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Moduli of stable sheaves, I

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Introduction. Let $f: X \to S$ be a smooth, projective, geometrically integral morphism of locally noetherian schemes with an f-very ample invertible sheaf $\mathcal{O}_X(1)$. In this situation

Definition ([3] and [8]). A coherent module F of rank r on the fibre over a geometric point s of S is said to be stable (or, semistable) (with respect to $\mathcal{O}_X(1)$) if and only if it is torsion free and for all proper coherent subsheaves E of rank $t (1 \leqq t \leqq r)$, the inequalities

$$P_E(m) = \chi(E(m))/t < P_F(m) = \chi(F(m))/r \text{ (or, } \leqq \text{, resp.)}$$

hold for all large integers m, where for a coherent moule H on X_s, $H(m) = H \otimes_{\mathcal{O}_S} \mathcal{O}_X(m)$ and $\chi(H(m)) = \sum_i (-1)^i \dim H^i(X_s, H(m))$.

For a numerical polynomial H and for a scheme T locally of finite type over S, set $\Sigma^H_{X/S}(T) = \{F \mid F$ is a coherent $\mathcal{O}_{X_T}$-module with the following property $(*)\}/\sim$, where $F_1 \sim F_2$ if and only if $F_1 \otimes_{\mathcal{O}_T} L \cong F_2$ with some invertible sheaf L on T;

$(*)$ F is T-flat and for all geometric points t of T, $F \otimes_{\mathcal{O}_T} k(t)$ is stable with respect to $\mathcal{O}_X(1) \otimes_{\mathcal{O}_S} \mathcal{O}_T$ and $\chi(F \otimes_{\mathcal{O}_T} k(t)(m)) = H(m)$.

Then an S-morphism $g: T' \to T$ defines a natural map $g^*: \Sigma^H_{X/S}(T) \to \Sigma^H_{X/S}(T')$. Clearly $\Sigma^H_{X/S}$ is a contravariant functor of the category (Sch/S) of schemes locally of finite type over S to (Sets). This functor is not necessarily a sheaf for the ètale topology in (Sch/S) even if f has a section. Hence $\Sigma^H_{X/S}$ is, in general, not representable. Neverthless $\Sigma^H_{X/S}$ may have a coarse moduli scheme (see [10]). In fact, we know that if $S = \operatorname{Spec}(k)$ with an algebraically closed field k and if $\dim X \leqq 2$, then our functor has a coarse moduli scheme ([12], [13], [7] and [3]) and moreover our main theorem (Theorem 5. 6) says that if S is an algebraic scheme over a field, then there exists a coarse moduli scheme $M_{X/S}(H)$ of $\Sigma^H_{X/S}$ which is locally of finite type over S.

As is stated in [7], to construct a coarse moduli scheme of $\Sigma^H_{X/S}$ by using "invariant theory", the problem is devided into three parts, that is, (1) boundedness, (2) openness and (3) existence of a geometric quotient of a scheme by an affine algebraic group. Though (2) is proved in [8], (1) is still an open problem except for some special cases; (a) the relative dimension of X over $S\leqq 2$ (see [7] or [3]) or (b) the rank of members of $\Sigma^{H}_{X/S}(T)$ is 2 ([9]). For this reason we introduce the notion of e-stable sheaves (Definition 3.1) and show that a stable sheaf is e-stable with some non-negative integer e and the property that a coherent sheaf is e-stable is bounded and open (§ 3). Thus $\Sigma^{H}_{X/S}$ is covered by open subfunctors $\Sigma^{H,e}_{X/S}$ ($e\geqq 0$) and each of $\Sigma^{H,e}_{X/S}$ is bounded. Hence our problem reduces to showing (3) for $\Sigma^{H,e}_{X/S}$. Thanks to the results of D. Mumford [10], M. Nagata [11] and W. Haboush [5], it is almost equivalent to the following;

What point in $Q=\mathrm{Quot}^{H}_{\mathcal{O}_X^{\oplus N}/X/S}$ is stable for a natural action on it of $SL(N)$ with respect to a suitable fixed invertible sheaf?

Since no direct answers to the above question are known, we construct a morphism of an open set for the étale topology of Q to a suitable scheme and measure stability of a point using its image by the morphism. Now we know two "measuring spaces". One is a product of Grassmann varieties (see [12] and [7]) and the other is a projective bundle over a finite union of connected components of $\mathrm{Pic}_{X/S}$ (see [3]). This is simpler than that because the latter needs only an open set $Q\times_S S'$ such that $X_{S'}\to S'$ has a section and the former does a rather finer covering. Thus our section 4 is devoted to generalizing and sharpening the techniques and the results in [3]. By virtue of our main theorem in § 4 (Theorem 4.17), our problem reduces to the following;

Does a point of Q corresponding to a stable sheaf enjoy the property (4.15.1)?

Proposition 3.6, which is an immediate corollary to Fundamental lemma in § 2, implies that the answer is affirmative.

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Notation and Convention. Throughout this paper a variety is a geometrically integral algebraic scheme over a field. For a coherent sheaf F on a k-variety X, $h^i(X,F)$ or simply $h^i(F)$ denotes $\dim_k H^i(X,F)$ and $\chi(F)$ does $\sum_i(-1)^i h^i(X,F)$. The rank of a coherent sheaf F on a variety X is the dimension of $E(x)=E\otimes_{\mathcal{O}_X}k(x)$ as a vector space over $k(x)$ with generic point x of X and is denoted by $r(E)$. For S-schemes Y and T, $Y(T)$ is the set of T-valued points of Y, that is, $Y(T)=\mathrm{Hom}_S(T,Y)$. In particular, if $T=\mathrm{Spec}(K)$ with K an algebraically closed field, then a point y in $Y(T)=Y(K)$ is said to be a geometric point of Y and K is denoted by $k(y)$. Thus a geometric point y of Y defines an S-morphism of $\mathrm{Spec}(k(y))$ to Y. Let $f:X\to S$ be a smooth, projective, geometrically integral morphism of locally noetherian schemes and let $\mathcal{O}_X(1)$ be an f-very ample invertible $\mathcal{O}_X$-module. For a geometric point s of S, X_s is the geometric fibre of X over s, that is, $X_s = X \times_S \mathrm{Spec}(k(s))$. For a coherent module E on a geometric fibre X_s of X, the degree of E with respect to $\mathcal{O}_X(1)$, which is denoted by $d(E, \mathcal{O}_X(1))$, is that of the first Chern class of E with respect to $\mathcal{O}_{X_s}(1) = \mathcal{O}_X(1) \otimes \mathcal{O}_S \mathcal{O}_{X_s}$. For integers a and b, (a, b) is the binomial coefficient $(a+b)!/a!b!$. Thus we have the equalities $(a, b) = (b, a)$ and $(a, b) = (a, b-1) + (a-1, b)$.

§ 1. Preliminaries.

First of all let us recall some results of the geometric invariant theory which will be used in § 4. Combining the results of D. Mumford [10], M. Nagata [11] and W. Haboush [5], we have

Theorem 1. 1. *Let X be an affine scheme over a field k, let G be a reductive affine algebraic k-group (i.e. the unipotent part of the radical of G is trivial) and let $\sigma: G \times_k X \to X$ be an action of G on X. Then there exist an affine k-scheme Y and a k-morphism ϕ of X to Y such that (Y, ϕ) is a good quotient of X by G (see [14] Definition 1.5) and ϕ is universally submersive. (Y, ϕ) is a geometric quotient of X by G if and only if the action σ is closed. Moreover if X is an algebraic k-scheme, then so is Y.*

To globalize the above result, we need the following notions due to D. Mumford ([10] p. 30 and p. 36).

Definition 1. 2. Let F be a coherent module on a scheme over a field k and let σ be an action of an algebraic k-group. A G-linearization of F is an isomorphism $\psi: \sigma^*(F) \xrightarrow{\sim} p_2^*(F)$ such that $(\mu \times 1_X)^*(\psi) = p_{23}^*(\psi)(1_G \times \sigma)^*(\psi)$, where $\mu: G \times_k G \to G$ is the group law and p_2 (or, p_{23}) is the projection of $G \times_k X$ to X (or, $G \times_k G \times_k X$ to $G \times_k X$, resp.).

Definition 1. 3. Let X, G, σ and p_2 be as above and let L be an invertible $\mathcal{O}_X$-module with a G-linearization ψ.

1) A geometric point x of X is said to be semi-stable if there exist a positive integer n and an invariant section s of $H^0(X, L^{\otimes n})$ (i.e. if ψ_n is induced by ψ, then $\psi_n(\sigma^*(s)) = p_2^*(s)$) such that $X_s = \{y \in X \mid s(y) \neq 0\}$ is affine and x is a geometric point of X_s.

2) A geometric point x of X is said to be stable if there exist a positive integer n and an invariant section of $H^0(X, L^{\otimes n})$ such that X_s is affine, the action of G on X_s is closed and x is a geometric point of X_s. A stable point x is said to be properly stable if the dimention of the stabilizer group at x is zero.

It is clear that there exists an open set $X^{ss}(L)$ ($X^s(L)$ or $X_0^s(L)$) in X such that the set of semi-stable (stable or properly stable, resp.) points is the set of geometric points of the open set.

Theorem 1.4. *Let X be an algebraic scheme over a field k and let G be a reductive affine algebraic k-group. If L is a G-linearized invertible $\mathcal{O}_X$-module, then there exists a good quotient (Y, ϕ) of $X^{ss}(L)$ by G. Moreover,*

(i) *Y is an algebraic k-scheme and ϕ is universally submersive,*

(ii) *there exists an ample invertble sheaf M on Y such that $\phi^*(M) = L^{\otimes n}$ for some integer n, hence Y is quasi-projective over k.*

(iii) *there exists an open subscheme Y' of Y such that $X^s(L) = \phi^{-1}(Y')$ and that $(Y', \phi | X^s(L))$ is a geometric quotient of $X^s(L)$ by G.*

Let X be a scheme proper over a field k, let G be a reductive affine algebraic k-group and let L be a G-linearized ample invertible sheaf on X. Pick a geometric point x of X. To study the stability of a fixed geometric point x, we may assume that k is algebraically closed and x is a closed point of X. For a one-parameter subgroup $\lambda : G_m \to G$, let us consider the morphism $f : G_m \ni \alpha \to \sigma(\lambda(\alpha), x) \in X$, where σ is the action of G on X. Since X is proper over k, f can be extended to a morphism $\bar{f}$ of A^1 to X. Clearly $\bar{f}(0)$ is a fixed point under the action of the one-parameter subgroup on X. Then the G-linearization on L induces an action of G_m on A^1 which is the dual space of $L \otimes k(\bar{f}(0))$. This action is given by a character χ of G_m; $\chi(\alpha) = \alpha^r$ for all $\alpha \in G_m(k)$. For this r, set $\mu_L(x, \lambda) = -r$. If we replace Theorem 1. 10 of [10] by Theorem 1. 4, the following is obtained by the same argument in Chap. 2, § 1 of [10].

Theorem 1.5. *Let X, G, L and x be as above. Then*

(i) *x is contained in $X^{ss}(L)$ if and only if $\mu^L(x, \lambda) \geqq 0$ for all one-parameter subgroups λ;*

(ii) *x is contained in $X_0{}^s(L)$ if and only if $\mu^L(x, \lambda) > 0$ for all one-parameter subgroups λ.*

We shall close this section by a lemma which will be used frequently in the sequal.

Lemma 1.6. *Let Y be a quasi-projective variety with a very ample invertible sheaf $\mathcal{O}_Y(1)$ and let F be a torsion free coherent $\mathcal{O}_Y$-module. Then for a general s in $H^0(Y, \mathcal{O}_Y(1))$, $F \otimes_{\mathcal{O}_Y} \mathcal{O}_H = \operatorname{coker}(F(-1) \cong F(-1) \otimes \mathcal{O}_Y \xrightarrow{1 \otimes s} F(-1) \otimes \mathcal{O}_Y(1) \cong F)$ is a torsion free $\mathcal{O}_H$-module, where H is the closed subscheme of Y defined by $s = 0$.*

Proof. For $F^\vee = \mathcal{H}om_{\mathcal{O}_Y}(F, \mathcal{O}_Y)$, there are an integer m and a surjective homomorphism $g : \mathcal{O}_Y(-m)^{\oplus r} \to F^\vee$ because $\mathcal{O}_Y(1)$ is ample. Then we obtain $h : F \xrightarrow{j} (F^\vee)^\vee \xrightarrow{g^\vee} \mathcal{O}_Y(m)^{\oplus r}$, where j is a canonical homomorphism. Since Y is an integral scheme, j induces an isomorphism on a non-empty open set of Y. Thus j is injective because F is torsion free. This implies that h is injective. Let E be the cokernel of h. Since Y is noetherian, $\operatorname{Ass}(E)$ is a finite set, whence for a general s in $H^0(Y, \mathcal{O}_Y(1))$, $\mathrm{Ass}(E) \cap \{s=0\} = \phi$. Moreover, we may assume that the closed subscheme H of Y defined by $s=0$ is integral. Using this s, we get the following exact commutative diagram;

$$\begin{tikzcd}
 & 0 & 0 & & \\
 & F \otimes \mathcal{O}_H \arrow[r, "\bar{h}"] \arrow[u] & \mathcal{O}_H(m)^{\oplus r} \arrow[r] \arrow[u] & E \otimes \mathcal{O}_H \arrow[r] & 0 \\
0 \arrow[r] & F \arrow[r, "h"] \arrow[u] & \mathcal{O}_Y(m)^{\oplus r} \arrow[r] \arrow[u] & E \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & F(-1) \arrow[r] \arrow[u, "1 \otimes s"] & \mathcal{O}_Y(m-1)^{\oplus r} \arrow[r] \arrow[u, "1 \otimes s"] & E(-1) \arrow[r] \arrow[u, "i"] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

Since $\mathrm{Ass}(E) \cap \{s=0\} = \phi$ implies that i is injective and since $\ker(\bar{h}) \cong \ker(i)$ by Snake lemma, we know the injectivity of $\bar{h}$. On the other hand, $\mathcal{O}_H(m)^{\oplus r}$ is torsion free $\mathcal{O}_H$-module, whence so is $F \otimes_{\mathcal{O}_Y} \mathcal{O}_H$. q.e.d.

§ 2. A fundamental lemma.

Let $f: X \to S$ be a smooth, projective, geometrically integral morphism of locally noetherian schemes and let $\mathcal{O}_X(1)$ be an f-very ample invertible sheaf on X. If S is connected, then the self-intersection number of $\mathcal{O}_{X_s}(1)$, or the degree of X_s with respect to $\mathcal{O}_{X_s}(1)$ is independent of the choice of a geometric point s of S, and we denote it by h. If F is a coherent $\mathcal{O}_{X_s}$-module of rank r, then

$$(2.1) \quad P_F(m) = \chi(F(m))/r = hm^n/n! + \{d(F, \mathcal{O}_X(1))/r - d(K_{X_s}, \mathcal{O}_X(1))/2\} m^{n-1}/(n-1)! + \text{terms of degree} \leq n-2,$$

where $n = \dim X_s$ and K_{X_s} is the canonical invertible sheaf of X_s. Since $K_{X_s} = \Omega^n_{X/S} \otimes_{\mathcal{O}_S} k(s)$, it is easy to see that $d(K_{X_s}, \mathcal{O}_X(1))$ is independent of s and we denote it by $c(X)$. Our aim of this section is to prove the following which plays an important role in the sequal.

Fundametal lemma 2.2. *Let S be a locally noetherian, connected scheme, $f: X \to S$ be a smooth, projective, geometrically integral morphism of relative dimension n and let $\mathcal{O}_X(1)$ be an f-very ample invertible sheaf on X. Assume that a_1 (or, e) is a negative (or, non-negative, resp.) integer with $a_1 + e < 0$, r is a positive integer and that a_i $(2 \leqq i \leqq n)$ are rational numbers. Set*

$$P(m) = hm^n/n! + \{a_1/r - c(X)/2\} m^{n-1}/(n-1)! + \sum_{i=2}^{n} a_i n^{n-i}.$$

Then there exist integers L and M such that if F is a torsion free coherent $\mathcal{O}_{X_s}$-module of rank $r' \leqq r$ with some geometric point s of S and if F has the properties (1), (2) and (3);

(1) *for general*[1] *non-singular curves* $C=D_1\cdot D_2\cdot\cdots\cdot D_{n-1}$, $D_i\in|\mathcal{O}_{X_s}(1)|$, *every coherent subsheaf* $E(\neq 0)$ *of* $F\otimes\mathcal{O}_C$ *has a degree* $\leqq r(E)(a_1+e)/r$,

(2) $\Delta^{n-1}P(m)\geqq\Delta^{n-1}P_F(m)$ *for all large integers* m, *where for a numerical polynomial* $g(m)$ *of one variable, we define that* $\Delta g(m)=g(m)-g(m-1)$ *and* $\Delta^k g(m)=\Delta(\Delta^{k-1}g(m))$,

(3) $h^0(F(m))\geqq r'P(m)$ *for some* $m\geqq L$,

then the following holds; $\quad d(F,\mathcal{O}_X(1))\geqq M$.

Before proving the above lemma, let us show some lemmas.

Lemma 2.3. *Assume that a coherent torsion free* $\mathcal{O}_{X_s}$*-module* F *has the property* (1) *of Lemma 2.2. Then we have*

(1) *for each* i $(0\leqq i<n)$, *there exists a non-empty open set* U_i *of* $V_i=\{D_1\cdot D_2\cdot\cdots\cdot D_i\,|\,D_1,\cdots,D_i\in|\mathcal{O}_{X_s}(1)|,\ D_1\cdot D_2\cdot\cdots\cdot D_i$ *is a smooth variety of dimension* $n-i\}$ ($V_i(1\leqq i<n)$ *is an open set ot of a Grassmann variety and* V_0 *is the point* X_s) *such that for every* $k(s)$*-rational point* Y_j *of* U_i, $F\otimes\mathcal{O}_{Y_i}$ *is torsion free and every coherent subsheaf* $E(\neq 0)$ *of* $F\otimes\mathcal{O}_{Y_i}$ *has a degree* $\leqq r(E)(a_1+e)/r$.

(2) *for every* $k(s)$*-rational point* Y_i *of* U_i *and for every coherent subsheaf* E *of* $F\otimes\mathcal{O}_{Y_i}$, $H^0(Y_i,E)=0$.

