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Good and bad field generators

By

Peter RUSSELL

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Let k be a field. A *field generator* in two variables over k is a polynomial $f \in k[x, y]$ such that $k(x, y)=k(f, g)$ for some rational function $g \in k(x, y)$. We continue the investigation of field generators begun in [1] and [2]. Using methods of [2], we first study in detail properties of the multiplicity tree at infinity of f once coordinate functions x, y have been chosen that are natural for f (see [2, 4.7]). Our original motivation for this had been an attempt to show that all field generators are *good* in the sense that a complementary generator g can be found in $k[x, y]$. However, a quite astonishing example of a bad field generator has been constructed by C. Jan in [1], and we instead use the numerical information obtained to determine, with the help of a computer, all bad field generators of degree ≤ 25, the degree of Jan's example. We find that field generators are good for degrees $d \leq 20$ and $d=22, 23, 24$, and that there is exactly one "type" of bad field generator for $d=21$ and $d=25$ (see 2.6 for a more precise statement). R. Ganong helped materially with the rather elaborate calculations needed to establish this and with the writing of an appendix in which some of the details are explained.

A good field generator f appears as part of a birational morphism φ: $\mathbf{A}_k^2 \rightarrow \mathbf{A}_k^2$ with $\varphi(\alpha, \beta)=(f(\alpha, \beta), g(\alpha, \beta))$ for $\alpha, \beta \in k$. We show that this is almost true in general. Namely, if f is a field generator, a complementary generator $g=a/b$ can always be found with $a, b \in k[x, y]$ such that $(a, b)k[x, y]=k[x, y]$. This means that the pencil of curves $\{g-\mu \mid \mu \in k\}$ has no base points at finite distance and that φ: $\mathbf{A}_k^2 \rightarrow \mathbf{P}_k^2$, $\varphi(\alpha, \beta)=(1, f(\alpha, \beta), g(\alpha, \beta))$, is a birational morphism.

1. We assume that k is algebraically closed in the sequel. This is done mainly to simplify arguments and could be avoided in most places. We use systematically the notation of [2]. Also, if S is a non-singular surface and $p \in S$, π_p: $S' \rightarrow S$ will denote the locally quadratic transformation (l.q.t.) with centre p and $E_p=\pi_p^{-1}(p)$ its exceptional fibre. E_0 will stand for the line at infinity of $k[x, y]$ i.e., $E_0=V(X_0)$ where (X_0, X_1, X_2) are homogeneous coordinates of $\mathbf{P}_k^2$ such that $x=X_1/X_0$, $y=X_2/X_0$.

Let f be a field generator, $d=\deg f$ and $\Lambda=\Lambda(f)$ the pencil of curves $\{V(f-\lambda)|\lambda\in k\}$ (see [2, 1.2]). By [2, 4.5] we may assume that x, y have been chosen so that either f is linear or f has exactly two points on E_0. We rule out the first possibility and may then choose $p_0=(0, 0, 1)$ and $q_0=(0, 1, 0)$ as the points at infinity of f. Let $\mu_1=\mu(p_0, \Lambda)$ and $\mu_2=\mu(q_0, \Lambda)$ (see [2, 2.5]). Then $(\Lambda, E_0)=\mu_1+\mu_2$ by [2, 3.7] and hence

$$\mu_1+\mu_2=d. \tag{1}$$

Put $E_1=E_{p_0}$ and $D_1=E_{q_0}$. Let S be a non-singular surface dominating $\mathbf{P}_k^2$ and D an irreducible divisor on S. We recall that $m(D)=\Sigma\mu(q, \Lambda)$, the sum extended over base points of Λ infinitely near (i.n.) to D (see [2, 2.7]). Let $\Lambda^{(S)}$ denote the proper transform of Λ on S and $\Lambda_\infty^{(S)}$ its member at infinity. We recall that $\varepsilon(D)$ is the multiplicity of D as a component of $\Lambda_\infty^{(S)}$ (see [2, 3.4]). We have $\varepsilon(E_0)=d$ and hence by [2, 3.5.4]

