

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