Proof. (2) is an immediate consequence of (1) because if $H^0(Y_i,E)\neq 0$, then $\mathcal{O}_{Y_i}$ is a subsheaf of E and because $a_1+e<0$. To prove (1) let us consider the universal family $X\to V_i\times_{k(s)}X_s$ of the subvarieties of X_s corresponding to the points of V_i. Set $F^{(i)}=p_{2i}{}^*(F)$ with the second projection $p_{2i}:X_i\to X_s$. It is easy to see that the first projection $p_{1i}:X_i\to V_i$ is a smooth, projective, geometrically integral morphism with a very ample invertible sheaf $L_i=p_{2i}{}^*(\mathcal{O}_{X_s}(1))$. Shrinking V_i if necessary, we may assume that $F^{(i)}$ is flat over V_i. By virtue of Proposition 2.1 of [8] and Lemma 1.6 there exists a non-empty open set V_i' of V_i such that for all points v of V_i', $F^{(i)}\otimes k(v)$ is torsion free. The property (1) of Lemma 2.2 for $F\otimes\mathcal{O}_C$ means just that $F\otimes\mathcal{O}_C$ is cotype $((a_1+e)/r-b/r',\cdots,(a_1+e)/r-b/r')$, where $b=d(F,\mathcal{O}_X(1))$. Thus U_{n-1} exists by virtue of Theorem 2.8 of [8]. Now let W_i be the subscheme of $V_{n-1}\times_{k(s)}V_i$ which defines the incidence correspondence between the open sets of the Grassmann varieties V_{n-1} and V_i. Since W_i is an open set of a flag variety, the second projection $q_{2i}:W_i\to V_i$ is flat. Hence for the first projection $q_{1i}:W_i\to U_{n-1}$, $U_i=q_{2i}(q_{1i}{}^{-1}(U_{n-1}))\cap V_i'$ is an open set of V_i. Note that for a $k(s)$-rational point Y_i of U_i, if one takes sufficiently general members $D_{i+1},\cdots,D_{n-1}$ in $|\mathcal{O}_{X_s}(1)|$, then $Y_i\cdot D_{i+1}\cdot\cdots\cdot D_{n-1}$ is contained in U_{n-1}. Assume that for a $k(s)$-rational point Y_i of U_i, $F\otimes\mathcal{O}_{Y_i}$ has a coherent subsheaf E with degree $>r(E)(a_1+e)/r$. If $D_{i+1},\cdots,D_{n-1}$ are sufficiently general members of $|\mathcal{O}_{X_s}(1)|$, then for $C'=Y_i\cdot D_{i+1}\cdot$

[1] $U=\{C=D_1\cdot\cdots\cdot D_{n-1}|D_i\in|\mathcal{O}_X(1)|,\ C$ is a non-singular curve$\}$ forms an open set of a Grassmann variety. We have only to assume that there exists a dense subset V in $U(k(s))$ such that every curve in V satisfies the condition (1).

$\cdots\cdot D_{n-1}$, $E\otimes\mathcal{O}_{C'}$ is a subsheaf of $F\otimes\mathcal{O}_{C'}$ (see the proof of Lemma 1. 6), the degree of $E\otimes\mathcal{O}_{C'}$ is equal to that of $E\otimes\mathcal{O}_{Y_i}$ and C' is $k(s)$-rational point of U_{n-1}. This is a contradiction. Therefore the above U_i's are the desired open sets.

q.e.d.

We need some numerical lemmas.

Lemma 2. 4.
(1) *Set* $P(n, m)=\sum_{i=0}^{m-1}(n-2, i)$, *then* $P(n, m)=(n-1, m-1)$.[2]
(2) *Set* $Q(n, m)=\sum_{i=0}^{m-1}(n-2, i)(m-i)$, *then* $Q(n, m)=(n, m-1)$.
(3) $\sum_{i=a}^{b}(i-t, c-1)=\sum_{i=0}^{c-1}(a-t-1, i)(b-a, c-i)$ *for all integers* a, b, c *and* t *with* $b\geqq a>t\geqq 0$ *and* $c>0$.

Proof. If one notes the equalities $P(n, m)=P(n-1, m)+P(n, m-1)$ and $Q(n, m)=Q(n-1, m)+Q(n, m-1)$, then (1) and (2) are proved easily by induction on $n+m$. Let us show (3) for every fixed t by induction on $a+c$. Set $R(a, b, c, t)=\sum_{i=a}^{b}(i-t, c-1)$ and $R'(a, b, c, t)=\sum_{i=0}^{c-1}(a-t-1, i)(b-a, c-i)$. Then, using (1), we obtain

$$R(t+1, b, c, t)=\sum_{i=t+1}^{b}(i-t, c-1)=\sum_{i=1}^{b-t}(c-1, i)$$
$$=P(c+1, b-t+1)-1=(b-t, c)-1.$$

On the other hand,

$$R'(t+1, b, c, t)=\sum_{i=0}^{c-1}(b-t-1, c-i)=\sum_{i=1}^{c}(b-t-1, i)$$
$$=P(b-t+1, c+1)-1=(b-t, c)-1.$$

Thus we have $R(t+1, b, c, t)=R'(t+1, b, c, t)$ for all b and c. Moreover, $R(a, b, 1, t)=b-a+1=R'(a, b, 1, t)$ for all a and b. Next assume that $a\geqq t+2$ and $c\geqq 2$. Then since

$$R(a, b, c, t)=\sum_{i=a-t}^{b-t}(c-1, i)=P(c+1, b-t+1)-P(c+1, a-t),$$

we have

$$R(a, b, c, t)=R(a, b, c-1, t)+R(a-1, b-1, c, t).$$

By the induction assumption we obtain

$$R(a, b, c, t)=R'(a, b, c-1, t)+R'(a-1, b-1, c, t).$$

Now let us prove that the right hand side of this equality is equal to $R'(a, b, c, t)$, which completes our proof.

$$R'(a, b, c-1, t)+R'(a-1, b-1, c, t)$$
$$=\sum_{i=0}^{c-2}(a-t-1, i)(b-a, c-i-1)+\sum_{i=0}^{c-1}(a-t-2, i)(b-a, c-i)$$
$$=\sum_{i=1}^{c-1}(a-t-1, i-1)(b-a, c-i)+\sum_{i=0}^{c-1}(a-t-2, i)(b-a, c-i)$$
$$=(b-a, c)+\sum_{i=1}^{c-1}\{(a-t-1, i-1)+(a-t-2, i)\}(b-a, c-i)$$

[2] See Notation and Convention.

$$=\sum_{i=1}^{c-1}(a-t-1, i)(b-a, c-i)+(b-a, c)$$
$$=\sum_{i=0}^{c-1}(a-t-1, i)(b-a, c-i)=R'(a, b, c, t). \qquad \text{q.e.d.}$$

Lemma 2.5. *For $f(x)=x^n$, the coefficient of x^{n-i-1} in $(\Delta^i f)(x-1)$ is $-(i+2)n!/2(n-i-1)!$.*

Proof. It is clear that for $g(x)=(x-1)^n$, $\Delta^i g(x)=(\Delta^i f)(x-1)$. Since $g(x)=\sum_{i=0}^n(-1)^i(n-i, i)x^{n-i}$, our lemma holds for $i=0$. Assume that our assertion holds for $i=j$. Then $\Delta^j g(x)=(n!/(n-j)!)x^{n-j}-((j+2)n!/2(n-j-1)!)x^{n-j-1}$ $+$terms of lower degrees. Hence

$$\begin{aligned}\Delta^{j+1}g(x)=&(n!/(n-j)!)\{x^{n-j}-(x-1)^{n-j}\}-((j+2)n!/2(n-j-2)!)x^{n-j-2}\\&+\text{terms of degrees}\leqq n-j-3=(n!/(n-j-1)!)x^{n-j-1}\\&-(n!/(n-j)!)((n-j)!/2(n-j-2)!)x^{n-j-2}\\&-((j+2)n!/2(n-j-2)!)x^{n-j-2}\\&+\text{terms of degrees}\leqq n-j-3=(n!/(n-j-1)!)x^{n-j-1}\\&-((j+3)n!/2(n-j-2)!)x^{n-j-2}+\text{terms of lower degrees.}\end{aligned}$$

Therefore our proof is completed by induction on i.

The following is due to M. F. Atiyah [1].

Lemma 2.6. *Let F be an indecomposable vector bundle on a non-singular projective curve of genus g and let d and r be the degree and the rank of F respectively. For a maximal splitting $(L_1, \cdots, L_r)$ of F, we have the following inequalities;*

$$\begin{aligned}d/r-(r-1)(3g-2)\leqq d(L_i)&\leqq d/r+(g-1)(r-1)+(i-1)g\\&\leqq d/r+(2g-1)(r-1),^{3)}\end{aligned}$$

where $d(L_i)$ denotes the degree of L_i.

Proof of Lemma 2.2. The idea of our proof is essentially the same as Gieseker's in the proof of Lemma 1.2 of [3]. The main part of our proof consists of an evaluation of $h^0(F(m))$.

As in [3] let H_m be the smallest coherent subsheaf of $F(m)$ such that $H^0(X_s, H_m)=H^0(X_s, F(m))$ and $F(m)/H_m$ is torsion free. Since $d(H_m, \mathcal{O}_X(1))\geqq 0$, the assumption (1) and (1) of Lemma 2.3 imply that $H_0=0$. Moreover, the exact commutative diagram

$$\begin{array}{ccccccc}0\longrightarrow & H^0(X_s, H_m) & \xrightarrow{\sim} & H^0(X_s, F((m)) & \longrightarrow & H^0(X_s, F(m)/H_m)\\ & \uparrow & & \uparrow & & \uparrow\\ 0\longrightarrow & H^0(X_s, H_m(-p)) & \rightarrow & H^0(X_s, F(m-p)) & \rightarrow & H^0(X_s, F(m-p)/H_m(-p))\\ & \uparrow & & \uparrow & & \uparrow\\ & 0 & & 0 & & 0\end{array}$$

yields

3) This inequality is sharper than Atiyah's original one. But the fact is not essential.

(2. 2. 1) $H^0(X_s, H_m(-p))=H^0(X_s, F(m-p))$ for all non-negative integers p.

We claim

(2. 2. 2) $H_m(p)$ is a subsheaf of H_{m+p} for all non-negative integers p.

In fact (2. 2. 1) implies that the subsheaf H_m' of $F(m)$ generated by $H^0(X_s, F(m))$ is that of $H_{m+p}(-p)$ generated by $H^0(X_s, H_{m+p}(-p))$. Let H_m'' be the inverse image of the torsion part of $H_{m+p}(-p)/H_m'$ by the natural homomorphism $H_{m+p}(-p)\to H_{m+p}(-p)/H_m'$. Then $F(m)/H_m''$ is torsion free because so are $F(m)/H_{m+p}(-p)$ and $H_{m+p}(-p)/H_m''$. Since $H^0(X_s, H_m'')$ $=H^0(X_s, H_{m+p}(-p))=H^0(X_s, F(m))$, we have that $H_m''=H_m$. This means that $H_m(p)$ is a subsheaf H_{m+p}.

Choose so general $k(s)$-rational members $D_1, \cdots, D_{u-1}$ of $|\mathcal{O}_{X_s}(1)|$ that each $Y_i=D_1\cdot\cdots\cdot D_i$ is contained in U_i of Lemma 2. 3 and that $H_m\otimes\mathcal{O}_{Y_i}$ is a subsheaf of $F(m)\otimes\mathcal{O}_{Y_i}$. The exact sequence

$$0\to H_m(-1)\to H_m\to H_m\otimes\mathcal{O}_{Y_1}\to 0$$

and (2. 2. 1) provides us with the inequality

$$\begin{aligned} h^0(F(m))=h^0(H_m)&\leqq h^0(H_m(-1))+h^0(H_m\otimes\mathcal{O}_{Y_1})\\ &=h^0(F(m-1))+h^0(H_m\otimes\mathcal{O}_{Y_1}). \end{aligned}$$

Summing up these inequalities from 0 to m, we obtain

$$h^0(F(m))\leqq\sum_{u=0}^{m}h^0(H_u\otimes\mathcal{O}_{Y_1}).$$

By virtue of Lemma 2. 3, (2) and the exact sequence

$$0\to H_u(j-1)\otimes\mathcal{O}_{Y_i}\to H_u(-j)\otimes\mathcal{O}_{Y_i}\to H_u(-j)\otimes\mathcal{O}_{Y_{i+1}}\to 0$$

the inequalities

$$h^0(H_u(-j)\otimes\mathcal{O}_{Y_i})\leqq\sum_{k=0}^{u-j}h^0(H_u(-j-k)\otimes\mathcal{O}_{Y_{i+1}})$$

are obtained. Thus we have

$$\begin{aligned} h^0(F(m))&\leqq\sum_{u=0}^{m}\sum_{j_1=0}^{u}h^0(H_u(-j_1)\otimes\mathcal{O}_{Y_2})\leqq\sum_{u=0}^{m}\sum_{j_1=0}^{u}\sum_{j_2=0}^{u-j_1}h^0(H_u(-j_1-j_2)\otimes\mathcal{O}_{Y_3})\\ &=\sum_{u=0}^{m}\sum_{\substack{0\leqq j_1, j_2\leqq u\\ j_1, j_2\geqq 0}}h^0(H_u(-j_1-j_2)\otimes\mathcal{O}_{Y_3})\leqq\cdots\\ &\leqq\sum_{u=0}\sum_{\substack{0\leqq j_1+\cdots+j_{n-2}\leqq u\\ j_1,\cdots,j_{n-2}\geqq 0}}h^0(H_u(-j_1-j_2-\cdots-j_{n-2})\otimes\mathcal{O}_{Y_{n-1}}). \end{aligned}$$

From this the following is obtained

(2. 2. 3) $h^0(F(m))\leqq\sum_{u=0}^{m}\sum_{j=0}^{u}(n-3, j)h^0(H_u(-j)\otimes\mathcal{O}_{Y_{n-1}})$.

Let g be the genus of the curve Y_{n-1}. Since S is connected, g is independent of the choice of s and Y_{n-1}. Let $m_1, \cdots, m_l$ be the integers such that $H_{m_t}\neq H_{m_t-1}(1)$. Clearly $l\leqq r'$. We denote the rank of H_m by r_m

In the first place, assume that $m<m_l$. Set $E=H_{m_l-1}(-m_l+1)\otimes\mathcal{O}_{Y_{n-1}}$.

By (2.2.2), $H_u(-j)$ is a subsheaf of $H_{m_l-1}(u-j-m_l+1)$ if $u<m_l$. Thus $H_u(-j)\otimes\mathcal{O}_{Y_{n-1}}$ is a subsheaf of $E(u-j)$. This and (2.2.3) assert

$$h^0(F(m))\leqq\sum_{u=0}^{m}\sum_{j=0}^{u}(n-3, j)h^0(E(u-j))=\sum_{t=0}^{m}\sum_{i=t}^{m}(n-3, i-t)h^0(E(t))$$
$$=\sum_{t=0}^{m}h^0(E(t))(\sum_{i=0}^{m-t}(n-3, i)).$$

By Lemma 2.4, (1) we have

$$(2.2.4)\quad h^0(F(m))\leqq\sum_{t=0}^{m}(n-2, m-t)h^0(E(t)),\ \text{if } m<m_l.$$

Write $E=E_1\oplus E_2\oplus\cdots\oplus E_u$ with indecomposable vector bundles E_i of rank ρ_i on Y_{n-1}. Since each E_i is a coherent subsheaf of $F\otimes\mathcal{O}_{Y_{n-1}}$, $d(E_i)=d_i\leqq\rho_i(a_1+e)/r$ by the assumption on Y_{n-1}. By Lemma 2.3, (2) we have that $h^0(E_i)=0$. We shall apply Lemma 2.6 to E_i. Let $(L_1^{(i)}, \cdots, L_{\rho_i}^{(i)})$ be a maximal splitting of E_i. If $h^0(E_i\otimes\mathcal{O}_{X_s}(j))\neq0$, then one of $L_k^{(i)}(j)=L_k^{(i)}\otimes\mathcal{O}_{X_s}(j)$ has a non-negative degree, whence $d_i/\rho_i+jh+(2g-1)(\rho_i-1)\geqq0$ by virtue of Lemma 2.6. Let t_i be the integer such that $\{-d_i/\rho_i-(2g-1)(\rho_i-1)\}/h+1>t_i\geqq\{-d_i/\rho_i-(2g-1)(\rho_i-1\}/h$ and let t_i' be the integer $\max(t_i, 0)$. Then we have

$$\sum_{t=0}^{m}(n-2, m-t)h^0(E_i(t))=\sum_{t=t_i'}^{m}(n-2, m-t)h^0(E_i(t))$$
$$=\sum_{t=t_i'}^{m}\{\rho_iht+d_i-\rho_i(g-1)+h^1(E_i(t))\}(n-2, m-t)$$
$$\leqq\sum_{t=t_i'}^{m}\{\rho_iht+d_i-\rho_i(g-1)\}(n-2, m-t)$$
$$+\sum_{k=1}^{\rho_i}\sum_{t=t_i'}^{m}h^1(L_k^{(i)}(t))(n-2, m-t),$$

where $E_i(t)=E_i\otimes\mathcal{O}_{X_s}(t)$. Since $d(L_n^{(i)}(t_i'))\geqq-(\rho_i-1)(3g-2)-(2g-1)(\rho_i-1)$ $=-(\rho_i-1)(5g-3)$, we have that for $t\geqq t_i'$, $h^1(L_k^{(i)}(t))\leqq\max\{(\rho_i-1)(5g-3)-(t-t_i')h+g-1, g\}\leqq\max\{(\rho_i-1)(5g-3)+g-1, g\}$. Since $\rho_i\leqq r'\leqq r$, there exists an integer A_1, which depends only on r and g, such that $h^1(L_k^{(i)}(t))\leqq A_1$ for all i, k, and t with $t\geqq t_i'$. On the other hand, since $d(L_k^{(i)}(t_i'+t))\geqq th-(\rho_i-1)(5g-3)$, we see that $h^1(L_k^{(i)}(t_i'+t))=0$ if $t>\{(\rho_i-1)(5g-3)+2g-2\}/h$. Combining above results, we obtain

$$(2.2.5)\quad \sum_{t=0}^{m}(n-2, m-t)h^0(E_i(t))\leqq\sum_{t=t_j'}^{m}\{\rho_iht+d_i-\rho_i(g-1)\}(n-2, m-t)$$
$$+A(n-2, m-t),\ \text{where } d(E_i)=d_i,\ r(E_i)=\rho_i \text{ and } A \text{ depends only on } r, h \text{ and } g.$$

Now let us come back to the computation of $h^0(F(m))$. By (2.2.5) we have

$$h^0(F(m))\leqq\sum_{t=0}^{m}(n-2, m-t)\sum_{i=1}^{u}h^0(E_i(t))$$
$$\leqq\sum_{i=1}^{u}\sum_{t=t_i'}^{m}(n-2, m-t)\{t\rho_ih+d_i-\rho_i(g-1)\}+\sum_{i-1}^{u}(n-2, m)A$$

$$\leqq \sum_{i=1}^{u} \sum_{t=t_i'}^{m} (n-2, m-t)\{t\rho_i h+\rho_i(a_1+e)/r-\rho_i(g-1)\}$$
$$+\sum_{i=1}^{u}(n-2, m)A.$$

Let t_0 be the integer such that $\{-(a_1+e)/r+(g-1)\}/h+1>t_0\geqq\{-(a_1+e)/r+(g-1)\}/h$. Then our computation proceeds as follows;

$$h^0(F(m))\leqq \sum_{t=t_0}^{m}(n-2, m-t)r(E)\{th+(a_1+e)/r-(g-1)\}$$
$$+r(E)(n-2, m)A=r(E)\sum_{t=0}^{m-t_0}(n-2, t)\{(m-t)h+(a_1+e)/r$$
$$-(g-1)\}+r(E)(n-2, m)A=r(E)[h(n, m-t_0)$$
$$+\{(a_1+e)/r-(g-1)\}(n-1, m-t_0)+(n-2, m)A].$$

Since $r(E)<r'$, we know

(2. 2. 6) If $m<m_l$, $h^0(F(m))\leqq g_{r'}(m)=(r'-1)[h(n, m-t_0)+\{(a_1+e)/r-g+1\}(n-1, m-t_0)]+(n-2, m)B$, where t_0 is an integer depending only on a_1, e, r, g and h, and where B depends only on r, h and g.