$$\varepsilon(E_1)=d-\mu_1=\mu_2, \tag{2}$$

$$\varepsilon(D_1)=d-\mu_2=\mu_1.$$

Now $m(E_0)=3d-2$ (see [2, 3.3]) and in view of (1) we obtain

$$m(E_1)+m(D_1)\leq 2d-2. \tag{3}$$

Let h_1 and l_1 be the number of i.n. base points of Λ on E_1 and D_1 respectively. Since $\varepsilon(E_1)>0$, $\varepsilon(D_1)>0$, we have $h_1\geq 1$ and $l_1\geq 1$ by [2, 3.5.6]. By [2, 3.5.8] and (2)

$$m(E_1)\geq h_1\mu_2, \tag{4}$$

$$m(D_1)\geq l_1\mu_1.$$

By (1), (3) and (4)

$$(l_1-1)\mu_1+(h_1-1)\mu_2\leq d-2<d. \tag{5}$$

It follows that $h_1=1$ or $l_1=1$. Say $h_1=1$. Then there is a unique base point, p_1 say, of Λ on E_1. Let $i\geq 1$. We define inductively $E_{i+1}=E_{p_i}$ as long as there is a unique base point p_i of Λ on E_i, and we find (uniquely) an integer s such that on E_{s+1} there are either zero base points of Λ or at least two. We have shown $s\geq 1$. Note $\mu(p_i, \Lambda)=(E_i, \Lambda)=\mu(p_{i-1}, \Lambda)$ for $i=1,\ldots, s$ and hence

(6) $$\mu(p_i, \Lambda)=\mu_1 \quad \text{for} \quad i=0,\dots, s.$$

We let $\nu=\varepsilon(E_{s+1})$ and obtain

(7) $$\nu=\mu_2-s\geq\mu_1\geq 0.$$

If h is the number of i.n. base points of Λ on E_{s+1} and $l=l_1$, then $h\nu+l\mu_1=h(\mu_2-s\mu_1)+l\mu_1\leq 2d-2-s\mu_1$ by (3), (6), (7) and [2, 3.5.8]. Hence

(8) $$(h-2)(\mu_2-s\mu_1)+(l-s-2)\mu_1\leq -2.$$

Recall that either $h=0$ or $h\geq 2$. In the first case, $\nu=0$ and $\mu_2=s\mu_1$ by [2, 3.5.6]. In either case $l-s-2<0$ by (7) and (8) and hence $l\leq s+1$. Let $q_{1,1},\dots, q_{1,l}$ be the base points of Λ on D_1 and $\mu_{2,i}=\mu(q_{1,i}, \Lambda)$. If S is obtained from $\mathbf{P}_k^2$ by 1.q.t. at p_0 and q_0 and, say, $q_{1,1}\in S$, then D_1 is the only component of $\Lambda_\infty^{(S)}$ containing $q_{1,1}$ and $\mu_{2,1}\leq\varepsilon(D_1)=\mu_1$ by [2, 3.5.3 and 3.5.4]. Since any $q_{1,i}$ is i.n. to some $q_{1,i_0}\in S$, we have

(9) $$\mu_{2,i}\leq\mu_1 \quad \text{for} \quad i=1,\dots, l.$$

On the other hand $\sum_{1\leq i\leq l}\mu_{2,i}=(\Lambda, D_1)=\mu_2$ and hence

(10) $$l\mu_1\geq\mu_2=\sum_{1\leq i\leq l}\mu_{2,i}\geq s\mu_1.$$

It follows that

(11) $$s\leq l\leq s+1 \quad \text{and} \quad 0\leq\nu=\mu_2-s\mu_1\leq\mu_1.$$

We consider three cases for future reference:

1.1 Suppose $s=l$. Then $\mu_2=s\mu_1$ and $\mu_{2,i}=\mu_1$ for $i=1,\dots, l$ (by (10)). Hence $\varepsilon(E_{s+1})=0$ and $\varepsilon(E_{q_{1,i}})=\mu_1-\mu_{2,i}=0$ for $i=1,\dots, l$. It follows that $p_0,\dots, p_s$, q_0, $q_{1,1},\dots, q_{1,l}$ account for all base points of Λ and that equality holds in (8). Hence $\mu_1=1$.

