An example of indecomposable vector bundle of rank $n-1$ on $P^n$.

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Introduction and notation

It is well known that there exists a vector bundle of rank $n-1$ on $P^n$ for $n$ odd, which is not direct sums of line bundles cf. [1]. In this paper we shall give an example of indecomposable vector bundle of rank $n-1$ on $P^n$ for each $n \geq 3$.

In this paper we shall use the following notation: $\mathcal{O}_{P^n}$ is the structure sheaf of $n$-dimensional projective space $P^n$ defined over an algebraically closed field $k$ of an arbitrary characteristic; $\mathcal{O}_{P^n}(1)$ is the line bundle associated with a hyperplane of $P^n$; $\mathcal{O}_{P^n}$ is the sheaf of germs of regular differential 1-forms; $T_{P^n}$ is the tangent bundle on $P^n$; $\tilde{E}$ is the dual vector bundle of a vector bundle $E$; $E(m)$ is the vector bundle $E \otimes \mathcal{O}_{P^n}(1)^{\otimes m}$; $c_i(E)$ is the $i$-th Chern class of $E$; $c(E) = 1 + c_1(E) + c_2(E) + \cdots$ is the Chern polynomial of $E$; $h = c_1(\mathcal{O}_{P^n}(1))$ i.e. the first Chern class of a hyperplane; $H^i(E)$ = $H^i(X, E)$ and $h^i(E)$ = dim$_k H^i(X, E)$ for a vector bundle $E$ on a complete nonsingular variety $X$ defined over $k$; $Gr(n, d)$ is the Grassmann variety which parametrizes $d$-dimensional linear subspaces of $P^n$; $Q(n, d)$ is the universal quotient bundle of $Gr(n, d)$; $L_x$ is the $d$-dimensional linear subspace of $P^n$ which is represented by a point $x$ of $Gr(n, d)$; $\omega_{s, o, \ldots, o}(A) = \{x \in Gr(n, d) | L_x \cap A \neq \emptyset\}$ is the special Schubert variety for an $n-d-s$ dimensional linear subspace $A$ of $P^n$; and $\omega_{s, o, \ldots, o}$ is the Schubert cycle associated with a $\omega_{s, o, \ldots, o}(A)$. 
Construction of the example

Lemma 1. \( \mathcal{O}_{p^*}(2) \) is generated by its global sections.

Proof. Consider the following commutative diagram with exact rows and exact columns.

\[
\begin{array}{ccccccccc}
0 & \to & H^0(\mathcal{O}_{p^*}(2)) \otimes \mathcal{O}_{p^*} & \to & H^0(\mathcal{O}_{p^*}(1)) \otimes \mathcal{O}_{p^*} & \to & \mathcal{O}_{p^*} & \to & 0 \\
& & \downarrow g_0 & & \downarrow g_1 & & \downarrow g_2 & & \\
& & 0 & & \mathcal{O}_{p^*}(2) & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & & \\
\end{array}
\]

It is easy to see that \( f \) and \( f' \) are surjections. Hence, the Snake lemma shows that \( g_0 \) is surjective. q.e.d.

By virtue of the proof of Lemma 1, we have

\[
h^s(\mathcal{O}_{p^*}(2)) = (n + 1) h^s(\mathcal{O}_{p^*}(1)) - h^s(\mathcal{O}_{p^*}(2)) = \frac{1}{2} n (n + 1).
\]

We denote Kernel \( g_0 \) by \( \tilde{E}_n \). Then, we have the following exact sequence of vector bundles

\[
0 \to T_{p^*}(-2) \to \bigoplus_{s=0}^n \mathcal{O}_{p^*} \to E_n \to 0
\]

where \( N_s = \frac{1}{2} n (n + 1) \) and rank \( E_n = N_n - n = \frac{1}{2} n (n - 1) \). Using the long exact sequences of cohomology groups

\[
\begin{align*}
0 &\to H^0(T_{p^*}(-2)) \to H^0(\bigoplus_{s=0}^n \mathcal{O}_{p^*}) \to H^0(E_n) \to H^1(T_{p^*}(-2)) \\
0 &\to H^s(\bigoplus_{s=0}^{n+1} \mathcal{O}_{p^*}(-1)) \to H^s(T_{p^*}(-2)) \to H^s(\mathcal{O}_{p^*}(-2)) = 0 \\
0 &\to H^1(\bigoplus_{s=0}^{n+1} \mathcal{O}_{p^*}(-1)) \to H^1(T_{p^*}(-2)) \to H^1(\mathcal{O}_{p^*}(-2)) = 0
\end{align*}
\]

we obtain \( h^s(T_{p^*}(-2)) = h^1(T_{p^*}(-2)) = 0 \) and \( h^0(E_n) = N_n \).

Theorem 2. \( E_n \) has an indecomposable quotient bundle \( E_n' \) of
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rank \( n - 1 \).

In order to prove the Theorem, we need the following four lemmas.