In the next place, we shall evaluate $h^0(F(m))$ for $m\geqq m_l$. We may assume that $F(m_l)\otimes\mathcal{O}_{Y_{n-1}}$ is generically generated by its global sections and that

(2. 2. 7) $d(F, \mathcal{O}_X(1))\leqq r'a_1/r-r'(e+1)$.

If we set

$$v(m)=\sum_{u=m_l}^{m}\sum_{j=0}^{u}(n-3, j)h^0(F(u-j)\otimes\mathcal{O}_{Y_{n-1}}),$$

then (2. 2. 3) implies

$$h^0(F(m))\leqq\sum_{u=0}^{m_l-1}\sum_{j=0}^{u}(n-3, j)h^0(H_u(-j)\otimes\mathcal{O}_{Y_{n-1}})$$
$$+\sum_{u=m_l}^{m}\sum_{j=0}^{u}(n-3, j)h^0(F(u-j)\otimes\mathcal{O}_{Y_{n-1}})\leqq g_{r'}(m_l-1)+v(m).$$

On the other hand,

$$v(m)=\sum_{i=m_l}^{m}\sum_{t=0}^{i}(n-3, i-t)h^0(F(t)\otimes\mathcal{O}_{Y_{n-1}})$$
$$=\sum_{t=m_l}^{m}\sum_{i=t}^{m}(n-3, i-t)h^0(F(t)\otimes\mathcal{O}_{Y_{n-1}})$$
$$+\sum_{t=1}^{m_l-1}\sum_{i=m_l}^{m}(n-3, i-t)h^0(F(t)\otimes\mathcal{O}_{Y_{n-1}}).$$

Thus we obtain

(2. 2. 8) If $m\geqq m_l$, then $h^0(F(m))\leqq g_{r'}(m_l-1)+v_1(m)+v_2(m)$, where $v_1(m)=\sum_{t=m_l}^{m}\sum_{i=t}^{m}(n-3, i-t)h^0(F(t)\otimes\mathcal{O}_{Y_{n-1}})$ and $v_2(m)=\sum_{t=1}^{m_l-1}\sum_{i=m_l}^{m}(n-3, i-t)h^0(F(t)\otimes\mathcal{O}_{Y_{n-1}})$.[4]

[4] If $n=1$ or 2, then $v_2(m)=0$.

Since $F(m_l)\otimes\mathcal{O}_{Y_{n-1}}$ is generically generated by its global sections, every member of a maximal splitting of $F(m_l)\otimes\mathcal{O}_{Y_{n-1}}$ has a non-negative degree. Thus if $m>m_l+(2g-2)/h$, then $h^1(F(m)\otimes\mathcal{O}_{Y_{n-1}})=0$. Moreover, $h^1(F(m)\otimes\mathcal{O}_{Y_{n-1}})\leqq r'g$ if $m_l\leqq m\leqq m_l+(2g-2)h$. These and the fact that $\chi(F(t)\otimes\mathcal{O}_{Y_{n-1}})=\Delta^{n-1}\chi(F(t))$ imply that if t_1 is the integer with $(2g-2)/h\leqq t_1<(2g-2)/h+1$, then

$$\begin{aligned} v_1(m) &= \sum_{t=m_l}^{m}\{\Delta^{n-1}\chi(F(t))+h^1(F(t)\otimes\mathcal{O}_{Y_{n-1}})\}\sum_{i=t}^{m}(n-3,i-t)\\ &\leqq \sum_{t=m_l}^{m}\Delta^{n-1}\chi(F(t))\sum_{i=0}^{m-t}(n-3,i)+\sum_{t=m_l}^{m_l+t_1}r'g\sum_{i=0}^{m-t}(n-3,i)\\ &= \sum_{t=m_l}^{m}(n-2,m-t)\Delta^{n-1}\chi(F(t))+\sum_{t=m_l}^{m_l+t_1}r'g(n-2,m-t)\\ &\leqq \sum_{t=m_l}^{m}(n-2,m-t)\{r'\Delta^{n-1}P(t)-\alpha\}+K(n-2,m-m_l), \end{aligned}$$

where $\alpha=r'a_1/r-d(F,\mathcal{O}_X(1))$ and $K=(t_1+1)rg$. By the assumption (2) we know that α is non-negative. Our aim is to show that α is bounded above. We claim

$$\sum_{t=m_l}^{m}(n-2,m-t)\Delta^{n-1}P(t)=P(m)-P(m_l-1)-\sum_{j=1}^{n-2}(m-m_l,j)\Delta^jP(m_l-1).$$

In fact, since

$$\sum_{t=m_l}^{m}(n-2,m-t)\Delta^{n-1}P(t)=\sum_{t=0}^{n-m_l}(n-2,t)\Delta^{n-1}P(m-t),$$

we have only to show that

$$\begin{aligned} \sum_{t=0}^{m-m_l}(n-2,t)\Delta^{n-1}P(m-t) &= \sum_{t=0}^{m-m_l}(n-2-i,t)\Delta^{n-1-i}P(m-t)\\ &\quad -\sum_{j=1}^{i}(n-1-j,m-m_l)\Delta^{n-1-j}P(m_l-1). \end{aligned}$$

Assume that this holds for i. Then

$$\begin{aligned} \sum_{t=0}^{m-m_l}(n-2,t)\Delta^{n-1}P(m-t) &= \Delta^{n-2-i}P(m)+\sum_{t=1}^{m-m_l}\{(n-2-i,t)\\ &\quad -(n-2-i,t-1)\}\Delta^{n-2-i}P(m-t)-(n-2-i,m-m_l)\Delta^{n-2-i}P(m_l-1)\\ &\quad -\sum_{j=0}^{i}(n-1-j,m-m_l)\Delta^{n-1-j}P(m_l-1)\\ &= \sum_{t=0}^{m-m_l}(n-3-i,t)\Delta^{n-2-i}P(m-t)-\sum_{j=1}^{i+1}(n-1-j,m-m_l)\Delta^{n-1-j}P(m_l-1). \end{aligned}$$

Thus our claim is proved by induction on i. Set $N=m-m_l+1$. Then we have

$$\sum_{t=m_l}^{m}(n-2,m-t)\Delta^{n-1}P(t)=P(m)-P(m_l-1)-\sum_{j=1}^{n-2}(N-1,j)\Delta^jP(m_l-1)$$

and

$$\sum_{t=m_l}^{m}(n-2,m-t)=\sum_{t=0}^{m-m_l}(n-2,t)=(m-m_l,n-1)=(N-1,n-1).$$

Therefore we obtain

(2. 2. 9) $v_1(m) \leqq r'\{P(m)-P(m_l-1)\}-\alpha(N-1, n-1)$
$-r'\sum_{j=1}^{n-2}(N-1, j)\Delta^j P(m_l-1)+K(N-1, n-2)$, where $N=m-m_l+1$ and where K depends only on h, r and g.

Let us compute $v_2(m)$. Using Lemma 2. 4, (3) for $a=m_l$, $b=m$, $c=n-2$ and $t=t$, we obtain

$$\begin{aligned} v_2(m) &= \sum_{t=1}^{m_l-1} h^0(F(t)\otimes\mathcal{O}_{Y_{n-1}})\sum_{i=0}^{n-3}(m_l-t-1, i)(m-m_l, n-2-i) \\ &= \sum_{t=1}^{m_l-1} h^0(F(t)\otimes\mathcal{O}_{Y_{n-1}})\sum_{i=1}^{n-2}(m_l-t-1, n-2-i)(N-1, i) \\ &= \sum_{i=1}^{n-2}(N-1, i)\sum_{t=1}^{m_l-1}(m_l-t-1, n-2-i)h^0(F(t)\otimes\mathcal{O}_{Y_{n-1}}) \\ &= \sum_{i=1}^{n-2}(N-1, i)f_i, \end{aligned}$$

where $f_i=\sum_{t=1}^{m_l-1}(m_l-t-1, n-2-i)h^0(F(t)\otimes\mathcal{O}_{Y_{n-1}})$.

Write $F\otimes\mathcal{O}_{Y_{n-1}}=F_1\oplus F_2\oplus\cdots\oplus F_u$ with F_i indecomposable vector bundles on Y_{n-1}. If $d_i=d(F_i)$, $\rho_i=r(F_i)$, then we may assume that $d_1\leqq\rho_1 a_1/r-e-1$ and $d_i\leqq\rho_i(a_1+e)/r$ for $2\leqq i\leqq u$ because of the assumption (1) and (2. 2. 7). As (2. 2. 5) we have

$$\begin{aligned} &\sum_{t=1}^{m_l-1}(m_l-t-1, n-2-i)h^0(F_j(t)) \\ &\quad\leqq \sum_{t=t_j'}^{m_l-1}\{\rho_j ht+d_j-\rho_j(g-1)\}(m_l-t-1, n-2-i)+A(m, n-2-i), \end{aligned}$$

where $t_j'=\max(1, t_j)$ with the integer t_j such that $-\{d_j/\rho_j+(2g-1)(\rho_j-1)\}/h\leqq t_j<-\{d_j/\rho_j+(2g-1)(\rho_j-1)\}/h+1$. For the integer t_0' (or, t_0'') defined by $-\{(a_1+e)/r+g-1\}/h+1>t_0'\geqq-\{(a_1+e)/r+g-1\}/h$ (or, $-\{a_1/r-(e+1)/\rho_1+g-1\}/h+1>t_0''\geqq-\{a_1/r-(e+1)/\rho_1+g-1\}/h$, resp.), put $t'=\max(1, t_0')$ and $t''=\max(1, t_0'')$. Then f_i is evaluated as follows;

$$\begin{aligned} f_i &\leqq \sum_{t=t_1'}^{m_l-1}\{\rho_1 ht+\rho_1 a_1/r-e-1-\rho_1(g-1)(m_l-t-1, n-2-i) \\ &\quad+\sum_{j=2}^{u}\sum_{t=t_j'}^{m_l-1}\{\rho_j ht+\rho_j(a_1+e)/r-\rho_j(g-1)\}(m_l-t-1, n-2-i) \\ &\quad+uA(m_l, n-2-i) \\ &\leqq \sum_{t=t'}^{m_l-1}\{r'ht+r'a_1/r-1-r'(g-1)\}(m_l-t-1, n-2-i) \\ &\quad-\sum_{t=t'}^{t''}\{\rho_1 ht+\rho_1 a_1/r-e-1-\rho_1(g-1)\}(m_l-t-1, n-2-i) \\ &\quad+uA(m_l, n-2-i). \end{aligned}$$

Furthermore,

$$-\sum_{t=t'}^{t''}\{\rho_1 ht+\rho_1 a_1/r-e-1-\rho_1(g-1)\}(m_l-t-1, n-2-i)$$
$$\leqq\{(e/r+(e+1)/\rho_1)/h+1\}\rho_1|a_1/r-(e-1)/\rho_1-(g-1)|(m_l, n-2-i)$$
$$\leqq A'(m_l, n-2-i),$$

where A' is an integer which depends only on e, r, h and g. Thus we obtain

$$f_i\leqq r'h\sum_{t=t'}^{m_l-1}(m_l-t-1, n-2-i)t$$
$$+\{r'a_1/r-1-r'(g-1)\}\sum_{t=t'}^{m_l-1}(m_l-t-1, n-2-i)+B'(m_l, n-2-i)$$
$$\leqq r'h\sum_{t=0}^{m_l-2}(t, n-2-i)(m_l-1-t)$$
$$+\{r'a_1/r-1-r'(g-1)\}\sum_{t=0}^{m_l-t'-1}(t, n-2-i)+B'(m_l, n-2-i)$$
$$=r'h(m_l-2, n-i)+\{r'a_1/r-1-r'(g-1)\}(m_l-t'-1, n-i-1)$$
$$+B'(m_l, n-i-2).$$

The last part in the above inequality can be regarded as a polynomial with respect to m_l. The leading term of the polynomial is $r'hm_l^{n-i}/(n-i)!$. Since $g-1=\{(n-1)h+c(X)\}/2$ by the adjunction formula, the term of degree $n-i-1$ of the polynomial is

$$[(\sum_{k=-1}^{n-i-2}k)r'h/(n-i)!+\{r'a_1/r-1$$
$$-r'((n-1)h+c(X))/2\}/(n-i-1)!]m_l^{n-i-1}$$
$$=[r'h\{(n-i-3)(n-i)/2(n-i)!-(n-1)/2(n-i-1)!\}$$
$$+(r'a_1/r-1-r'c(X)/2)/(n-i-1)!]m_l^{n-i-1}$$
$$=\{-r'h(i+2)/2+r'a_1/r-1-r'c(X)/2\}m_l^{n-i-1}/(n-i-1)!.$$

Therefore

(2.2.10) $v_2(m)\leqq\sum_{i=1}^{n-2}(N-1, i)g_i(m_l)$, where $g_i(m_l)$ is a polynomial with respect to m_l of the following form; $g_i(m_l)=r'hm_l^{n-i}/(n-i)!+\{-r'h(i+2)/2+r'a_1/r-1-r'c(X)/2\}m_l^{n-i-1}/(n-i-1)!+$terms of degree$<n-i-1$ and the coefficients of $g_i(m)$ depend only on a_1, e, r, r', h, n and g.

Since by virtue of Lemma 2.5

$$\Delta^i P(m_l-1)=hm_l^{n-i}/(n-i)!+\{-h(i+2)/2+a_1/r$$
$$-c(X)/2\}m_l^{n-i-1}/(n-i-1)!+\text{terms of degree}<n-i-1,$$

the inequalities (2.2.8), (2.2.9) and (2.2.10) imply

(2.2.11) If $m\geqq m_l$, then $h^0(F(m))-r'P(m)\leqq g_{r'}(m_l-1)-r'P(m_l-1)-\alpha(N-1, n-1)+\sum_{i=1}^{n-2}\phi_i^{(r')}(m_l)(N-1, i)$, where $\phi_i^{(r')}(m_l)$ is a polynomial with respect to m_l of the following form;

$\phi_i^{(r')}(m_l)=-m_l^{n-i-1}/(n-i-1)!+$terms of degree$<n-i-1$ and the coefficients of $\phi_i^{(r')}(m_l)$ depend only on a_1, e, r, r', h, n and g.

Since the leading term of $g_{r'}(m)$ is $(r'-1)hm^n/n!$ and that of $r'P(m)$ is $r'hm^n/n!$, there exists an integer L such that

(2. 2. 12) $g_{r'}(m-1)-r'P(m-1)<0$ and $\phi_i^{(r')}(m)<0$ for all r', i and $m\geqq L$.

(2. 2. 6) says that if $m\geqq L$ and $h^0(F(m))-r'P(m)\geqq 0$, then m must be greater than m_l-1. If one takes this L in advance and assumes that F has the properties (1), (2) and (3) for L, then the above fact and the property (3) imply that $h^0(F(m))-r'P(m)\geqq 0$ for some $m\geqq m_l$. Assume that $m_l\geqq L$ and F satisfies the assumption (2. 2. 7). Choose an integer m such that $m\geqq m_l$ and $h^0(F(m))-r'P(m)\geqq 0$. Then (2. 2. 11) and (2. 2. 12) assert that

$$0\leqq h^0(F(m))-r'P(m)\leqq g_{r'}(m'-1)-r'P(m_l-1)-\alpha(N-1, n-1)$$
$$+\sum_{i=1}^{n-2}\phi_i^{(r')}(m_l)(N-1, i)<0$$

This is a contradiction, whence $m_l<L$. Therefore if F enjoys the properties (1), (2), (3) and (2. 2. 7) for this L, then $F(L)$ is generically generated by its global sections. Thus $d(F(L), \mathcal{O}_X(1))\geqq 0$, which implies that $d(F, \mathcal{O}_X(1))\geqq -rLh$. Therefore $M=\min\{-rLh,\ a_1-r(e+1)\}$ is the desired integer.

q.e.d.

§ 3. *e*-stable sheaves.

In this section we shall assume that $f: X\to S$ is a projective, smooth, geometrically integral morphism with relative dimension n and an f-very ample invertible sheaf $\mathcal{O}_X(1)$ and S is connected and noetherian. To construct the moduli schemes of stable sheaves we cover the family of stable sheaves by subfamilies which are open and bounded. For this purpose let us introduce the following notion.

Definition 3. 1. Let e be a non-negative integer. A stable (or, semi-stable) sheaf F (with respect to $\mathcal{O}_X(1)$) of rank r on a geometric fibre X_s of X over S is said to be e-stable (or, e-semi-stble, resp.) (with respect to $\mathcal{O}_X(1)$) if for general non-singular curves[5] $C=D_1\cdot D_2\cdot\cdots\cdot D_{n-1}$, $D_i\in|\mathcal{O}_{X_s}(1)|$, every coherent subsheaf E of $F\otimes\mathcal{O}_C$ of rank $t(1\leqq t\leqq r-1)$ has a degree$\leqq\{td(F, \mathcal{O}_X(1))+e\}/r$.

Remark 3. 2. The condition on $F\otimes\mathcal{O}_C$ in the above definition means that $F\otimes\mathcal{O}_C$ is of cotype (β) with $\beta_t=e/rt$ or equivalently it is of type (α) with $\alpha_t=te/(r-t)^2r$ (see [8] Definition 1. 1). Note that $\alpha_1\leqq\alpha_2\leqq\cdots\leqq\alpha_{r-1}$.

To show that the family of e-stable sheaves is bounded we need

Lemma 3. 3. *Let $\Phi_{X/S}(e, H)$ be a family of classes of coherent sheaves*

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5) See the footnote (1).

on the fibres of X over S such that if $F\in\Phi_{X/S}(e, H)$, then F is a torsion free module of rank r on a geometric fibre X_s of X over S, for general curves $C=D_1\cdot D_2\cdot\cdots\cdot D_{n-1}$, $D_i\in|\mathcal{O}_{X_s}(1)|$, $F\otimes\mathcal{O}_C$ is of cotype (β) with $\beta_t=e/rt$ and the Hilbert polynomial of F is H. Then $\Phi_{X/S}(e, H)$ is bounded.