1.2 Suppose $\nu=\mu_1$. Then $l=s+1$, $\mu_2=(s+1)\mu_1$ and $\mu_{2,i}=\mu_1$ for $i=1,\dots, l$ (by (10)).

1.3 Suppose $0<\nu<\mu_1$. Then $h\geq 2$ and $l=s+1$. Let $p_{1,1},\dots, p_{1,h}$ be the base points of Λ on E_{s+1} and $\mu_{1,i}=\mu(p_{1,i}, \Lambda)$, $i=1,\dots, h$. Then

(12) $$\mu_{1,i}\leq\nu=\varepsilon(E_{s+1}), \qquad i=1,\dots, h \quad \text{and}$$

(13) $$h\nu \geq \sum_{1\leq i\leq h} \mu_{1,i} = \mu_1.$$

(We repeat the argument made for the $q_{1,i}$ above.) Finally, by (8) and (13)

(14) $$(h-2)\nu+2\leq \mu_1 \leq h\nu.$$

If $\mu_{2,i}=\mu_1$ for some i, then f can be simplified by a suitable birational endomorphism of $k[x, y]$ as we will show now. Let $\tau\colon \mathbf{A}_k^2\to\mathbf{A}_k^2$ be the birational morphism defined by $A=k[u, v]\simeq k[xy, y]\subset k[x, y]=B$ (i.e., $\tau(\alpha, \beta)=(\alpha\beta, \beta)$ for $\alpha, \beta\in k$), and let $\tilde{\tau}\colon S_1=\mathbf{P}_k^2\to\mathbf{P}_k^2=S_2$ be the induced birational map. The following facts are easily checked: The fundamental points of $\tilde{\tau}$ are $p_0=(0, 0, 1)$, $q_0=(0, 1, 0)$ and the point $q_{1,1}\in E_{q_0}$ corresponding to the direction of $L=V(X_2)$ at q_0. Let S be the surface obtained from $\mathbf{P}_k^2$ by l.q.t. at $p_0, q_0, q_{1,1}$, let $\rho\colon S\to S_2$ be the birational morphism induced by $\tilde{\tau}$ and let a prime denote taking proper transform on S. Then $\rho(L')=(1, 0, 0)$, $\rho(E'_0)=\rho(E'_{q_0})=(0, 1, 0)$, no other curve on S contracts to a point on S_2 and $\rho(E'_{p_0})=$ line at infinity of S_2. Now let f be as above and suppose $q_{1,1}$ is a base point of Λ with $\mu_{2,1}=\mu(q_{1,1}, \Lambda)=\mu_1$. Then $(L', \Lambda)=(L, \Lambda)-\mu_2-\mu_1=0$, that is, a general member of Λ does not meet L'. As we have seen above, L' is the only curve on S that contracts to a point at finite distance on S_2 and hence the transform of Λ on S_2 has no base points at finite distance. It follows that $\tau^{-1}(\tau(V(f-\lambda)))=V(f-\lambda)$, or $((f-\lambda)B\cap A)B=(f-\lambda)B$, for almost all $\lambda\in k$. Since $(f-\lambda)B\cap A$ is principal, $f-\lambda\in A$ and $f(x, y)=\tilde{f}(xy, y)$ with $\tilde{f}\in k[u, v]$. We have $\deg\tilde{f}=d-\mu_1=\mu_2$. In fact, if x is chosen so that $V(X_1)$ is tangent to Λ at p_0, the degree form of $\tilde{f}$ is $u^{\mu_1}v^{\mu_2-\mu_1}$. (Note that the degree form of f is $x^{\mu_1}y^{\mu_2}$.) By an obvious induction argument we find:

1.4 Suppose Λ has r fundamental points of multiplicity μ_1 on D_1. Then there exists $p(y)\in k[y]$ of degree r such that $f(x, y)=\tilde{f}(xp(y), y)\in k[xp(y), y]$. We have $\deg\tilde{f}=d-r\mu_1$, $\tilde{f}$ is a field generator and if $\tilde{f}$ is good so is f.

If we apply the argument given above to the birational morphism induced by $k[x, xy]\subset k[x, y]$, we find:

1.5 Suppose $\mu_1=\mu_2$. Then $s=1$ and $\nu=0$. If x is chosen so that $V(X_1)$ is tangent to Λ at p_0, then $f(x, y)=\tilde{f}(x, xy)\in k[x, xy]$.