**Lemma 3.** \( c_n(E_n) = 0 \) and \( c_{n-1}(E_n) \neq 0 \).

**Proof.** Indeed the exact sequences

\[
0 \to T_{p^n}(-2) \to \bigoplus \mathcal{O}_{p^n} \to E_n \to 0
\]

\[
0 \to \mathcal{O}_{p^n}(-2) \to \bigoplus \mathcal{O}_{p^n}(-1) \to T_{p^n}(-2) \to 0
\]

shows that \( c(E_n) \cdot c(T_{p^n}(-2)) = 1 \) and

\[
c(T_{p^n}(-2)) \cdot c(\mathcal{O}_{p^n}(-2)) = c(\bigoplus \mathcal{O}_{p^n}(-1)).
\]

Hence, we have

\[
c(E_n) = c(T_{p^n}(-2))^{-1} = (1 - 2h)(1 - h)^{n-1} = \left(\sum_{i=1}^{n \choose i} h^i\right)(1 - 2h).
\]

Therefore, \( c_n(E_n) = \left(\frac{2n}{n} - 2\left(\frac{2n - 1}{n - 1}\right)\right)h^n = 0 \) and

\[
c_{n-1}(E_n) = \left(\frac{2n - 1}{n - 1} - 2\left(\frac{2n - 2}{n - 2}\right)\right)h^{n-1} \neq 0. \quad \text{q.e.d.}
\]

**Lemma 4.** Let \( E \) be a vector bundle of rank \( r \) on a complete nonsingular variety \( X \). Suppose that \( E \) is generated by its global sections and \( c_i(E) = 0 \) for a positive integer \( s \leq r \). Then \( E \) has a trivial vector bundle of rank \( r - s + 1 \) as a subbundle.

**Proof.** Since \( E \) is generated by its global sections, there exists an exact sequence of vector bundles

\[
\bigoplus \mathcal{O}_{p^n} \to E \to 0
\]

where \( m + 1 = h^*(E) \). Then, there is a canonical morphism \( f: X \to \text{Gr}(m, m - r) \) such that \( E = f^*Q(m, m - r) \). Since \( 0 = c_s(E) = f^*c_s(Q(m, m - r)) = f^*\omega_{s, s, \ldots, s} \), we see that \( f(X) \cdot \omega_{s, s, \ldots, s} = 0 \). Hence, there exists a linear subspace \( A \) of dimension \( r - s \) of \( P^n \) such that
Lemma 5. Let \( n > s > d \geq 0 \) and let \( f \) be a morphism from \( P^s \) to \( \text{Gr}(s, d) \), then \( f(P^n) \) consists only of one point. cf. [2].

Lemma 6. (i) Let \( E \) be a nontrivial vector bundle of rank \( r \) on \( P^s \). If \( E \) is generated by its global sections, then \( h^0(E) \geq n + 1 \).

(ii) Let \( E \) be a vector bundle which has no trivial vector bundle as a direct summand. Assume that \( E \) is generated by its global sections and that \( h^0(E) \leq 2n + 1 \). Then, \( E \) is indecomposable.

Proof. (i). Since \( E \) is generated by its global sections, there exists an exact sequence of vector bundles

\[ \oplus \mathcal{O}_{P^n} \to E \to 0 \]

where \( m + 1 = h^0(E) \). Then, there exists a canonical morphism \( f : P^n \to \text{Gr}(m, m - r) \) such that \( E = f^*Q(m, m - r) \). Since \( E \) is nontrivial vector bundle, we see that \( f(P^n) \) is not one point. Hence, we have \( m \geq n \), by virtue of Lemma 5.

(ii). (ii) follows from (i). q.e.d.

Proof of Theorem 2. Since \( E_n \) is generated by its global sections and \( c_n(E_n) = 0 \), we have the exact sequence of vector bundles

\[ 0 \to F \to E_n \to E_n' \to 0 \]

where \( F \) is a trivial vector bundle of rank \( \frac{1}{2}n(n-1) - n + 1 \) and \( E_n' \) is the quotient bundle of rank \( n - 1 \), by virtue of Lemma 4. From the exact sequence of cohomology groups

\[ 0 \to H^0(F) \to H^0(E_n) \to H^0(E_n') \to H^1(F) = 0 \]

we obtain that \( h^0(E_n') = h^0(E_n) - h^0(F) = 2n - 1 \). The fact that \( c_{n-1}(E_n') = c_{n-1}(E_n) \neq 0 \) shows that \( E_n' \) has no trivial vector bundle as a direct summand. Since \( E_n \) is generated by its global sections, so is \( E_n' \). These results shows that \( E_n' \) is indecomposable, by virtue of Lemma 6 (ii). q.e.d.
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Remark. Canonically $Gr(n, 1)$ is embedded in $P^{n-1}$. By this embedding $\omega_{n-1,1}(P) = \{ x \in Gr(n, 1) \mid L_x \supseteq P \}$ is $n - 1$ dimensional linear subspace of $P^{n-1}$. Hence, we have a map $\varphi: P^n \to Gr(N_n - 1, n - 1)$. On the other hand, by virtue of the exact sequence (1), we have a morphism $\mathcal{V}: P^n \to Gr(N_n - 1, n - 1)$. In this sense $\varphi$ and $\mathcal{V}$ are projectively equivalent, i.e. there exists a collineation $f: P^n \to P^n$ such that $\varphi = \mathcal{V} \circ f$.

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Bibliography