Proof. Let F be a coherent sheaf of on a geometric fibre X_s of X over S. Assume that F is contained in $\Phi_{X/S}(e, H)$. Then, as in Lemma 2.3, we can find $k(s)$-rational members $D_1, D_2, \cdots, D_{n-1}$ in $|\mathcal{O}_{X_s}(1)|$ such that (1) $Y_0=X_s$, $Y_1=D_1, \cdots, Y_{n-1}=D_1\cdot D_2\cdot\cdots\cdot D_{n-1}$ are non-singular, (2) $F\otimes\mathcal{O}_{Y_i}$ is a torsion free $\mathcal{O}_{Y_i}$-module and that every coherent $\mathcal{O}_{Y_i}$-submodule E of $F\otimes\mathcal{O}_{Y_i}$ of rank $t(1\leqq t\leqq r-1)$ has a degree$\leqq\{td(F, \mathcal{O}_X(1))+e\}/r$. On the other hand, there exists an integer m, which depends only on H and e, such that $td(F(m), \mathcal{O}_X(1))+e<0$ for all t. Then $h^0(Y_i, F(m)\otimes\mathcal{O}_{Y_i})=0$ for all $i=0, 1, \cdots, n-1$. This and Theorem 1.13 of [6] complete our proof.

Let $\mathfrak{S}_{X/S}(e, H)$ be the family of classes of coherent sheaves on the fibres of X over S such that F is contained in $\mathfrak{S}_{X/S}(e, H)$ if and only if F is e-semi-stable and the Hilbert polynomial of F is H.

Corollary 3.3.1. *For each e, H, $\mathfrak{S}_{X/S}(e, H)$ is bounded.*

From now on we assume

(3.4) *for all geometric points s of S and all $i>0$, $H^i(X_s, \mathcal{O}_{X_s}(1))=0$.*

If one replaces $\mathcal{O}_X(1)$ by $\mathcal{O}_X(m)$ with m a sufficiently large integer, then the assumption (3.4) is satisfied. Moreover, a coherent $\mathcal{O}_{X_s}$-module F is stable (or, semi-stable) with respect to $\mathcal{O}_X(1)$ if and only if it is so with respect to $\mathcal{O}_X(m)$. Thus, without losing any generalities, we may assume that $\mathcal{O}_X(1)$ satisfies (3.4).

Lemma 3.5. *Under the assumption 3.4, the property that a coherent sheaf is e-stable (or, e-semi-stable) is open.*

Proof. The assumption (3.4) implies that $f_*(\mathcal{O}_X(1))=E$ is a locally free $\mathcal{O}_S$-module (E.G.A., Ch. III, 7.9.10). Moreover, X is a closed subscheme of $\mathbf{P}(E)$, $\mathcal{O}_X(1)\cong\mathcal{O}_{\mathbf{P}(E)}(1)\otimes\mathcal{O}_X$ and $H^0(\mathbf{P}(E)_s, \mathcal{O}_{\mathbf{P}(E)_s}(1))\cong H^0(X_s, \mathcal{O}_{X_s}(1))$ for all geometric points s of S. Put $r(E)=N$. Let us consider the Grassmanian $Y=\mathrm{Grass}_{N-n}(E)$ and the closed subscheme $\mathbf{P}(Q)$ of $\mathbf{P}(E)\times_S Y$ with Q the universal quotient bundle. Let X' be the scheme theoretic intersection $\mathbf{P}(Q)$ and $X\times_S Y$ in $\mathbf{P}(E)\times_S Y$. It is clear that X' is a fibre bundle over X, and hence the first projection $p:X'\to X$ is smooth. Using the Jacobian criterion for smoothness, we know that for the second projection $q:X'\to Y$, there exists an open set U of Y such that (i) $q_U:X'_U\to U$ is smooth and geometrically integral, (ii) $\dim X'_u=1$ for all points u of U and (iii) $U\cap Y_s\neq\phi$ for all geometric points s of S (see E.G.A., Ch. IV, 17.13.2 and 17.13.4, (i)).

$$\begin{tikzcd}
P(Q)\cap(X\times_S Y)=X' \arrow[rr, "p"] \arrow[ddd, "q"'] & & X \arrow[ddd, "f"] \\
& X'_U \arrow[ul, hook] \arrow[d] & \\
& U \arrow[dl, hook] \arrow[dr] & \\
Y \arrow[rr] & & S
\end{tikzcd}$$

Set $F'=P^*(F)$ for the given S-flat coherent $\mathcal{O}_X$-module F. For a geometric point s of S, U_s is a variety over $k(s)$ and q_{U_s} is proper. Thus there exists a non-empty open set V_s of U_s such that $F'\otimes_{\mathcal{O}_S}k(S)$ is flat over V_s. Since F is f-flat and since p is a flat morphism, F' is flat over S. Therefore, applying Theorem 11. 3. 10 of E. G. A., Ch. IV, we see that F' is q-flat at every point of $q^{-1}(V_s)$. Replacing U by an open set of U, we may assume that F'_U is flat over U and U enjoys the properties (i), (ii) and (iii) above. By virtue of Theorem 2. 8 of [8] and Remark 3. 2, the property that a coherent sheaf is of cotype (β) with $\beta_t=e/rt$ is open. Thus there exists an open set U' such that fos every algebraically closed field k, $U'(k)=\{u\in U(k)\,|\,F'\otimes_{\mathcal{O}_Y}k(u)$ is of cotype $(\beta)\}$. It is easy to see that for a geometric point s of S, U'_s is non-empty if and only if for general curves $C=D_1\cdot D_2\cdot\cdots\cdot D_{n-1}$, $D_i\in|\mathcal{O}_{X_s}(1)|$, $F\otimes\mathcal{O}_C$ is cotype (β). Sine Y is flat over S, the image W of U' in S is open. On the other hand, we can find an open set W' such that for every algebraically closed field k, $W'(k)=\{s\in S(k)\,|$ $F\otimes_{\mathcal{O}_S}k(s)$ is stable (or, semi-stable, resp.)$\}$ ([8] Theorem 2. 8). Then it is obvious that the open set $W\cap W'$ is the desired one in S. q.e.d.

The following, which plays a key role in the sequal, is a corollary to Lemma 2. 2 and Corollary 3. 3. 1.

Proposition 3. 6. *For each $\mathfrak{S}_{X/S}(e, H)$, there exists an integer N such that*

1) *for all $F\in\mathfrak{S}_{X/S}(e, H)$, $m\geqq N$ and $i>0$, $F(m)$ is generated by its global sections and $h^i(F(m))=0$,*

2) *if F is contained in $\mathfrak{S}_{X/S}(e, H)$ and if it is stable, then for all $m\geqq N$ and all coherent subsheaves E of F with $0\neq E\subsetneq F$,*

$$h^0(E(m))/r(E)<h^0(F(m))/r(F),$$

3) *if F is contained in $\mathfrak{S}_{X/S}(e, H)$ and if it is not stable, then for all $m\geqq N$ and all coherent subsheaves E of F with $0\neq E\subsetneq F$,*

$$h^0(E(m))/r(E)\leqq h^0(E(m))/r(E)$$

and, moreover, there exists a coherent, non-trivial subsheaf E_0 of F such that $h^0(E_0(m))/r(E_0)=h^0(F(m))/r(F)$ for all $m\geqq N$.

Proof. By virtue of Corollary 3. 3. 1 there exists an integer N_1 such that (1) holds for $N=N_1$. By taking $\mathfrak{S}_{X/S}(e, H)(m_0)=\{F(m_0)\,|\,F\in\mathfrak{S}_{X/S}(e, H)\}$

instead of $\mathfrak{S}_{X/S}(e, H)$ with m_0 a sufficiently negative integer, we may assume that $d(F, \mathcal{O}_X(1))+e<0$ for every $F\in\mathfrak{S}_{X/S}(e, H)$. Let us apply Lemma 2. 2 to the case that $P(m)=\chi(F(m))/r(F)=H(m)/r(F)$, $r=r(F)$ and $e=e$, where F is a member of $\mathfrak{S}_{X/S}(e, H)$. Then we obtain the integers L and M satisfying the conditions in Lemma 2. 2 because $a_1+e=d(F, \mathcal{O}_X(1))+e<0$. We may assume that $L\geqq N_1$. Let $\mathscr{B}$ be the family of classe of coherent sheaves on the fibres of X over S such that E is contained in $\mathscr{B}$ if and only if (a) E is a coherent subsheaf of a member F of $\mathfrak{S}_{X/S}(e, H)$, (b) F/E is torsion free and (c) $h^0(E(m))\geqq r(E)P(m)=r(E)h^0(F(m))/r(F)$ for some $m\geqq L$. Then every member E of $\mathscr{B}$ enjoys the properties (1), (2) and (3) in Lemma 2. 2 for $F=E$. Thus the set $\{d(E, \mathcal{O}_X(1))\,|\,E\in\mathscr{B}\}$ is bounded below by M. Since $\mathfrak{S}_{X/S}(e, H)$ is bounded, the condition (b) above and Corollary 1. 2. 1 of [8] imply that $\mathscr{B}$ is bounded. Therefore, there exists an integer $N\geqq L$ such that for all $i>0$, $m\geqq N$ and $E\in\mathscr{B}$, $h^i(E(m))=0$. This and the definition of the stable (or, semi-stable) sheaves imply that $h^0(E(m))/r(E)<h^0(F(m))/r(F)$ (or, $\leqq$, resp.) for all $m\geqq N$ and all coherent subsheaves E of F such that $E\neq 0$, F/E is torsion free and F is contained in $\mathfrak{S}_{X/S}(e, H)$. Pick a coherent subsheaf E of $F\in\mathfrak{S}_{X/S}(e, H)$ with $0\neq E\neq F$. There exists a coherent subsheaf E' of F such that $r(E)=r(E')$, $E\subseteqq E'$ and F/E' is torsion free. Thus if $r(E)<r$, then $h^0(E(m))/r(E)\leqq h^0(E'(m))/r(E')<h^0(F(m))/r(F)$ (or, $\leqq$, resp.) for all $m\geqq N$. If $h^0(F(m))=h^0(E(m))$, then $H^0(F(m))=H^0(E(m))$ and hence $E(m)=F(m)$ because $F(m)$ is generated by its global sections. Thus if $r(E)=r$ and if $E\neq F$, then we have also that $h^0(E(m))/r(E)<h^0(F(m))/r$. Finally assume that a member F of $\mathfrak{S}_{X/S}(e, H)$ is not stable, then we can find a coherent, non-trivial subsheaf E_0 of F such that $\chi(E_0(m))/r(E_0)=\chi(F(m))/r(F)$ and F/E_0 is torsion free. It is easy to see that E_0 is contained in $\mathscr{B}$. Hence for all $m\geqq N$, and all $i>0$, $h^i(E_0(m))=0$. Thus $h^0(E_0(m))/r(E_0)=h^0(F(m))/r$ for all $m\geqq N$. N is therefore the desired integer. q.e.d.

§ 4. Techniques of Gieseker.

In this section we shall recall and generalize the results of D. Gieseker [3] on the quotient of an algebraic scheme by an algebraic group.

From now on k denotes a field of characteristic $p\geqq 0$. Let V be an N-dimensional vector space over k and let V' be another finite dimensional vector space over k. For $G=SL(N, k)$, $\hat{\sigma}_0$ denotes a natural dual action of G on V; $\hat{\sigma}_0: V\to V\otimes_k k[G]$. For an integer r with $1\leqq r\leqq N$, set $W=\mathrm{Hom}_k(\bigwedge^r V, V')$, then $\hat{\sigma}_0$ provides us with a dual action $\hat{\sigma}$ of G on $W^\vee$, where $W^\vee$ is the dual vector space of W. Fix a basis $e_1, e_2, \cdots, e_N$ of V and a basis $f_1, f_2, \cdots, f_M$ of V'. Then for suitable functions $\{x_{ij}\}$ defining a system of coordinates of $SL(N, k)$, $\hat{\sigma}$ can be written as follows;

$$\begin{aligned}&\hat{\sigma}(e_{i_1}\wedge\cdots\wedge e_{i_r}\otimes f_j{}^\vee)\\&=\sum_{j_1<\cdots<j_r}\sum_{\tau\in S_r} e_{j_1}\wedge\cdots\wedge e_{j_r}\otimes f_j{}^\vee\otimes(\mathrm{sgn}(\tau)x_{i_1j_{\tau(1)}}\cdot\cdots\cdot x_{i_rj_{\tau(r)}})\end{aligned}$$

where $\{f_j^\vee\}$ is the dual basis of $\{f_j\}$ and S_r is the r-th symmetric group. Thus we obtain an action σ of G on $\mathbf{P}(W^\vee)$ and a G-linearization on the hyperplane bundle L on $\mathbf{P}(W^\vee)$. Since the center of G acts trivially on $\mathbf{P}(W^\vee)$, σ induces an action $\bar{\sigma}$ of $\bar{G}=PGL(N, k)$ on $\mathbf{P}(W^\vee)$. and $\bar{G}$-linearizations on $L^{\otimes\alpha N}$ for all integers α. For an algebraically closed field K containing k, a non-zero element T of $W\otimes_k K$ defines a K-linear injection $K\ni a\to aT\in W\otimes_k K$, whence T induces a K-linear surjection $W^\vee\otimes_k K\to K$. The last map yields a K-rational point of $\mathbf{P}(W^\vee)$, which is also denoted by T. T can be regarded as an alternative multilinear map of $V\otimes_k K$ to $V'\otimes_k K$. For vectors $v_1, \cdots, v_r$ in $V\otimes_k K$, the value of T at $v_1\wedge v_2\wedge\cdots\wedge v_r$ is denoted by $T(v_1, \cdots, v_r)$. Now let us employ the following notion due to Gieseker.

Definition. Let K be an algebraically closed field containing k and let T be a non-zero element of $W\otimes_k K$ or a K-rational point of $\mathbf{P}(W^\vee)$. Vectors $v_1,\cdots,v_d$ in $V\otimes_k K$ are said to be T-independent if there exist vectors $v_{d+1},\cdots,v_r$ in $V\otimes_k K$ such that $T(v_1, \cdots, v_r)\neq 0$. A vector v in $V\otimes_k K$ is said to be T-dependent on $v_1,\cdots,v_d$ if $T(v_1,\cdots,v_d,v,w_{d+2},\cdots,w_r)=0$ for all vectors $w_{d+2},\cdots,w_r$ in $V\otimes_k K$. The vector subspace of $V\otimes_k K$ formed by vectors which are T-dependent on $v_1, \cdots, v_d$ will be called the T-span of $v_1, \cdots, v_d$.

By the same argument as in Proposition 2. 3 and Proposition 2. 4 of [3] and by Theorem 1. 5 we obtain

Proposition 4. 1. *Let K be an algebraically closed field containing k.*

1) *A point T in $\mathbf{P}(W^\vee)(K)$ is properly stable (or, semi-stable) with respect to the action σ and the G-linearized invertible sheaf L if whenever $v_1, \cdots, v_d$ in $V\otimes_k K$ are T-independent and U is the T-span of $v_1, \cdots, v_d$, then* $\dim U<dN/r=(d/r)\dim V$. *(or,* $\dim U\leqq dN/r$, *resp.)*

2) *For a point T in $\mathbf{P}(W^\vee)(K)$, assume that there exist a vector subspace U of $V\otimes_k K$ and an integer d such that $T(v_1, \cdots, v_d, v_{d+1}, \cdots)=0$ whenever $v_1, \cdots, v_{d+1}$ are in U and that* $\dim U>dN/r$. *Then the T is not semi-stable.*

Let $f:X\to S$ and $\mathcal{O}_X(1)$ be as in § 3. Moreover, assume that S is an algebraic k-scheme. Fix a numerical polynomial

$$H(m)=rhm^n/n!+\{a_1-rc(X)2\}m^{n-1}/(n-1)!+\text{terms of degree}<n-1,$$

where r is a positive integer, h is the degree of X with respect of $\mathcal{O}_X(1)$ and where $c(X)$ is the degree of the canonical divisor on a fibre of X over S. Let Q be a union of some of connected components of $\mathrm{Quot}^H_{\mathcal{O}_X^{\oplus N}/X/S}$ and let $X_Q=X\times_S Q$. The universal quotient sheaf on X_Q is denoted by $\phi:V\otimes_k\mathcal{O}_{X_Q}\to F$. Fix a basis $e_1, \cdots, e_N$ of V and functions $\{x_{ij}\}$ defining a system of coordinates of $SL(N, k)$. Define a homomorphism of $\mathcal{O}_{G\times_k Q}$-modules ψ of $V\otimes_k\mathcal{O}_{G\times_k Q}$ to itself as follows;

$$\phi(e_i\otimes 1)=\sum_{j=1}^{N} e_j\otimes x_{ij}.$$

Set $\tilde{\phi}=p_2{}^*(\phi)$ and $\tilde{F}=p_2{}^*(F)$ with $p_2:G\times_k X_Q\to X_Q$ the second projection. Then we obtain the homomorphism $\tilde{\phi}\lambda:V\otimes_k\mathcal{O}_{G\times_k X_Q}\to F$, where λ is the base change of ϕ by X. By virtue of the universality of (F,ϕ) and the connectedness of G, we obtain a morphism $\tau:G\times_k Q\to Q$ and an isomorphism $\lambda':\tilde{\tau}^*(F)\xrightarrow{\sim}\tilde{F}$ such that the following diagram is commutative;

$$\begin{tikzcd}
V\otimes_k\mathcal{O}_{G\times_k X_Q} \arrow[r, "\tilde{\phi}"] \arrow[d, "\lambda"'] & \tilde{F} \arrow[r] \arrow[d, "\wr\,\lambda'"] & 0 \\
V\otimes_k\mathcal{O}_{G\times_k X_Q} \arrow[r, dash] & \tilde{\tau}^*(F) \arrow[r] & 0
\end{tikzcd}$$

where $\tilde{\tau}:G\times_k X_Q\to X_Q$ is the base extension of τ. Let $X_1=G\times_k X_Q$, $X_2=G\times_k G\times_k X_Q$, $p_{23}:X_2\to X_1$ be the projection to the second and the third factors and let $\mu:G\times_k G\to G$ be the group multiplication. Then we know easily

$$(1_G\times\tau)^*(\lambda)(e_i\otimes 1\otimes 1\otimes 1)=\sum_{j=1}^{N} e_j\otimes x_{ij}\otimes 1\otimes 1$$

$$p_{23}{}^*(\lambda)(e_i\otimes 1\otimes 1\otimes 1)=\sum_{j=1}^{N} e_j\otimes 1\otimes x_{ij}\otimes 1$$

$$(\mu\times 1_{X_Q})^*(\lambda)(e_i\otimes 1\otimes 1\otimes 1)=\sum_{j=1}^{N} e_j\otimes\Big(\sum_{k=1}^{N} x_{ik}\otimes x_{kj}\Big)\otimes 1,$$

whence we have

$$p_{23}{}^*(\lambda)(1_G\times\tilde{\tau})^*(\lambda)=(\mu\times 1_{X_Q})^*(\lambda).$$

Consider the following commutative diagram;

$$\begin{tikzcd}
(\tilde{\tau}(\mu\times 1_{X_Q}))^*(V\otimes_k\mathcal{O}_{X_1}) \arrow[rr, "(\tilde{\tau}(\mu\times 1))^*(\phi)"'] \arrow[d, "(\mu\times 1)^*(\lambda)"] & & (\tilde{\tau}(\mu\times 1_{X_Q}))^*(F) \\
(p_2(\mu\times 1_{X_Q}))^*(V\otimes_k\mathcal{O}_{X_1})=p_3{}^*(V\otimes_k\mathcal{O}_{X_1}) & & \\
 & =(p_2p_{23})^*(V\otimes_k\mathcal{O}_{X_1}) \arrow[r, "p_3{}^*(\phi)"] & p_3{}^*(F) \arrow[uu, "(\mu\times 1)^*(\lambda')"'] \\
(p_2(1_G\times\tilde{\tau}))^*(V\otimes_k\mathcal{O}_{X_1})= & (\tilde{\tau}p_{23})^*(V\otimes_k\mathcal{O}_{X_1}) \arrow[r, "(\tilde{\tau}p_{23})^*(\phi)"] \arrow[u, "p_{23}{}^*(\lambda)"] & (\tilde{\tau}p_{23})^*(F) \arrow[u, "p_{23}{}^*(\lambda')"'] \\
(\tilde{\tau}(1_G\times\tilde{\tau}))^*(V\otimes_k\mathcal{O}_{X_1}) \arrow[rr, "(\tilde{\tau}(1_G\times\tilde{\tau}))^*(\phi)"] \arrow[u, "(1_G\times\tilde{\tau})^*(\lambda)"'] & & (\tilde{\tau}(1_G\times\tilde{\tau}))^*(F) \arrow[u, "(1_G\times\tilde{\tau})^*(\lambda')"']
\end{tikzcd}$$

where $p_3:X_2\to X_Q$ is the third projection. Note that all the sheaves of the left hand side of the above diagram are canonically isomorphic to $V\otimes_k\mathcal{O}_{X_2}$. The equality

$$\begin{aligned}
&(\mu\times 1_{X_Q})^*(\lambda')(\tilde{\tau}(\mu\times 1_{X_Q}))^*(\phi)=p_3{}^*(\phi)(\mu\times 1_{X_Q})^*(\lambda)\\
&\quad=p_3{}^*(\phi)p_{23}{}^*(\lambda)(1_G\times\tilde{\tau})^*(\lambda)=p_{23}{}^*(\lambda')(1_G\times\tilde{\tau})^*(\lambda')(\tilde{\tau}(1_G\times\tilde{\tau}))^*(\phi)
\end{aligned}$$

implies that

$$\tilde{\tau}(\mu\times 1_{X_Q})=\tilde{\tau}(1_G\times\tilde{\tau}) \text{ and}$$
$$(\mu\times 1_{X_Q})^*(\lambda')=p_{23}{}^*(\lambda')(1_G\times\tilde{\tau})(\lambda')$$

because of the universality of (Q, F, ϕ). These facts mean that τ (or, $\tilde{\tau}$) is an action of G on Q (or, X_Q, resp.) and, moreover, ϕ (λ or λ') defines a G-linearization on $V\otimes_k\mathcal{O}_{G\times_k X_Q}(V\otimes_k\mathcal{O}_{X_Q}$ or F, resp.). It is obvious that the structure morphism $P:Q\to S$ is a G-morphism with the the trivial action of G on S.