The preceeding results are summarized in

1.6 Theorem: *Let $f\in k[x, y]$ be a field generator, $d=\deg f$ and $\Lambda=\Lambda(f)$. Assume $p_0=(0, 0, 1)$ and $q_0=(0, 1, 0)$ are the points at infinity of f with $\mu_1=\mu(p_0, \Lambda)\leq\mu(q_0, \Lambda)=\mu_2$. Then:*

(1) $\mu_1+\mu_2=d$.

(2) *There is a unique integer* $s\geq 1$ *such that*

(*i*) *there are* $s+1$ *base points* $p_0,\dots, p_s$ *of* Λ *i.n. to* p_0 *with* $\mu(p_i, \Lambda)=\mu_1$ *for* $i=0,\dots, s$,

(*ii*) *if* h *is the number of base points of* Λ *on* $E_{s+1}=E_{p_s}$, *then* $h=0$ *or* $h\geq 2$.

(3) *Let* $v=\mu_2-s\mu_1$. *Then* $0\leq v\leq\mu_1$. *If* $v=\mu_1$, *then* $f(x, y)=\bar{f}(xp(y), y)\in k[xp(y), y]$, *where* $p(y)\in k[y]$ *is of degree* $s+1$ *and* $\bar{f}$ *is a field generator of degree* μ_1.

(4) *Suppose* $h\neq 0$. *Then* $(h-2)v+2\leq\mu_1\leq hv$. *If* $p_{1,1},\dots, p_{1,h}$ *are the base points of* Λ *on* E_{s+1}, *then* $\mu_{1,i}=\mu(p_{1,i}, \Lambda)\leq v$ *for* $i=1,\dots, h$ *and* $\sum_{1\leq i\leq h}\mu_{1,i}=\mu_1$.

(5) *Let* $q_{1,1},\dots, q_{1,l}$ *be the base points of* Λ *on* $D_1=E_{q_0}$. *Then* $s\leq l\leq s+1$, $\mu_{2,i}=\mu(q_{1,i}, \Lambda)\leq\mu_1$ *for* $i=1,\dots, l$ *and* $\sum_{1\leq i\leq l}\mu_{2,i}=\mu_2$. *If* $s=l$, *then* $\mu_{2,i}=\mu_1$ *for* $i=1,\dots, l$. *If* $\mu_{2,i}=\mu_1$ *for* r *values of* i, *then* $f(x, y)=\bar{f}(xp(y), y)\in k[xp(y), y]$, *where* $p(y)\in k[y]$ *is of degree* r *and* $\bar{f}$ *is a field generator of degree* $d-r\mu_1$.

(6) *If* $\mu_1=\mu_2$ *and* x *is chosen so that* $V(x)$ *is tangent to* f *at* p_0, *then* $f(x, y)=\bar{f}(x, xy)\in k[x, xy]$, *where* $\bar{f}$ *is a field generator of degree* μ_1.

2. Let $f\in k[x, y]$ be a field generator, $K=k(f)$ and C_f the complete regular curve over K with function field $k(x, y)$. Note that f is a good field generator if and only if $k(x, y)=K(g)$ for some $g\in K[x, y]$. Since $C_f\simeq\mathbf{P}_k^1$, this is the case if and only if there is a place rational over K (the place given by the degree function on $K[g]$) among the places at infinity of C_f (see [2, section 1]). The places at infinity of C_f may be found by resolving via 1.q.t. the non-regular points (all at infinity) of the plane curve $V(f(x, y)-t)$ with t transcendental over k, or, which is the same, the non-regular points of the generic member Λ_η of $\Lambda=\Lambda(f)$ (see [2, 2.8]). As was pointed out in [2, 2.9], the non-regular points of Λ_η are among the base points of Λ. Hence

2.1 C_f has a rational place at infinity if and only if there is a base point q of Λ at which Λ has a simple branch with variable tangent, i.e., the leading form of a local equation for Λ at q has a variable linear factor (see [2, 2.6]).

Clearly, the above condition is satisfied if there exists a base point q' of Λ such that $\mu(q', \Lambda)=1$, and we find:

2.2 If f is a bad field generator, then $\mu(q, \Lambda)>1$ for all base points q of Λ.

Let $\mu_1,\dots, \mu_1$ ($(s+1)$-times), $\mu_2, \mu_{1,1},\dots, \mu_{1,h}, \mu_{2,1},\dots, \mu_{2,l}, \mu_{3,1},\dots, \mu_{3,r}$ be the multiplicities of the base points of Λ, with s, l, h, μ_1, μ_2 and the $\mu_{1,i}$, $\mu_{2,j}$ as in 1.6 while $\mu_{3,1},\dots, \mu_{3,r}$ represent the remaining multiplicities. When searching for all bad field generators f of degree d, omitting those obtained from field generators $\tilde{f}(u, v)$ of smaller degree by substitutions of the form $u=xp(y)$, $v=y$ with $p(y) \in k[v]$, we may assume, appealing to 1.6, 2.2 and [2, 3.1 and 3.3]:

2.3 (1) μ_1, μ_2 and all $\mu_{i,j}$ are integers ≥ 2.
(2) $(s+1)\mu_1^2+\mu_2^2+\Sigma\mu_{i,j}^2=d^2$,
$(s+1)\mu_1+\mu_2+\Sigma\mu_{i,j}=3d-2$.
(3) $\mu_1<d/2$, $\mu_2=d-\mu_1$ and $\mu_2=s\mu_1+v$, where v is an integer and $0\leq v <\mu_1$.
(4) $v=0$ and $h=0$ or $(h-2)v+2\leq\mu_1\leq hv$, $\mu_{1,j}\leq v$ for $j=1,\ldots, h$ and $\Sigma\mu_{1,j} =v$.
(5) $l=s+1$, $\mu_{2,j}<\mu_1$ for $j=1,\ldots, l$ and $\Sigma\mu_{2,j}=\mu_2$.
(6) $\mu_{3,j}<\mu_1$ for $j=1,\ldots, r$. (An upper bound for r is easily determined.)

A computer programmed to find all sequences satisfying 2.3 for $d\leq 25$ came up with about 80 solutions. All but two, however, are ruled out as the multiplicity sequence of a field generator by fairly straightforward arguments. (An example is given in the appendix.) The remaining ones are:

2.4 $\mu_1=9$, $\mu_2=12$, $\mu_{1,1}=\mu_{1,2}=\mu_{1,3}=3$, $\mu_{2,1}=8$, $\mu_{2,2}=4$, $\mu_{3,1}=4$, $\mu_{3,2}=\mu_{3,3}=\mu_{3,4} =2$.

2.5 $\mu_1=9$, $\mu_2=16$, $\mu_{1,1}=6$, $\mu_{1,2}=3$, $\mu_{2,1}=\mu_{2,2}=8$, $\mu_{3,1}=\mu_{3,2}=3$, $\mu_{3,3}=\cdots=\mu_{3,6}=2$.

$f(x, y) = y^3(xy+1)^9 + 4x^7y^9 + 25x^6y^8 + 66x^5y^7 + 6x^5y^6 + 95x^4y^6 + 23x^4y^5 + 80x^3y^5 + 34x^3y^4 + 4x^3y^3 + 39x^2y^4 - 6x^2y^3 + 7x^2y^2 + 10xy^3 - 52xy^2 + 3xy + x + y^2 - 29y$ is an example of a bad field generator with 2.4 as multiplicities at infinity. (If t is transcendental over k, $V(f-t)$ is a curve of genus 0 over $k(t)$. Also, $x=t$, $y=0$ is a rational point of $V(f-t)$. Hence f is a field generator over the prime field of k.) C_f has exactly two places at infinity, one of degree 2 and one of degree 3 over $k(f)$. Jan's example has 2.5 as multiplicities at infinity. (See [1, chapter III]. We would like to point out that the assumption $\operatorname{char} k=0$ made there is unnecessary.)

A field generator f with two points at infinity of multiplicities μ_1 and μ_2 is of the form $f(x, y)=x^{\mu_1}y^{\mu_2}+g(x, y)$, where $\deg g<\mu_1+\mu_2$, $\deg_x g\leq\mu_1$ and $\deg_y g\leq\mu_2$. (This follows easily from [2, 3.7].) One sees immediately that any nonlinear substitution $x=a(u, v)$, $y=b(u, v)$ increases the degree of f by at least $\min\{\mu_1, \mu_2\}$. We therefore conclude

2.6 A bad field generator of degree ≤ 25 has 2.4 or 2.5 as sequence of multiplicities at infinity. Field generators of degree $d\leq 20$ and $d=22, 23, 24$ are good.