The following is a generalization of Lemma 4. 1 of [3].

Lemma 4. 2. *Let U be the largest open set X_Q over which F is locally free. Then there exists a G-linearized invertible sheaf L on X_Q and a G-homomorphism $\gamma:\overset{r}{\wedge}F\to L$ which is an isomorphism on U.*

Proof. Since $p:Q\to S$ is a G-morphism, $\mathcal{O}_{X_Q}(1)=(1_X\times p)^*(\mathcal{O}_X(1))$ carries a G-linearlzation. If one notes that in the diagram

$$\begin{tikzcd}[column sep=huge]
G\times_k G\times_k X_Q \arrow[r, shift left=2, "1_G\times\tilde{\tau}"] \arrow[r, "\tilde{p}_{23}"] \arrow[r, shift right=2, "\mu\times 1_{X_Q}"'] \arrow[d] & G\times_k X_Q \arrow[r, shift left=1, "\tilde{p}_2"] \arrow[r, shift right=1, "\tilde{\tau}"'] \arrow[d] & X_Q \arrow[d, "q"] \\
G\times_k G\times_k Q \arrow[r, shift left=2, "1_G\times\tau"] \arrow[r, "p_{23}"] \arrow[r, shift right=2, "\mu\times 1_Q"'] & G\times_k Q \arrow[r, shift left=1, "p_2"] \arrow[r, shift right=1, "\tau"'] & Q
\end{tikzcd}$$

every square made by corresponding morphisms is cartesian and every morphism in the lower row is flat ($\tau:G\times_k Q\xrightarrow{(1_G,\,\tau)}G\times_k Q\xrightarrow{p_2}Q$ and $(1_G, \tau)$ is an isomorphism etc.), then it is easy to see that for every G-linearized $\mathcal{O}_{X_Q}$-module E, $q_*(E)$ has a G-linearization, whence so does $q^*q_*(E)$ (see E.G.A., Ch. III, 1. 4. 15). Moreover, the canonical map $q^*q_*(E)\to E$ is a G-homomorphism. Now let us apply the above observation to $F(m)=F\otimes\mathcal{O}_{X_Q}(m)$. Then as in the proof of Proposition 2. 1 of [8] we have a resolution of F by locally free, G-linearized $\mathcal{O}_{X_Q}$-modules;

$$0\longrightarrow E_n\xrightarrow{f_n}E_{n-1}\xrightarrow{f_{n-1}}\cdots\xrightarrow{f_2}E_1\xrightarrow{f_1}E_0\xrightarrow{f_0}F\longrightarrow 0$$

where all the f_i are G-homomorphisms. Set

$$L=(\det E_0)\otimes(\det E_1)^{-1}\otimes\cdots\otimes(\det E_n)^{(-1)^n},$$

then L carries a G-linearization. This L is the desired invertible sheaf. Since F is q-flat, so is $\ker(f_i)$. Thus $\ker(f_i)$ and F are locally free on U and for all points x of Q,

$$0\longrightarrow E_n\otimes_{\mathcal{O}_Q}k(x)\xrightarrow{f_n\otimes k(x)}\cdots\longrightarrow E_1\otimes_{\mathcal{O}_Q}k(x)\xrightarrow{f_1\otimes k(x)}$$
$$E_0\otimes_{\mathcal{O}_Q}k(x)\xrightarrow{f_0\otimes k(x)}F\otimes_{\mathcal{O}_Q}k(x)\longrightarrow 0$$

is exact. We know therefore that for $K_0=\ker(f_0\otimes k(x))$ and $y\in q^{-1}(x)$, $\mathrm{hd}(K_{0,y})\leqq\max\{\dim(\mathcal{O}_{q^{-1}(x),y})-1, 0\}$. This implies that if U' is the largest open set of X_Q over which $\ker(f_0)$ is locally free, then (1) all the $\ker(f_i)$ are locally free on U', (2) $\mathrm{codim}(X_y-U'_y, X_y)\geqq 2$ for all points y of Q and (3) $U'\supseteq U$. Since $\ker(f_i)$ are naturally G-linearized, U' is G-stable. Let us cover U' by a family of affine open subsets $\{U_j\}$ such that every $K_{ij}=\ker(f_i)|U_j$ is a free $\mathcal{O}_{U_j}$-module. Let $\{a_1(i, j), \cdots, a_{r_i}(i, j)\}$ be a free basis of K_{ij} ($i\geqq 0$) and let $b_1(i+1, j), \cdots, b_{r_i}(i+1, j)$ be elements of $\Gamma(U_j, E_{i+1})$ whose images to K_{ij} are $a_1(i, j), \cdots, a_{r_i}(i, j)$ respectively. Then the set $\{a_1(i, j), \cdots, a_{r_i}(i, j), b_1(i, j), \cdots, b_{r_{i-1}}(i, j)\}$ forms a free basis of $\Gamma(U_j, E_i)$ ($i\geqq 1$). Take $s_1, \cdots, s_r$ from $\Gamma(U_j, F)$ and pull them back to $t_1, \cdots, t_r$ in $\Gamma(U_j, E_0)$. Let $\gamma_j(s_1\wedge\cdots\wedge s_r)$ be the element of $\Gamma(U_j, L)$ defined as follows

$$(t_1\wedge\cdots\wedge t_r\wedge a_1(0, j)\wedge\cdots\wedge a_{r_0}(0, j))$$
$$\otimes(b_1(1, j)\wedge\cdots\wedge b_{r_0}(1, j)\wedge a_1(1, j)\wedge\cdots\wedge$$
$$a_{r_1}(1, j))^{-1}\otimes\cdots\otimes(b_1(n-1, j)\wedge\cdots\wedge b_{r_{n-2}}(n-1, j)\wedge a_1(n-1, j)\wedge\cdots\wedge$$
$$a_{r_{n-1}}(n-1, j))^{(-1)^{n-1}}\otimes(b_1(n, j)\wedge\cdots\wedge b_{r_{n-1}}(n, j))^{(-1)^n}.$$

Then it is clear that $\gamma_j(s_1\wedge\cdots\wedge s_r)$ is independent of the choice of $t_1,\cdots,t_r, a_l(i, j)$, $b_m(i, j)$. Thus we obtain a map of $\overset{r}{\wedge}(F|_{U_j})$ to $L|_{U_j}$ and moreover, γ_j coincides with $\gamma_{j'}$ on $U_j\cap U_{j'}$. Patching them together, we get a homomorphis $\gamma_{U'}$: $\overset{r}{\wedge}F|_{U'}\to L_{U'}$ which is an isomorhism on U. By the uniqueness of γ_j, we see that $\gamma_{U'}$ is a G-homomorphism. In order to extend the $\gamma_{U'}$ to a homomorphism on the whole space X_Q, we need

Claim: For all points x of X_Q, $\mathrm{depth}(\mathcal{O}_{X_Q,x})\geqq\dim(\mathcal{O}_{q^{-1}q(x),x})$.

In fact, by virtue of E.G.A., Ch. IV, 17.5.8 we have

$$\dim(\mathcal{O}_{X_Q,x})-\mathrm{depth}(\mathcal{O}_{X_Q,x})=\dim(\mathcal{O}_{Q,q(x)})-\mathrm{depth}(\mathcal{O}_{Q,q(x)})$$

because q is smooth. Thus

$$\mathrm{depth}(\mathcal{O}_{X_Q,x})\geqq\dim(\mathcal{O}_{X_Q,x})-\dim(\mathcal{O}_{Q,q(x)})=\dim(\mathcal{O}_{q^{-1}q(x),x}).$$

Since $\mathrm{codim}(X_y-U'_y, X_y)\geqq 2$ for all points y of Q, the above claim implies that $\mathrm{depth}(\mathcal{O}_{X_Q,x})\geqq 2$ for all points x of X_Q-U'. By this and E.G.A., Ch. IV 5.10.5 we know

$$\eta_*(L|_{U'})=L$$

where $\eta: U'\to X_Q$ is the natural open immersion. Thus $\gamma_{U'}$ can be extended to a homomorphism on X_Q as follows

$$\gamma: \overset{r}{\wedge}F\xrightarrow{\alpha}\eta_*(\overset{r}{\wedge}F|_{U'})\xrightarrow{\eta_*(\gamma_{U'})}\eta_*(L|_{U'})=L$$

where α is a natural G-homomorphism. Since τ, p_2 are flat, we have

$$\tau^*\eta_*(L|_{U'})\cong(1_G\times\eta)_*(\tau_{U'})^*(L|_{U'})$$
$$p_2{}^*\eta_*(L|_{U'})\cong(1_G\times\eta)_*(p_{2,U'})^*(L|_{U'})$$
$$\tau^*\eta_*(\overset{r}{\wedge}F|_{U'})\cong(1_G\times\eta)_*(\tau_{U'})^*(\overset{r}{\wedge}F|_{U'})$$
$$p_2{}^*\eta_*(\overset{r}{\wedge}F|_{U'})\cong(1_G\times\eta)_*(p_{2,U'})^*(\overset{r}{\wedge}F|_{U'})$$

which imply that $\eta_*(\gamma_{U'})$ is a G-homomorphism. q.e.d.

Following D. Gieseker we denote L in the above lemma by $\det F$. From now on we assume

(4. 3) *for all invertible sheaves A on geometric fibres X_y of X_Q which is numerically equivalent to* $(\det F)\otimes_{\mathcal{O}_Q}k(y)$, *$h^0(A)$ is constant and $h^i(A)=0$ for all positive integers i.*

We also assume for a moment

(4. 4) *f has a section $\varepsilon: S\to X$.*

Since f is projective, smooth and geometrically integral, the Picard scheme $\mathrm{Pic}_{X/S}$ exists and moreover, the assumption (4. 4) implies that we have a unique Poincaré sheaf L on $X\times_S\mathrm{Pic}_{X/S}$ such that $(\varepsilon\times 1_{\mathrm{Pic}_{X/S}})^*(L)\cong\mathcal{O}_{\mathrm{Pic}_{X/S}}$. Let ν be the morphism of Q to $\mathrm{Pic}_{X/S}$ defined by $\det F$ and let P be a union of a finite number of connected components of $\mathrm{Pic}_{X/S}$ such that (1) $\nu(Q)\subseteq P$ and (2) $h^0(L\otimes_{\mathcal{O}_P}k(z))$ is constant and $h^i(L\otimes_{\mathcal{O}_P}k(z))=0$ for all geometric points z of P and all positive integers i. ν can be regarded as a morphism of Q to P and we shall use the notation L instead of $L|_P$. By the universality of L, we see that $(1_X\times\nu)^*(L)\cong(\det F)\otimes q^*(M)$ for some invertible sheaf M on Q.

Lemma 4. 5. *ν is a G-morphism with the trivial action of G on P.*

Proof. Since $\det F$ is a G-linearized $\mathcal{O}_{X_Q}$-module, there exists an isomorphism $\tau^*(\det F)\xrightarrow{\sim}p_2{}^*(\det F)$. Hence we see that

$$\begin{aligned}\tilde{\tau}^*(1_X\times\nu)^*(L)&\xrightarrow{\sim}\tilde{p}_2{}^*(1_X\times\nu)^*(L)\otimes\tilde{\tau}^*q^*(M)^\vee)\otimes\tilde{p}_2{}^*q^*(M)\\&\cong\tilde{p}_2{}^*(1_X\times\nu)^*(L)\otimes(1_G\times q)^*(\tau^*(M^\vee)\otimes p_2{}^*(M))\end{aligned}$$

which implies that $\nu\tau=\nu p_2$. Therefore ν is a G-morphism. q.e.d.

By virtue of E.G.A., Ch. III, 7. 9. 10, the assumption (2) on (P, L) yields the following;

$E=\pi_*(L)$ is locally free and $\nu^*(E)=\nu^*\pi_*(L)\cong q_*(1_X\times\nu)^*(L)\cong q_*(\det F)\otimes M$, *where $\pi: X\times_S P\to P$ is the projection.*

Now set

$$(4.6)\quad \begin{aligned} Z &= \mathbf{P}(\mathscr{H}om_{\mathcal{O}_P}(\overset{r}{\wedge} V\otimes_k \mathcal{O}_P, E)^\vee) \\ \bar{Z} &= \mathbf{P}(\mathscr{H}om_{\mathcal{O}_Q}(\overset{r}{\wedge} V\otimes_k \mathcal{O}_Q, q_*(\det F)\otimes M)^\vee) \end{aligned}$$

Then the dual action $\hat{\sigma}_0 : V \to V\otimes_k k[G]$ induces a G-action on Z and a G-linearization on the tautological line bundle $\mathcal{O}_Z(1)$. Since

$$\begin{aligned} \nu^*(\mathscr{H}om_{\mathcal{O}_P}(\overset{r}{\wedge} V\otimes_k \mathcal{O}_P, E)^\vee) &\cong \mathscr{H}om_{\mathcal{O}_Q}(\nu^*(\overset{r}{\wedge} V\otimes_k \mathcal{O}_P), \nu^*(E))^\vee \\ &\cong \mathscr{H}om_{\mathcal{O}_Q}(\overset{r}{\wedge} V\otimes_k \mathcal{O}_Q, q_*(\det F)\otimes M)^\vee \\ &\cong \mathscr{H}om_{\mathcal{O}_Q}(\overset{r}{\wedge} V\otimes_k \mathcal{O}_Q, q_*(\det F))^\vee \otimes M^\vee, \end{aligned}$$

we have that $\bar{Z}\cong Z\times_P Q$ and $\bar{Z}\cong \mathbf{P}(\mathscr{H}om_{\mathcal{O}_Q}(\overset{r}{\wedge} V\otimes_k \mathcal{O}_Q, q_*(\det F))^\vee)$. Thus G acts on $\bar{Z}$ and the projections $\bar{Z}\to Z$ and $\bar{Z}\to Q$ are G-morphisms. Moeover, this action is just one induced by the dual action $\hat{\sigma}_0$. On the other hand, using the canonical G-homomorphism $\gamma : \overset{r}{\wedge} F \to \det F$ in Lemma 4.2, we obtain a G-homomorphism

$$\tilde{\gamma} : \overset{r}{\wedge} V\otimes_k \mathcal{O}_Q = q_*(\overset{r}{\wedge} V\otimes_k \mathcal{O}_{X_Q}) \xrightarrow{q_*(\overset{r}{\wedge}\phi)} q_*(\overset{r}{\wedge} F) \xrightarrow{q_*(\gamma)} q_*(\det F).$$

Pick a point y of Q and consider $(\tilde{\gamma}\otimes k(y)) : \overset{r}{\wedge} V\otimes_k k(y) \to q_*(\det F)\otimes_{\mathcal{O}_Q} k(y)$. The assumption (4.3) provides us with a caonical isomorphism $q_*(\det F)\otimes_{\mathcal{O}_Q} k(y) \xrightarrow{\sim} H^0(q^{-1}(y), (\det F)\otimes \mathcal{O}_Q k(y))$. Thus if s_i denotes the image of e_i by $\Gamma(\phi\otimes_{\mathcal{O}_Q} k(y))$: $V\otimes_k k(y)\cong H_0(q^{-1}(y), V\otimes_k \mathcal{O}_{q^{-1}(y)}) \to H^0(q^{-1}(y), F\otimes_{\mathcal{O}_Q} k(y))$, then $(\tilde{\gamma}\otimes k(y))(e_{i_1}\wedge \cdots \wedge e_{i_r})$ coincides with $s_{i_1}\wedge\cdots\wedge s_{i_r}$ on the largest open set U_y over which $F\otimes_{\mathcal{O}_Q} k(y)$ is locally free. Since U_y is not empty and sice $s_1, \cdots, s_N$ generate $F\otimes_{\mathcal{O}_Q} k(y)$, $\tilde{\gamma}\otimes k(y)$ is not zero. This means that for the G-homomorphism $\delta : \mathcal{O}_Q \to \mathscr{H}om_{\mathcal{O}_Q}(\overset{r}{\wedge} V\otimes_k \mathcal{O}_Q, q_*(\det F))$ defined by $\tilde{\gamma}$, the dual of δ, $\delta^\vee : \mathscr{H}om_{\mathcal{O}_Q}(\overset{r}{\wedge} V \otimes_k \mathcal{O}_Q, q_*(\det F))^\vee \to \mathcal{O}_Q$ is surjective. We obtain therefore a G-morphism $Q\to \bar{Z}$ which is a section of the projection $\bar{Z}\to Q$. Consequently, composing this section and the projection $\bar{Z} = Z\times_P Q \to Z$, a G-morphism $\mu : Q\to Z$ is obtained. Moreover, the following diagram is commutative

$$(4.7)\qquad \begin{tikzcd} Q \arrow[r, "\mu"] \arrow[dr, "\nu"'] & Z \arrow[d, "p"] \\ & P \end{tikzcd}$$

where p is the natural projection.