It is shown in the appendix that there is a unique irreducible family of bad field generators f of degree 21. The main point is that 2.4 almost completely determines the positions of the multiple points of f, the only difficulty arising from $p_{1,1}, p_{1,2}, p_{1,3}$ on E_2 with multiplicities $\mu_{1,1}=\mu_{1,2}=\mu_{1,3}=3$ (the notation is as in section 1). These points can be chosen distinct or infinitely near in various combinations (more precisely, one has the choice of a divisor of degree 3 on E_2), and it is not clear a priori whether the f corresponding to a generic choice of three distinct points specializes correctly when two or more points are made to coincide. Jan's example exhibits a very similar behaviour and most likely is again a member of a unique irreducible family of bad field generators of degree 25.

3. Let $f, g \in k(x, y)$. We call (f, g) a *generating pair* if $k(f, g)=k(x, y)$. Associated with any generating pair (f, g) there are birational maps

$$\begin{tikzcd} U=\mathbf{A}^2_k \arrow[r, "\varphi"] \arrow[d, phantom, "\cap"] & \mathbf{A}^2_k=V \arrow[d, phantom, "\cap"] \\ \mathbf{P}^2_k \arrow[r, "\tilde{\varphi}"] & \mathbf{P}^2_k \end{tikzcd} \tag{1}$$

(with $\varphi(\alpha, \beta)=(f(\alpha, \beta), g(\alpha, \beta))$ for $\alpha, \beta \in k$, and $\tilde{\varphi}\,|\,\mathbf{A}^2_k=\varphi$). From (1) we deduce commutative diagrams

$$\begin{tikzcd} & \Gamma \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ \mathbf{A}^2_k \arrow[rr, "\varphi"] & & \mathbf{A}^2_k \end{tikzcd} \tag{2}$$

and

$$\begin{tikzcd} & Z \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbf{P}^2_k \arrow[rr, "\tilde{\varphi}"] & & \mathbf{P}^2_k \end{tikzcd} \tag{3}$$

where Γ is the graph of φ and π_1 (resp. π_2) is the composite of the l.q.t. with centres at the fundamental points of $\tilde{\varphi}$ (resp. $\tilde{\varphi}^{-1}$).

Now suppose $f \in k[x, y]$. Then the coordinate ring of Γ is $A=k[x, y, g]$ and φ_1, φ_2 are given by the inclusions

$$k[x, y] \subset A \supset k[f, g].$$

Write $g=a/b$ with $a, b \in k[x, y]$ and $GCD(a, b)=1$. Then the fundamental points of $\tilde{\varphi}$ on $\mathbf{A}_k^2$ are precisely the common zeros of a and b, or, which is the same, the base points at finite distance of the pencil $\{V(a+\lambda b)|\lambda \in k\}$. The fact that A is a simple extension of $k[x, y]$ has the following nice consequence for the structure of φ. I am indebted to W. Heinzer for pointing this out to me.

3.1 Propositon: *Let $f \in k[x, y]$ be a field generator and (f, g) a generating pair. Write $g=a/b$ with $a, b \in k[x, y]$ and $GCD(a, b)=1$. Suppose $M \subset k[x, y]$ is a maximal ideal such that $a, b \in M$. Then there is a unique height one prime $J \subset A=k[x, y, g]$ such that $J \cap k[x, y]=M$. If $f \in M$ (note $f-\lambda \in M$ for some $\lambda \in k$), then $A_J=k[f, g]_{(f)}$.*

Proof: One sees easily that $A/MA=k[\bar{g}]$ (where $\bar{g}$ is the residue class of $g \bmod MA$) is isomorphic to a polynomial ring in one variable over k. Hence $J=MA$ is prime, and the only height one prime of A contracting to M. Let (R, M_R) be a valuation subring of $k(x, y)$ (M_R=maximal ideal of R) such that $M_R \cap A=J$. Put $M'=M_R \cap k[f, g]$. Note that $R/M_R \supset A/J=k[\bar{g}]$, i.e., g is residually transcendental for R. This shows that M' is not maximal, for otherwise $g-\mu \in M' \subset M_R$ for some $\mu \in k$. Hence $f \in M \subset M_R$ implies $M'=fk[f, g]$ and $k[f, g]_{(f)} \subset A_J$. Since $k[f, g]_{(f)}$ is a valuation ring, equality holds.