To analize the morphism μ we need

Lemma 4.8. *Let $f : X\to S$ be a projective, geometrically integral morphism, E be a locally free $\mathcal{O}_X$-module and let both E_1 and E_2 be quotient coherent $\mathcal{O}_X$-modules of E. Suppose that for a point s of S, $E_1\otimes_{\mathcal{O}_S} k(s)$ and $E_2\otimes_{\mathcal{O}_S} k(s)$ have the same Hilbert polynomial.*

1) *If* $S=\mathrm{Spec}(K)$ *with* K *a field,* E_1 *is torsion free and if for a non-empty open set* U *of* X, $E_1|_U=E_2|_U$ *as quotient sheaves of* $E|_U$, *then* E_2 *is isomorphic to* E_1 *as quotient sheaves of* E.

2) *If* $S=\mathrm{Spec}(A)$ *with* A *an artinian local ring,* f *is smooth,* E_1 *and* E_2 *are* f*-flat, for the unique point* s *of* S, *both* $E_1\otimes_{\mathcal{O}_S}k(s)$ *and* $E_2\otimes_{\mathcal{O}_S}k(s)$ *are torsion free and if* $E_1|_U=E_2|_U$ *as quotient sheaves of* $E|_U$ *for an open set with* $\mathrm{codim}(X-U, X)\geqq 2$, *then* E_2 *is isomorphic to* E_1 *as quotient sheaves of* E.

Proof. Let F_i be the kernel of the homomorphism $\phi_i: E\to E_i$, J be the coherent subsheaf of E generated by F_1 and F_2 and let $\bar{E}=E/J$.

1) Since $J\supseteq F_1$, there exists a natural homomorphism $\alpha: E_1\to\bar{E}$. Our assumption implies that $\mathrm{Supp}(\ker(\alpha))\subseteq X-U$, and hence $\ker(\alpha)$ is a torsion sheaf. By this and the fact that E_1 is torsion free, we get that $\ker(\alpha)=0$, which means that $J=F_1$. Thus F_1 contains F_2. Hence we have the following exact commutative diagram;

$$\begin{tikzcd} & & & E \arrow[dl, "\phi_2"'] \arrow[dr, "\phi_1"] & & \\ 0 \arrow[r] & F_1/F_2 \arrow[r] & E_2 \arrow[rr] \arrow[dl] & & E_1 \arrow[r] \arrow[dr] & 0 \\ & 0 & & & & 0 \end{tikzcd}$$

Then $F_1/F_2=0$ because the Hilbert polynomial of F_1/F_2 is 0. Thus E_2 is isomorphic to E_1 as quotient sheaves of E.

2) Since f is projective and smooth, E_1 and E_2 are f-flat and since $E_1\otimes_{\mathcal{O}_S}k(s)$ and $E_2\otimes_{\mathcal{O}_S}k(s)$ are torsion free, we obtain the following exact sequences;

$$0\longrightarrow E^{(i)}_{n-1}\xrightarrow{f^{(i)}_{n-1}} E^{(i)}_{n-2}\longrightarrow\cdots\longrightarrow E^{(i)}_1\xrightarrow{f^{(i)}_1} E\xrightarrow{\phi_i} E_i\longrightarrow 0,$$

where $E_j^{(i)}$ are locally free $\mathcal{O}_X$-modules and $n=\dim X$ (see the proof of Proposition 2. 1 of [8]). Furthermore, for a point x of X with $\dim(\mathcal{O}_{X,x})=d$, $\ker(f^{(i)}_{d-2})_x$ is a free $\mathcal{O}_{X,x}$-module. Since $F_i=\ker(\phi_i)$, $\mathrm{hd}_{\mathcal{O}_{X,x}}(F_{i,x})\leqq\max\{\dim(\mathcal{O}_{X,x})-2, 0\}$. As is claimed in the proof of Lemma 4. 2, $\mathrm{depth}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x})\geqq\dim(\mathcal{O}_{X,x})$. On the other hand, we know the equality

$$\mathrm{depth}_{\mathcal{O}_{X,x}}(F_{i,x})+\mathrm{hd}_{\mathcal{O}_{X,x}}(F_{i,x})=\mathrm{depth}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}) \text{ (see [2] Theorem 3. 7).}$$

Thus we have that $\mathrm{depth}(F_{i,x})\geqq\min\{2, \dim(\mathcal{O}_{X,x})\}$. This and the assumption that $\mathrm{codim}(X-U, X)\geqq 2$ imply that for all points x in $X-U$, $\mathrm{depth}_{\mathcal{O}_{X,x}}(F_{i,x})\geqq 2$. Therefore if $j: U\to X$ is the natural open immersion, then $j_*(F_i|_U)=F_i$, which means that $F_1=F_2$ as subsheaves of E because $j_*(E|_U)=E$. Thus we see that E_2 is isomorphic to E_1 as quotient sheaves of E. q.e.d.

Let R be the open set of Q such that for every algebraically closed field K, $R(K)=\{x\in Q(K)\,|\,F\otimes_{\mathcal{O}_Q}k(x)$ is torsion free$\}$ (see Proposition 2. 1 of [8]). Clearly R is a G-stable open set of Q.

Proposition 4.9. *$\mu|_R$ is an immersion. To be more precise, there exists a G-stable open set Z_0 of Z such that μ induces a closed immersion of R to Z_0.*

Proof. Let K be an algegraically closed field containing k. Pick two points x_1 and x_2 in $Q(K)$. Suppose that x_1 is contained in $R(K)$ and that $\mu(K)(x_1)=\mu(K)(x_2)$. If s is the point in $S(K)$ over which x_1 and x_2 lie. Then both $E_1=F\otimes_{\mathcal{O}_Q}k(x_1)$ and $E_2=F\otimes_{\mathcal{O}_Q}k(x_2)$ are quotient sheaves of $V\otimes_k\mathcal{O}_{X_K}$, where $X_K=X_s\otimes_{k(s)}K$. If U is a non-empty open set of X_K over which E_1 and E_2 are locally free, then $\phi\otimes_{\mathcal{O}_Q}k(x_i):V\otimes_k\mathcal{O}_{X_K}\to E_i\to 0$ defines a morphism α_i of U to Grass(N, r) such that $\phi\otimes_{\mathcal{O}_Q}k(x_i)|_U:V\otimes_k\mathcal{O}_U\to E_i|_U$ is the pull back of the unversal quotient bundle by α_i. The assumption that $\mu(K)(x_1)=\mu(K)(x_2)$ means that $s_{i_1}^{(1)}\wedge\cdots\wedge s_{i_r}^{(1)}=s_{i_1}^{(2)}\wedge\cdots\wedge s_{i_r}^{(2)}$ in $H^0(X_K, (\det F\otimes_{\mathcal{O}_Q}k(x_1))=H^0(X_K, (\det F)\otimes_{\mathcal{O}_Q}k(x_2))$, where $s_j^{(i)}$ is the image of e_j in $H^0(X_K, E_i)$ by $\Gamma(\phi\otimes_{\mathcal{O}_Q}k(x_i))$. This asserts that $\alpha_1=\alpha_2$, and hence $E_1|_U=E_2|_U$ as quotient sheaves of $V\otimes_k\mathcal{O}_U$. Since E_1 is torsion free, E_1 is isomorphic to E_2 as quotient sheaves of $V\otimes_k\mathcal{O}_{X_K}$ by virtue of Lemma 4.8, (1). Thus $x_1=x_2$. We obtain therefore

(4.9.1) $\mu(R)\cap\mu(Q-R)=\phi$,

(4.9.2) if x_1 and x_2 are mutually distinct points in $R(K)$, then $\mu(K)(x_1)\neq\mu(K)(x_2)$.

Since Q is proper over S and since Z is separated over S, μ is a proper morphism (E.G.A., Ch. II, 5.4.3). Thus if one sets $Z_0=Z-\mu(Q-R)$, then Z_0 is G-stable open set in Z because $Q-R$ is a G-stable closed set in Q and μ is a G-morphism. (4.9.1) implies that $\mu^{-1}(Z_0)=R$, whence $\mu':R\to Z_0$ induced by μ is proper. This and (4.9.2) say that μ' is a finite morphism and for every algebraically closed field K, $\mu'(K):R(K)\to Z_0(K)$ is injective. Take a point x in R and an artinian local ring A. Let $\tilde{x}_1$ and $\tilde{x}_2$ be A-valued points of R whose images of the unique point of Spec(A) are x. Assume that $\mu(A)(\tilde{x}_1)=\mu(A)(\tilde{x}_2)$. Let $\tilde{E}_i=F\otimes_{\mathcal{O}_Q}A$, where Spec$(A)$ is regarded as a Q-scheme by the A-valued point $\tilde{x}_i$. Then $\tilde{E}_1$ and $\tilde{E}_2$ are quotient coherent sheaves of $V\otimes_k\mathcal{O}_{X_A}$ with the same $X_A=X\times_S\mathrm{Spec}(A)$. Since $\tilde{E}_1$ is flat over Spec(A) and since for the maximal ideal $\mathfrak{m}$ of A, $\tilde{E}_i\otimes_A A/\mathfrak{m}$ is torsion free, there exists an open set U' in X_A such that both $\tilde{E}_1$ and $\tilde{E}_2$ are locally free on U' and that codim$(X-U', X)\geqq 2$ (see Corollary 1.3.1 of [8]). By the same reason as above, the assumption that $\mu(A)(\tilde{x}_1)=\mu(A)(\tilde{x}_2)$ yields an isomorphism of $\tilde{E}_1|_{U'}$ to $\tilde{E}_2|_{U'}$ as quotient sheaves of $V\otimes_k\mathcal{O}_{U'}$. Now if we apply Lemma 4.8, (2) to this situation, then we see that $\tilde{E}_1$ is isomorphic to $\tilde{E}_2$ as quotient sheaves of $V\otimes_k\mathcal{O}_{X_A}$. Therefore $\mu'(A):R(A)\to Z_0(A)$ is injective, and hence μ' is an unramified morphism. For a point x of R, set $z=\mu'(x)$. Then since $\mathcal{O}_{R,x}$ is unramified over $\mathcal{O}_{Z,z}$, $k(x)=\mathcal{O}_{R,x}/\mathfrak{m}_x$ is a separablly algebraic extension of $k(z)=\mathcal{O}_{Z,z}/\mathfrak{m}_z$ and $\mathfrak{m}_x=\mathfrak{m}_z\mathcal{O}_{R,x}$, where $\mathfrak{m}_x$ and $\mathfrak{m}_z$ are the maximal ideals of $\mathcal{O}_{R,x}$ and $\mathcal{O}_{Z,z}$, respectively. This implies that $k(x)=k(z)$ because for every algebraically closed field K, $\mu'(K)$ is injective. On the other hand, since μ' is finite and injective, $\mathcal{O}_{R,x}$ is a finite module over $\mathcal{O}_{Z,z}$. Combining these results and Nakayama's lemma, we see that $\mathcal{O}_{Z,z} \to \mathcal{O}_{R,x}$ is surjective. q.e.d.

Now we shall remove the assumption (4. 4) from the above results. Since $f: X \to S$ is smooth, there exists an étale, surjective morphism $v: S' \to S$ such that $f' = f \times_S S': X' \times_S S' \to S'$ has a section ε' (E.G.A., Ch. IV, 17. 16. 3). we have the following commutative diagram:

$$\begin{tikzcd}
X_Q \times_S (S' \times_S S') = X'_{Q'} \times_{X_Q} X'_{Q'} \arrow[r, shift left, "1_X \times_S \pi_1"] \arrow[r, shift right, "1_X \times_S \pi_2"'] \arrow[d, "q''"] & X'_{Q'} \arrow[r, "1_X \times_S \pi"] \arrow[d, shift right, "q'"'] & X_Q \arrow[d, "q"] \\
Q \times_S (S' \times_S S') = Q' \times_Q Q' = Q'' \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[u, shift left=2, "\bar{\varepsilon}_2''"] \arrow[u, shift left=4, "\bar{\varepsilon}_1'"] & Q' \arrow[r, "\pi"] \arrow[u, shift right, "\bar{\varepsilon}'"'] & Q
\end{tikzcd}$$

where $Q' = Q \times_S S'$, π is the projection, π_1 (or, π_2) is the first (or, the second, resp.) projection and where $\bar{\varepsilon}'$, $\bar{\varepsilon}_1''$ and $\bar{\varepsilon}_2''$ are the natural sections induced by ε'. Since $f': X' \to S'$ and $f'' = f \times_S S'': X'' = X' \times_X X'' \to S'' = S' \times_S S'$ have sections, we can construct P' and Z' (or, P'' and Z'') for X', S' and $\det F'$ (or, X'', S'' and $\det F''$, resp.) as in (4. 6) under the assumption that (4. 3) holds for Q and F, where $F' = (1_X \times_S \pi)^*(F)$ and $F'' = (1_X \times_S \pi_1)^*(F')$. We can find a subscheme P of $\mathrm{Pic}_{X/S}$ such that $P' = P \times_S S'$ and $P'' = P \times_S S'' = P' \times_P P'$. Let L' and L'' be universal invertible sheaves on $X \times_S P'$ and $X \times_S P''$, respectively. If u_1 and u_2 are the projections of P'' to P', then $(1_X \times_S u_1)^*(L') \cong L'' \otimes_{\mathcal{O}_{P''}} M_1$ and $(1_X \times_S u_2)^*(L') \cong L'' \otimes_{\mathcal{O}_{P''}} M_2$ for some invertible sheaves M_1 and M_2 on P''. Thus we get an isomorphism $\alpha: Z_1'' \xrightarrow{\sim} Z'' \xrightarrow{\sim} Z_2''$, where Z_i'' is the base change of Z' by $u_i: P'' \to P'$. If m is the dimension of Z' over P', then $(\overset{m}{\bigwedge} \Omega_{Z'/P'})^{-1}$ is a P'-ample invertible sheaf. Since $u_i^*(\Omega_{Z'/P'})$ is canonically isomorphic to $\Omega_{Z''/P''}$, we obtain a canonical isomorphism $\xi: u_1^*((\overset{m}{\bigwedge} \Omega_{Z'/P'})^{-1}) \xrightarrow{\sim} u_2^*((\overset{m}{\bigwedge} \Omega_{Z'/P'})^{-1})$. It is clear that (α, ξ) defines descent data of $(Z', (\overset{m}{\bigwedge} \Omega_{Z'/P'})^{-1})$ for the étale, surjective morphism $u: P' \to P$. Thanks to the descent theory of quasi-projective schemes ([4] VIII, Proposition 7. 8), there exists a couple of a $\mathbf{P}^m$-bundle $p: Z \to P$ in the étale topology on P and a p-ample invertible sheaf H on Z such that $Z \times_P P' \cong Z'$ and $H \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z'} \cong (\overset{m}{\bigwedge} \Omega_{Z'/P'})^{-1}$. Since the actions of G on Z' and Z'' come from the dual action $\hat{\sigma}_0$ of G on V, the descent theory of morphisms provides us with an action of G on Z and a G-linearization on H.

$$\begin{tikzcd}
Z'' \arrow[r, shift left, "\pi_1'"] \arrow[r, shift right, "\pi_2'"'] \arrow[d, "p''"'] & Z' \arrow[r, "\pi'"] \arrow[d, "p'"] & Z \arrow[d, "p"] \\
P'' \arrow[r, shift left, "u_1"] \arrow[r, shift right, "u_2"'] & P' \arrow[r, "u"] & P
\end{tikzcd}$$

Clearly π' and p are G-morphisms with the trivial action of G on P.

On the other hand, we have G-homomorphisms $\tilde{\gamma}:\overset{r}{\bigwedge}V\otimes_k\mathcal{O}_Q\to q'_*(\det F)$ and $\tilde{\gamma}':\overset{r}{\bigwedge}V\otimes_k\mathcal{O}_{Q'}\to q'_*(\det F')$ (see the construction of the morphism μ before (4. 7)). Since π is flat, it is easy to see that $\det F'\cong(1_X\times_S\pi)^*(\det F)$, $q'_*(\det F')\cong\pi^*q_*(\det F)$ and $\pi^*(\tilde{\gamma})\cong\tilde{\gamma}'$. We have therefore that for $F''=(1_X\times_S\pi_1)^*(F')\cong(1_X\times_S\pi_2)^*(F')$ and for $\gamma_i'':\overset{r}{\bigwedge}V\otimes_k\mathcal{O}_{Q''}\to q''_*(\det F'')$ $(i=1, 2)$,

$$\begin{aligned}\det F''&\cong(1_X\times_S\pi_1)^*(\det F')\cong(1_X\times_S\pi_1)^*(1_X\times_S\pi)^*(\det F)\\&\cong(1_X\times_S\pi_2)^*(1_X\times_S\pi)^*(\det F)\cong(1_X\times_S\pi_2)^*(\det F'),\\ q''_*(\det F'')&\cong\pi_1{}^*q'_*(\det F')\cong\pi_1{}^*\pi^*q_*(\det F)\\&\cong\pi_2{}^*q'_*(\det F')\text{ and}\\ \gamma_1''&\cong\pi_1{}^*(\gamma')\cong\pi_1{}^*\pi^*(\gamma)\cong\pi_2{}^*\pi^*(\gamma)\cong\gamma_2''.\end{aligned}$$

As in (4. 7) we get the mophisms $\mu':Q'\to Z'$ and $\mu_i'':Q''\to Z''$ for $\tilde{\gamma}'$ and $\tilde{\gamma}_i''$ $(i=1, 2)$, respectively. The above three isomorphisms show that μ_i'' is the base change of μ' by i-th projection of S'' to S' and $\mu_1''\cong\mu_2''$. By virtue of the descent theory again, a morphism of Q to Z is obtained. Since $\mu\times_S S'=\mu'$ and since μ' is a G-morphism, μ is also a G-morphism.

Summarizing the above results, we have

Proposition 4. 10. *Assume that* (*4. 3*) *holds for Q and F. Then there exist an open and closed subscheme P of* $\mathrm{Pic}_{X/S}$ *of finite type over S and a* $\mathbf{P}^m$*-bundle $p:Z\to P$ in the étale topology on P such that*

1) *G acts on Z and there exists a p-ample, G-linearized invertible sheaf H on Z,*

2) *there exists a G-morphism $\mu:Q\to Z$ with $\mu|R$ an immersion,*

3) *if $u:S'\to S$ is an étale, surjective morphism such that $f'=f\times_S S'$ has a section, then $Z\times_S S'$ and $\mu\times_S S'$ are the same defined in* (*4. 7*).

Proof. By virtue of Proposition 4. 9, $(\mu|_R)\times_S S'$ is an immersion and it is quasi-compact. Then $\mu|_R$ is an immersion because S' is faithfully flat and quasi-compact (E.G.A. Ch. IV, 2. 7. 1). q.e.d.

Our next task is to analize the sets of stable points of Z and R. Let us begin with some general remarks.

Lemma 4. 11. *Let G be a geometrically reductive affine algebraic group over k and let A and A' be k-algebras with dual actions of G. If $\phi:A\to A'$ is a surjective G-homomorphism and if x is an element of A'^G, then there exists a positive integer t such that x^t is contained in $\phi(A^G)$.*

For a proof, see [11] 5. 1. B.

Lemma 4. 12. *Let $f:X\to Y$ be a projective morphism of algebraic k-schemes. Assume that a reductive affine algebraic group G over k acts on*

X and that f is a G-morphism with the trivial action of G on Y. Let L (or, M) be a G-linearized ample invertible sheaf on X (or, Y, resp.). Then there exists a non-negative integer α_0 such that for all $\alpha \geqq \alpha_0$, $X_0{}^s(L \otimes f^*(M^{\otimes \alpha})) = \bigcup_{y \in Y}(X_y)_0{}^s(L \otimes \mathcal{O}_Y k(y))$ and $X^{ss}(L \otimes f^*(M^{\otimes \alpha})) = \bigcup_{y \in Y}(X_y)^{ss}(L \otimes \mathcal{O}_Y k(y))$.

Proof. The inclusion $X_0{}^s(L \otimes f^*(M^{\otimes \alpha})) \subseteq \bigcup_{y \in Y}(X_y)_0{}^s(L \otimes \mathcal{O}_Y k(y)) = S_1$ and $X^{ss}(L \otimes f^*(M^{\otimes \alpha})) \subseteq \bigcup_{y \in Y}(X_y)^{ss}(L \otimes \mathcal{O}_Y k(y)) = S_2$ are obvious. Pick a closed point y of Y and a geometric point x of $(X_y)^{ss}(L \otimes \mathcal{O}_Y k(y))$. We may assume that $\{Y_u \mid u \in H^0(Y, M),\ Y_u \text{ is affine}\}$ covers Y, where $Y_u = \{z \in Y \mid u(z) \neq 0\}$. Choose a member u of $H^0(Y, M)$ such that y is a point of Y_u and Y_u is an affine scheme Spec(B). Set $X' = f^{-1}(Y_u)$. By a Leray's spectral sequence and the fact Y_u is affine, there exists a positive integer n_0 such that for all $n \geqq n_0$, $H^1(X', I_y \otimes L^{\otimes n}) = 0$, where I_y is the definig ideal of X_y in $\mathcal{O}_{X'}$. Let us consider graded G-algebras $A = B \oplus (\oplus_{i \geqq 1} B_i)$ and $A' = k(y) \oplus (\oplus_{i \geqq 1} B_i')$, where $B_i = H^0(X', L^{\otimes in_0})$ and $B_i' = H^0(X', (L \otimes \mathcal{O}_Y k(y))^{\otimes in_0})$. Then we get a surjective, graded G-homomorphism $\psi: A \to A'$. The assumption that x is a point of $(X_y)^{ss}(L \otimes \mathcal{O}_Y k(y))$ implies that there exists an element a of $B_i'^G$ such that $(X_y)_a$ is affine and x is a point of $(X_y)_a$. By virtue of Lemma 4. 11, a^t is contained in $\psi(A^G)$ for a positive integer t. Since ψ is graded, we can find an element b in $B_{it}{}^G$ such that $\psi(b) = a^t$. X'_b is an affine scheme because Y_u is affine. Moreover, $X'_b \cap X_y = (X_y)_a$. For a large integer α_x, $b \otimes u^{\otimes itn_0\alpha_x}$ can be regarded as an element of $H^0(X, (L \otimes f^*(M^{\otimes \alpha_x}))^{\otimes itn_0})^G$. Then for $s = b \otimes u^{\otimes (itn_0\alpha_x + 1)}$, $X_s = X'_b$. Thus we see that for all large integers α, x is a geometric point of $X^{ss}(L \otimes f^*(M^{\otimes \alpha}))$. Furthermore, since $X_s \subseteq S_2$, S_2 is an open set of X. Therefore the above argument shows that for all large integers α, there exists a positive integer n and sections $s_1, \cdots, s_m$ in $H^0(X, (L \otimes f^*(M^{\otimes \alpha}))^{\otimes n})^G$ such that $S_2 = \bigcup X_{s_i}$ and all the X_{s_i} are affine. This means that S_2 is contained in $X^{ss}(L \otimes f^*(M^{\otimes \alpha}))$. If x is a geometric point of $(Y_y)_0{}^s(L \otimes \mathcal{O}_Y k(y))$, then X_s can be so chosen that the G-orbit $o(x)$ of x is closed in $X_s \otimes_k k(x)$. Since the action of G at x is regular, there exist a positive integer j and a G-invariant section s' of $(L \otimes f^*(M^{\otimes \alpha}))^{\otimes j}$ such that x is a point of $X_{s'}$, $X_{s'}$ is affine and that the action of G on $X_{s'}$ is closed (see Amplification 1. 11 of [10]). We see therefore that for all large integers α, x is a geometric point of $X_0{}^s(L \otimes f^*(M^{\otimes \alpha}))$. Since $X_{s'} \subseteq S_1$, S_1 is open in X. These results show that for all large integers α, there exist a positive integer n' and G-invariant sections $s'_1, \cdots, s'_{m'}$ of $(L \otimes f^*(M^{\otimes \alpha}))^{\otimes n'}$ such that $S_1 = \bigcup X_{s_i'}$, all the $X_{s_i'}$ are affine and that the action of G on each $X_{s_i'}$ is closed. Therefore S_1 is a subset of $X_0{}^s(L \otimes f^*(M^{\otimes \alpha}))$. q.e.d.

We shall apply the above lemma to the following situation. Let H be the G-linearized invertible sheaf on Z obtained in Proposition 4. 10, M be an S-ample invertible sheaf on P and let $\{U_i\}$ be a finite affine open covering of S. Lemma 4. 12 for $X = Z_{U_i}$, Y_{U_i}, $f = p_{U_i}$, $L = H_{U_i} = H \mid Z_{U_i}$ and $M = M \mid P_{U_i}$ implies that if one replaces H by $H\otimes p^*(M^{\otimes\alpha})$ for a sufficiently large integer α, then for all i,

$$
(4.13)\quad \begin{aligned}(Z_{U_i})_0{}^s(H_{U_i}) &= \bigcup_{y\in P_{U_i}} (Z_y)_0{}^s(H\otimes_{\mathcal{O}_P}k(y))\\ (Z_{U_i})^{ss}(H_{U_i}) &= \bigcup_{y\in P_{U_i}} (Z_y)^{ss}(H\otimes_{\mathcal{O}_P}k(y))\end{aligned}
$$

For the invertible sheaf $\mathcal{O}_{Z_y}(1)$ corresponding to the hyperplanes in $Z_y=\mathbf{P}^m_{k(y)}$, $H\otimes_{\mathcal{O}_P}k(y)$ is isomorphic to $\mathcal{O}_{Z_y}(m+1)$. Thus Proposition 4.12 provides us with a criterion for stability of a geometric point of Z_{U_i}. On the other hand, Proposition 1.18 of [10] says that

$$
(4.14)\quad (R_{U_i})_0{}^s(\mu^*(H_{U_i})|_{R_{U_i}})\supseteq(\mu|_R)^{-1}\{(Z_{U_i})_0{}^s(H_{U_i})\}
$$

The following which is due to D. Gieseker is an interpretation of Proposition 4.1 in the words of sheaves.

Lemma 4.15. *Suppose that a geometric point y of R_{U_i} satisfies the condition*

(4.15.1) *$\Gamma(\phi\otimes k(y)):V\otimes_k k(y)\to H^0(X_y, F\otimes_{\mathcal{O}_Q}k(y))$ is bijective and for all proper coherent subsheaves E $(\neq 0)$ of $F\otimes_{\mathcal{O}_Q}k(y)$ generated by a subset of $H^0(X_y, F\otimes_{\mathcal{O}_Q}k(y))$, the following inequality holds;*

$$
h^0(X_y, E)<r(E)h^0(X_y, F\otimes_{\mathcal{O}_Q}k(y))/r.
$$

Then y is a geometric point of $(\mu|_R)^{-1}\{(Z_{U_i})_0{}^s(H_{U_i})\}$.

Proof. The point $z=\mu(y)$ can be regarded as a $k(y)$-linear map T_z of $\bigwedge^r V\otimes_k k(y)$ to $U=H^0(X_y, (\det F)\otimes_{\mathcal{O}_Q}k(y))$. If z is not stable in Z_{U_i}, then (4.13) shows that T_z is not stable, and hence there exist a subspace W of $V\otimes_k k(y)$ and a T_z-independent set of vectors $\{v_1, \cdots, v_d\}$ in W such that every vector in W is T_z-dependent on $v_1, \cdots, v_d$ and that $\dim W\geqq dN/r=dh^0(X_y, F\otimes_{\mathcal{O}_Q}k(y))/r$ by virtue of Propsition 4.1. Let E be the subsheaf of $F\otimes_{\mathcal{O}_Q}k(y)$ generated by $\{\Gamma(\phi\otimes k(y))(w)\mid w\in W\}$. Then it is easily seen that $r(E)=d$ and $h^0(X_y, E)\geqq \dim W$. This contradicts to the assumption (4.15.1). q.e.d.

The following is an easy generalization of Theorem 1.4.

Lemma 4.16. *Let $f:X\to S$ be a projective morphism of algebraic k-schemes. Assume that a reductive affine algebraic k-group G acts on X and f is a G-morphism with the trivial action of G on S. Let $\mathcal{O}_X(1)$ be a G-linearized f-ample invertible sheaf on X and let $X_0{}^s(\mathcal{O}_X(1))$ (or, $X^{ss}(\mathcal{O}_X(1))$) be $\bigcup_i(X_{U_i})_0{}^s(\mathcal{O}_{X_{U_i}}(1))$ (or, $\bigcup_i(X_{U_i})^{ss}(\mathcal{O}_{X_{U_i}}(1))$, resp.), where $\{U_i\}$ is a finite affine open covering of S (note that they are independent of $\{U_i\}$). Then a good quotient (Y, g) of $X^{ss}(\mathcal{O}_X(1))$ by G exists. Moreover,*

(i) *g is affine and universally submersive,*

(ii) *for the natural morphism $h:Y\to S$, there exists an h-ample invertible sheaf M on Y such that $g^*(M)=\mathcal{O}_X(m)$ for some positive integer m,*
(iii) *there exists an open subset Y' of Y such that $X_0^s(\mathcal{O}_X(1))=g^{-1}(Y')$ and $(Y', g|_{X_0^s(\mathcal{O}_X(1))})$ is a geometric quotient of $X_0^s(\mathcal{O}_X(1))$ by G.*

Proof. Since $\{U_i\}$ is a finite covering and since all the X_{U_i} are noetherian schemes, there exist a positive integer m and G-invariant sections $s_1^{(i)},\cdots, s_{r_i}^{(i)}$ in $H^0(X_{U_i}, \mathcal{O}_X(m))$ such that all the $(X_{U_i})_{s_j^{(i)}}=U_i^{(i)}$ are affine and $\bigcup_j U_j^{(i)}=(X_{U_i})^{ss}(\mathcal{O}_{X_{U_i}}(1))$. By virtue of Theorem 1. 1, there exists a good quotient $V_j^{(i)}$ of $U_j^{(i)}$ by G. Since for all affine open set $U'=\mathrm{Spec}(A)$ of $V_j^{(i)}$, $\Gamma(U_j^{(i)}\times_{V_j^{(i)}}U', \mathcal{O}_X)^G=\Gamma(U_j^{(i)}, \mathcal{O}_X)^G\otimes_{\Gamma(V_j^{(i)},\mathcal{O}_{V_j^{(i)}})}A=A$ (see [10] p. 9, Remark 7) and since $\mathrm{Spec}(\Gamma(U_j^{(i)}\times_{V_j^{(i)}}U', \mathcal{O}_X)^G)$ is a good quotient of $U_j^{(i)}\times_{V_j^{(i)}}U'$ by G, U' is a good quotient of $U_j^{(i)}\times_{V_j^{(i)}}U'$ by G. Thus we obtain

(4. 16. 1) for all open set U' of $V_j^{(i)}$, U' is a good quotient of $U_j^{(i)}\times_{V_j^{(i)}}U'$ by G.

Hence we can construct a good quotient Y_i of $(X_{U_i})^{ss}(\mathcal{O}_{X_{U_i}}(1))$ as in the proof of Theorem 1. 10 of [10]. Moreover, we see, by the same argument as above, that for all open sets U' of U_i, $Y_i\times_S U'$ is a good quotient of $X\times_S U'$. Thus for $U_{ij}=U_i\cap U_j$, $Y_i\times_S U_{ij}$ is a good quotient of $X\times_S U_{ij}$ by G. Hence we can patch Y_i together and obtain a good quotient (Y, g) of X by G. Furthermore, $s_j^{(i)}/s_{j'}^{(i')}$ is induced by a function $\sigma_{j,j'}^{(i,i')}$ of $\Gamma(V_j^{(i)}\cap V_{j'}^{(i')}, \mathcal{O}_Y)$ by virtue of (4. 16. 1). Clearly $\{\sigma_{j,j'}^{(i,i')}\}$ forms a Čzech 1-cocycle for the covering $\{V_j^{(i)}\}$ of Y and in the sheaf $\mathcal{O}_Y^*$. Thus we get an invertible sheaf M on Y such that $g^*(M)\cong\mathcal{O}_X(m)$. The proof of the fact that $M|_{Y_i}$ is ample is completely same as that in the proof of Theorem 1. 10 of [10]. The rest of the proof is similar to that of Theorem 1. 10 of [10].

Now we come to our main theorem of this section.

Theorem 4. 17. *Assume that (4. 3) holds for Q and F. Let U be a G-stable subscheme of R such that every geometric point of U satisfies the condition (4. 15. 1). Then there exist an S-scheme Y and an S-morphism $g:U\to Y$ such that (Y, g) is a geometric quotient of U by G and Y is quasi-projective over S.*

Proof. Since $\mu_U:U\to Z$ is an immersion and U is noetherian, $\mu|_U$ is quasi-affine. Thus, for a finite affine open covering $\{U_i\}$ of S, the morphism $\mu_i=(\mu|_U)\times_S U_i$ of $V_i=U\times_S U_i$ to $Z_i=Z\times_S U_i$ is quasi-affine. Then Proposition 1. 18 of [10] implies that $(V_i)_0^s(\mu_i^*(H|_{Z_i}))$ contains $\mu_i^{-1}\{(Z_i)_0^s(H|_{Z_i})\}$. On the other hand, Lemma 4. 15 and our assumption assert that V_i is a subset of $\mu_i^{-1}\{(Z_i)_0^s(H|_{Z_i})\}$. Thus $U=(\mu|_U)^{-1}\{Z_0^s(H)\}=U_0^s((\mu|_U)^*(H))$ under the notation of Lemma 4. 16. Therefore we obtain, by virtue of Lemma 4. 16, a geometric quotient (Y, g) and an S-ample invertible sheaf M on Y such $g^*(M)=(\mu|_U)^*(H^{\otimes m})$ for some positive integer m. q.e.d.

Remark 4. 18. Since the center of $SL(N, k)$ acts trivially on Q, the above results can be regarded as those for the action of $PGL(N, k)$ and also for the action of $GL(N, k)$.

§ 5. Construction of moduli of stable sheaves.

As in § 4, let $f: X \to S$ be a smooth, projective, geometrically integral morphism of algebraic k-schemes and let $\mathcal{O}_X(1)$ be an f-vary ample invertible sheaf on X which satisfies the condition (3. 4). In this section, combining the results of preceding sections, we shall construct coarse moduli schemes of stable sheaves on the fibres of X over S. Without losing any generality, we may assume that S is connected. Let n be the relative dimension of X over S, h be the degree of $\mathcal{O}_X(1)$ and let $c(X)$ be the degree of $\overset{n}{\wedge}\Omega_{X/S}$ with respect to $\mathcal{O}_X(1)$. For a positive integer r, let H be a numerical polynomial;

$$H(m) = rhm^n/n! + \{a_1 - rc(X)/2\}m^{n-1}/(n-1)! + \text{terms of degree} < n-1.$$

To fix ideas let us introduce the following contravariant functor $\Sigma^H_{X/S}$ of the category (Sch/S) of locally noetherian S-schemes to the category of sets (Sets):

For $T \in (\mathrm{Sch}/S)$, $\Sigma^H_{X/S}(T) = \{E \mid E$ has the properties (5. 1. 1) and (5. 1. 2)$\}/\sim$, where $\sim$ is such an equivalence relation that $E \sim E'$ if and only if $E \cong E' \otimes_{\mathcal{O}_T} L$ for some invertible sheaf L on T.

(5. 1. 1) E is a T-flat, coherent $\mathcal{O}_{X \times_S T}$-module.

(5. 1. 2) For all geometric points t of T, the Hilbert polynomial of $E \otimes_{\mathcal{O}_T} k(t)$ with respect to $\mathcal{O}_{X_t}(1)$ is H and $E \otimes_{\mathcal{O}_T} k(t)$ is stable with respect to $\mathcal{O}_X(1) \otimes_{\mathcal{O}_S} \mathcal{O}_T$.

$\Sigma^H_{X/S}$ is not necessarily a sheaf for the ètale topology in (Sch/S) even if f has a section. The aim of this section is to show that $\Sigma^H_{X/S}$ has, neverthless, a coarse moduli scheme.

To construct the moduli scheme of $\Sigma^H_{X/S}$, we need a subfunctor $\Sigma^{H,e}_{X/S}$ of $\Sigma^H_{X/S}$:

For $T \in (\mathrm{Sch}/S)$, $\Sigma^{H,e}_{X/S}(T) = \{E \in \Sigma^H_{X/S}(T) \mid$ for all gemetric points t of T, $E \otimes_{\mathcal{O}_T} k(t)$ is e-stable$\}$.

For an integer m, set $\Sigma^H_{X/S}(m)(T) = \{E \otimes p_1^*(\mathcal{O}_X(m)) \mid E \in \Sigma^H_{X/S}(T)\}$ and $\Sigma^{H,e}_{X/S}(m)(T) = \{E \otimes p_1^*(\mathcal{O}_X(m)) \mid E \in \Sigma^{H,e}_{X/S}(T)\}$, where p_1 is the first projection of $X \times_S T$ to X. Then $\Sigma^H_{X/S}$ (or, $\Sigma^{H,e}_{X/S}$) is isomorphic to $\Sigma^H_{X/S}(m)$ (or, $\Sigma^{H,e}_{X/S}(m)$, resp.). Thus we may replace $\Sigma^H_{X/S}$ and $\Sigma^{H,e}_{X/S}$ by $\Sigma^H_{X/S}(m)$ and $\Sigma^{H,e}_{X/S}(m)$, respectively. By virtue of Corollary 3. 3. 1 and Proposition 3. 6, we can find an integer m_e such that for all integers $m \geq m_e$, all geometric points s of S and for all E in $\Sigma^{H,e}_{X/S}(\mathrm{Spec}(k(s)))$,

(5. 2. 1) *$E \otimes \mathcal{O}_{X_s}(m)$ is generated by its global sections and $h^i(X_s, E \otimes \mathcal{O}_{X_s}(m)) = 0$ if $i > 0$,*

(5. 2. 2) *if an invertible sheaf L on X_s has the same Hilbert polynomial as $\det(E \otimes \mathcal{O}_{X_s}(m)) = c_1(E \otimes \mathcal{O}_{X_s}(m))$, then $h^i(X_s, L) = 0$ for*

all positive integers i.

(5. 2. 3) *for all coherent subsheaves E' of E with $0\neq E'\neq E$,*

$$h^0(X_S, E'\otimes\mathcal{O}_{X_s}(m))<r(E')h^0(X_S, E\otimes\mathcal{O}_{X_s}(m))/r.$$

We may assume that $m_e\geqq m_{e'}$ if $e>e'$. Let $H_e(m)=H(m+m_e)$, then the Hilbert polynomial of a member of $\sum_{X/S}^{H}(m_e)(\mathrm{Spec}(k(s)))$ is H_e. Set $N_e=H(m_e)$, then the condition (5. 2. 1) implies that for every member E of $\sum_{X/S}^{X,e}(m_e)(\mathrm{Spec}(k(s)))$, $h^0(X_s, E)=N_e$.

Now let us consider $\tilde{Q}=\mathrm{Quot}_{\mathcal{O}_X^{\oplus N_e}/X/S}^{H_e}$ and the universal qoutient sheaf $\phi: V_e\otimes_k\mathcal{O}_{X\times_S\tilde{Q}}\to F_e$, where V_e is an N_e-dimensional vector space over k. Then, by virtue of Lemma 3. 5, for each integer with $0\leqq e'\leqq e$, there exists an open set $R_{e,e'}$ such that a geometric point y of $\tilde{Q}$ is contained in $R_{e,e'}$ if and only if

(5. 3. 1) $\Gamma(\phi\otimes k(y)): V_e\to H^0(X_y, F_e\otimes\mathcal{O}_{\tilde{Q}}k(y))$ is bijective,

(5. 3. 2) $F_e\otimes\mathcal{O}_{\tilde{Q}}k(y)$ is contained in $\sum_{X/S}^{H,e'}(m_e)(\mathrm{Spec}(k(y)))$.

For every geometric point s of S and for every E of $\sum_{X/S}^{H,e'}(m_e)(\mathrm{Spec}(k(s)))$, there exists a surjective homomorphism $\alpha: V_e\otimes_k\mathcal{O}_{X_s}\to E$ such that $\Gamma(\alpha): V_e\otimes_k k(s)\to H^0(X_s, E)$ is bijective by virtue of (5. 2. 1). By the universality of $(\tilde{Q}, \phi, F_e)$, α corresponds to a geometric point y of $\tilde{Q}$ lying over s. Clearly y is a geometric point of $R_{e,e'}$. Thus we obtain a surjective map $R_{e,e'}(k(s))\to\sum_{X/S}^{H,e'}(m_e)(\mathrm{Spec}(k(s)))$. On the other hand, for a natural action $\bar{\tau}$ of $\bar{G}=PGL(N_e, k)$ on $\tilde{Q}$, $R_{e,e'}$ is $\bar{G}$-stable and if two geometric points y_1 and y_2 of $\tilde{Q}$ are in the same orbit of $\bar{G}$, then clearly $F_e\otimes\mathcal{O}_{\tilde{Q}}k(y_1)\cong F_e\otimes\mathcal{O}_{\tilde{Q}}k(y_2)$ (see § 4). Conversely assume that for geometric points y_1 and y_2 in $R_{e,e'}(k(s))$ with s a geometric point of S, there exists an isomorphism $\beta: F_e\otimes\mathcal{O}_{\tilde{Q}}k(y_1)\xrightarrow{\sim}F_e\otimes\mathcal{O}_{\tilde{Q}}k(y_2)$. Then $\Gamma(\phi\otimes k(y_2))^{-1}\Gamma(\beta)\Gamma(\phi\otimes k(y_1))$: $V_e\otimes_k k(s)\xrightarrow{\sim}H^0(X_s, F_e\otimes\mathcal{O}_{\tilde{Q}}k(y_1))\xrightarrow{\sim}H^0(X_s, F_e\otimes\mathcal{O}_{\tilde{Q}}k(y_2))\xrightarrow{\sim}V_e\otimes_k k(s)$ is a linear isomorphism which defines a $k(s)$-rational point $\bar{g}$ of $\bar{G}$. Hence we see that $\bar{\tau}(\bar{g}, y_1)=y_2$, whence y_1 and y_2 are in the same orbit of $\bar{G}$. We get therefore a natural bijection

(5. 4) $R_{e,e'}(k(s))/\bar{G}(k(s))\xrightarrow{\sim}\sum_{X/S}^{H,e'}(m_e)(\mathrm{Spec}(k(s)))\xrightarrow{\sim}\sum_{X/S}^{H,e'}(\mathrm{Spec}(k(s)))$.

Let $\{Q_1, \cdots, Q_t\}$ be the set of connected components of $\tilde{Q}$ having a non-empty intersection with $R_{e,e'}$. Then since the image of Q_i to $\mathrm{Pic}_{X/S}$ by the morphism defined by $(\det F_e)|_{X\times_S Q_i}$ is contained in a connected component of $\mathrm{Pic}_{X/S}$, for every geometric point y of Q_i, $(\det F_e)\otimes\mathcal{O}_{\tilde{Q}}k(y)$ has the same Hilbert polynomial as $(\det F_e)\otimes\mathcal{O}_{\tilde{Q}}k(y_0)$ where y_0 is a geometric point of $Q_i\cap R_{e,e'}$. Thus each Q_i enjoys the property (4. 3) by virtue of the assumption (5. 2. 2). Theorem 4. 17 and the assumption (5. 2. 3) provides us with a geometric quotient $(M_{e,e'}^{(i)}, g_{e,e'}^{(i)})$ of $Q_i\cap R_{e,e'}$ by $\bar{G}$. Set $M_{e,e'}=\coprod_i M_{e,e'}^{(i)}$ and $g_{e,e'}=\coprod_i g_{e,e'}^{(i)}$, then $(M_{e,e'}, g_{e,e'})$ is a geometric quotient of $R_{e,e'}$ by $\bar{G}$ and $M_{e,e'}$ is quasi-projective over S.

Proposition 5. 5. *$M_{e,e'}$ is a coarse moduli scheme of $\sum_{X/S}^{H,e'}$, that is,*

(i) *for all geometric points s of S, there exists a bijective map* θ_s *of* $\sum_{X/S}^{H,e'}(\mathrm{Spec}(k(s))$ *to* $M_{e,e'}(k(s))$,

(ii) *for* $T\in(\mathrm{Sch}/S)$ *and* $E\in\sum_{X/S}^{H,e'}(T)$, *there exists a morphism* $f_E^{e,e'}$ *of* T *to* $M_{e,e'}$ *such that* $f_E^{e,e'}(t)=\theta_s(E\otimes_{\mathcal{O}_T}k(t))$ *for all points t in* $T(k(s))$. *Moreover, for a morphim* $g:T'\to T$ *in* (Sch/S),

$$f_E^{e,e'}\cdot g=f_{(1_X\times g)^*(E)}^{e,e'}$$

(iii) *if* $M'\in(Sch/S)$ *and maps* $\theta_s':\sum_{X/S}^{H,e'}(\mathrm{Spec}(k(s)))\to M'(k(s))$ *satisfy the above condition (ii), then there exists a unique S-morphism* ϕ *of* $M_{e,e'}$ *to* M' *such that* $\phi(k(s))\cdot\theta_s=\theta_s'$ *and* $\phi\cdot f_E^{e,e'}=f'_E$ *for all geometric points s of S and for all* $E\in\sum_{X/S}^{H,e'}(T)$, *where* f'_E *is the morphism given by the condition (ii) for* M'.

Proof. The proof is essentially the same as that of Theorem 4.11 of [7]. The condition (i) is just (5.4). The restriction of ϕ and F_e to $X\times_S R_{e,e'}$ are denoted by $\phi_{e,e'}$ and $F_{e,e'}$. Then the triple $(R_{e,e'},\phi_{e,e'},F_{e,e'})$ has the following universal property:

(5.5.1) For all T in (Sch/S), E in $\sum_{X/S}^{H,e'}(T)$, and for all surjective homomorphisms $\alpha:V_e\otimes_k\mathcal{O}_{X\times_S T}\to E$ such that for all geometric points t of T, $\Gamma(\alpha\otimes k(t)):V_e\otimes_k k(t)\to H^0(X_t,E\otimes_{\mathcal{O}_T}k(t))$ is bijective, there exists a unique morphism h_α of T to $R_{e,e'}$ such that $(1_X\times_S h_\alpha)^*(F_{e,e'})\cong E$ and $(1_X\times_S h_\alpha)^*(\phi_{e,e'})\cong\alpha$.

Assume that $T\in(\mathrm{Sch}/S)$ and $E\in\sum_{X/S}^{H,e'}(T)$ are given. Set $E'=E\otimes p_1^*(\mathcal{O}_X(m_e))$ with the first projection p_1 of $X\times_S T$ to X, then E' is a member of $\sum_{X/S}^{H,e'}(m_e)(T)$, and hence $h^i(X_t,E'\otimes_{\mathcal{O}_T}k(t))=0$, $i>0$ and $E'\otimes_{\mathcal{O}_T}k(t)$, is generated by its global sections for all geometric points t of T. By these and the fact that the second projection p_2 of $X\times_S T$ to T is proper and E' is T-flat imply that $E''=(p_2)_*(E')$ is a locally free $\mathcal{O}_T$-module of rank N_e and the natural homomorphism $\beta:p_2^*(E'')\to E'$ is surjective. Let us cover T by a family of open sets $\{T_\lambda\}$ such $E''|_{T_\lambda}$ is free. Take a basis $\{e^\lambda_1,\cdots,e^\lambda_{N_e}\}$ of each $E''|_{T_\lambda}$. Using this basis, we obtain a surjective homomorphism

$$\beta_\lambda:V_e\otimes_k\mathcal{O}_{X\times_S T}\longrightarrow p_2^*(E'')|_{T_\lambda}\xrightarrow{\beta|_{T_\lambda}}E'|_{T_\lambda}.$$

Moreover, for all geometric points t of Y, $E''\otimes_{\mathcal{O}_T}k(t)\xrightarrow{\sim}H^0(X_t,E'\otimes_{\mathcal{O}_T}k(t))$, and hence $\Gamma(\beta_\lambda\otimes k(t)):V_e\otimes_k k(t)\to H^0(X_t,E'\otimes_{\mathcal{O}_T}k(t))$ is bijective. Therefore the universal property (5.5.1) gives us a unique morphism $h_\lambda:T_\lambda\to R_{e,e'}$ such that $(1_X\times_S h_\lambda)^*(F_{e,e'})\cong E'|_{T_\lambda}$ and $\beta_\lambda\cong(1_X\times_S h_\lambda)^*(\phi_{e,e'})$. Since a change of basis of $E''|_{T_\lambda}$ is represented by a T_λ-valued point of $GL(N_e,k)$ and since $M_{e,e'}$ is a geometric quotient of $R_{e,e'}$ by an action of $GL(N_e,k)$, the morphism $f_\lambda=g_{e,e'}\cdot h_\lambda$ is independent of the choice of a basis of $E''|_{T_\lambda}$. Hence $f_\lambda=f_\mu$ on $T_\lambda\cap T_\mu$. We get therefore a morphism $f_E^{e,e'}$ of T to $M_{e,e'}$.[6] Next assume that a morphism g of T' to T in (Sch/S) is given. The fact that $h^i(X_t,E'\otimes_{\mathcal{O}_T}k(t))=0$,

[6] It is clear that $f_{E\otimes_{\mathcal{O}_T}L}^{e,e'}=f_E^{e,e'}$ for every invertible sheaf L on T.

$i>0$ for all geometric points t of T implies that $g^*(E'')\cong(p_2')_*(1_X\times_S g(^*)E')$, where p_2' is the projection $X\times_S T'\to T'$. Thus if we define $\beta_\lambda':V_e\otimes_k\mathcal{O}_{X\times T'}\to(1_X\times_S g)^*(E')|_{T_\lambda'}$ on $T_\lambda'=g^{-1}(T_\lambda)$ by using the basis $\{g^*(e^\lambda_1),\cdots, g(e^\lambda_{N_e})\}$ of $g^*(E'')|_{T_\lambda'}$, then $\beta_\lambda'=(1_X\times_S(g|_{T_\lambda'}))^*(\beta_\lambda)$. Similarly to the above, β_λ' defines a morphism $h_\lambda':T_\lambda'\to R_{e,e'}$. It is obvious that $h_\lambda'=h_\lambda\cdot g$. Therefore, $f_E^{e,e'}\cdot g=f^{e,e'}_{(1_X\times g)^*(E)}$, which completes the proof that $M_{e,e'}$ has property (ii). In order to prove (iii), let us consider the following diagram;

$$\begin{array}{ccc} G\times_k R_{e,e'} & \xrightarrow{\tau'} & R_{e,e'} \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle f'_{F_{e,e'}}} \\ R_{e,e'} & \xrightarrow{f'_{F_{e,e'}}} & M' \end{array}$$

where τ' is the action of $G=SL(N_e,k)$ on $R_{e,e'}$ induced by the τ in § 4 and where q_2 is the projection. Since $F_{e,e'}$ carries a G-linearization, $(\tau')^*(F_{e,e'})$ is isomorphic to $q_2{}^*(F_{e,e'})$, which implies that $f'_{F_{e,e'}}\cdot q_2=f'_{F_{e,e'}}\cdot\tau'$. Thus there exists a unique mophism $\phi:M_{e,e'}\to M'$ with $\phi\cdot g_{e,e'}=f'_{F_{e,e'}}$ because $(M_{e,e'}, g_{e,e'})$ is a geometric quotient of $R_{e,e'}$ by G. By the functoriality of $f_E^{e,e'}$ and f'_E and by the universality of $R_{e,e'}$, we see that $\phi\cdot f_E^{e,e'}=f'_E$ for all E in $\Sigma^{H,e'}_{X/S}(T)$. It is clear that $\phi(k(s))\cdot\theta_s=\theta_s'$. q.e.d.

Since both $M_{e_1,e'}$ and $M_{e_2,e'}$ are coarse moduli schemes of the same functor $\Sigma^{H,e'}_{X/S}$, we obtain a unique isomorphism $\phi^{e'}_{e_1,e_2}:M_{e_1,e'}\to M_{e_2,e'}$ such that $\phi^{e'}_{e_1,e_2}\cdot f_E^{e_1,e'}=f_E^{e_2,e'}$. Since $M_{e,e'}$ is an open subscheme of $M_{e,e}$, $M_{e',e'}$ can be regarded an open subscheme of $M_{e,e}$ through $\phi^{e'}_{e,e'}$. Taking the inductive limit of $\{M_{e,e}\}$, an S-scheme $M_{X/S}(H)$ is obtained. Since each $M_{e,e}$ is quasi-projective over S, $M_{X/S}(H)$ is locally of finite type and separated over S.

Theorem 5. 6. *The functor $\Sigma^H_{X/S}$ has a coarse moduli scheme $M_{X/S}(H)$ in (Sch/S). Moreover, $M_{X/S}(H)$ is separated and locally of finite type over S.*

Proof. For all geometric points s of S, $\bigcup_e\Sigma^{H,e}_{X/S}(\mathrm{Spec}(k(s))=\Sigma^H_{X/S}(\mathrm{Spec}(k(s)))$ by virtue of Corollary 1. 2. 1 of [8]. Thus Proposition 5. 5 implies that $M_{X/S}(H)$ enjoys the property (i) of coarse moduli schemes for $\Sigma^H_{X/S}$. To show the property (ii), take a T in (Sch/S) and an E in $\Sigma^H_{X/S}(T)$. By virtue of Lemma 3. 5, there exists an ascending sequence of open sets $\{T_e\}_{e\geq 0}$ of T such that $\bigcup_e T_e=T$ and that a geometric point t is in T_e if and only if $E\otimes_{\mathcal{O}_T}k(t)$ is e-stable. Set $E_e=E|_{X\times_S T_e}$. Let us consider a pair of $T_{e'}\subseteq T_e$ $(e'\leqq e)$. Proposition 5. 5 provides us with morphisma $f^{e',e'}_{E_{e'}}:T_{e'}\to M_{e',e'}$ and $f^{e,e'}_{E_{e'}}:T_{e'}\to M_{e,e'}$ such that $\phi^{e'}_{e',e}\cdot f^{e',e'}_{E_{e'}}=f^{e,e'}_{E_{e'}}$. By the construction of $f^{e,e'}_{E_{e'}}$, we see that $j\cdot f^{e,e'}_{E_{e'}}=f^{e,e}_{(1_X\times i)^*(E_e)}$ for the open immersions $i:T_{e'}\to T_e$ and $j:M_{e,e'}\to M_{e,e}$. Thus we get $j\cdot\phi^{e'}_{e',e}\cdot f^{e',e'}_{E_{e'}}=f^{e,e}_{E_e}\cdot i$, whence a morphism $f_E:T\to M_{X/S}(H)$ is obtained. For the morphism $g:T'\to T$ in (Sch/S), $g(T_e')$ is contained in T_e, where T_e' for T' is the same as T_e for T. Thus the functoriality of f_E is an immediate consequence of that of $f^{e,e}_{E_e}$. Finally let us show the property (iii). Assume that $\{M', f_E', \theta_s'\}$

has the property (ii). Then it enjoys the property (ii) for $\sum_{X/S}^{H,e}$. Thus we get a morphism $\psi_e: M_{e,e} \to M'$. If $e' \geqq e$, then $(\psi_{e'}|_{M_{e,e}}) \cdot f_E^{e,e} = f_E'$, and hence the uniqueness of ψ_e implies that $\psi_e = \psi_{e'}|_{M_{e,e}}$. We have therefore a unique morphism $\psi: M_{X/S}(H) \to M'$ such that $\psi \cdot f_E = f_E'$. q.e.d.

We shall close this article by the following remark.

Remark 5. 7. 1) Let $S' \to S$ be a morphism of algebraic k-schemes and let $X' = X \times_S S'$. Then $M_{X/S}(H) \times_S S' = M_{X'/S'}(H)$. If the characteristic of k is zero, then this is easy because the geometric quotient in Theorem 4. 17 is a universal one (see [10]). In general case, this is a corollary to the fact that $R_{e,e'}$ is a principal fibre bundle over $M_{e,e'}$ by the group $\bar{G}$ (see the forthcoming paper [9]).

2) Is $M_{X/S}(H)$ of finite type over S? This is equivalent to the following question: Is the family of classes of stable sheaves with a fixed Hilbert polynomial on the fibres of X over S bounded? This is true if the relative dimension of X over S is 1 or 2 (see [1], [7] and [3]) or if $r=2$ (see [9]).

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