3.1.1 Corollary: *$A=k[x, y, g]$ is integrally closed.*

Proof: Since $A_b \simeq k[x, y]_b$, the proposition implies that A_P is a valuation ring for all height one primes P of A. Also,

$$A \simeq k[X, Y, W]/b(X, Y)W-a(X, Y)$$

is a complete intersection. Hence A is normal by [3, III, Prop. 9].

3.1.2 Corollary: *Suppose $M \subset k[x, y]$ is maximal and $a, b, f \in M$. Then M is the only maximal ideal with this property. Also, $f \notin M^2$ and $a+\lambda b \notin M^2$ for almost all $\lambda \in k$.*

Proof: By the proposition, $M=fk[f, g]_{(f)} \cap k[x, y]$. Since $JA_J=fA_J$, $f \notin M^2$. Let $N=(a, b, g+\lambda)A$ with $\lambda \in k$. By 3.1.1, A_N is regular and hence $a+\lambda b \notin M^2$ for almost all $\lambda \in k$.

3.2 Theorem: *Let $f \in k[x, y]$ be a field generator. Then there exist $a, b \in k[x, y]$ such that $k(f, a/b)=k(x, y)$ and $(a, b)k[x, y]=k[x, y]$.*

Proof: We may assume that f is a bad field generator. Let $g=a/b$ be a complementary generator with $GCD(a, b)=1$. We proceed by induction on $j(a, b)=\dim_k(k[x, y]/(a, b)k[x, y])$. Suppose $j(a, b)\geq 1$. Let M be a maximal ideal such that $a, b\in M$. We may assume $f\in M$. Also, replacing b by $a+\lambda b$ with suitable $\lambda\in k$ if necessary, we may assume that b is irreducible. Let $f_1\in k[x, y]$ be irreducible such that $f_1|f$ and $f_1\in M$. Put $V_{f_1}=k[x, y]_{(f_1)}$ and $V_f=k[f, g]_{(f)}$. Both V_f and V_{f_1} are principal valuation subrings of $k(x, y)$ and $V_f\neq V_{f_1}$ (since $fV_f\cap k[x, y]=M\neq f_1k[x, y]=f_1V_{f_1}\cap k[x, y]$). Hence either $V_{f_1}\not\supset k[f, g]$ (i.e., V_{f_1} has no centre on $k[f, g]$) or $M'=f_1V_{f_1}\cap k[f, g]$ is a maximal ideal. The first possibility we can rule out. It implies $f_1|b$, so $b|f$ (since b is irreducible) and f is a good field generator (since $(f, fa/b)$ is a generating pair). Hence $V_{f_1}\supset k[f, g]$ and M' is maximal, which implies $g-\mu\in M'$ for some $\mu\in k$. Then $f_1|a-\mu b$, and replacing a by $a-\mu b$ we may assume $f_1|a$. Let $h=GCD(f, a)$ and write $a=ha', f=hf'$. We claim that $(a', f')k[x, y]=k[x, y]$, and this proves the theorem. For then $(f, a'/f'b)$ is a generating pair and $j(a', f'b)=j(a', b)<j(a, b)=j(a', b)+j(h, b)$.

To establish the claim, consider a maximal ideal $N\subset k[x, y]$ such that $a', f\in N$. (If none exist, we are done.) If $N=M$, then $f'\notin N$ by 3.1.2. Also, $f, a', f'b\in N$, and applying 3.1.2 to the generating pair $(f, a'/f'b)$ we find that N is the only maximal ideal of $k[x, y]$ such that $a', f\in N$. Hence $(a', f')k[x, y]=k[x, y]$. So suppose $N\neq M$. Then $b\notin N$ (otherwise $a, b, f\in N$ and $N=M$ by 3.1.2) and $k[x, y]_N\supset k[f, g]$ (i.e., the rational map φ of (1) is defined at N). Let f_2 be an irreducible factor of f such that $f_2\in N$ and put $V_{f_2}=k[x, y]_{(f_2)}$. Again $f_2V_{f_2}\cap k[f, g]$ is a maximal ideal (we repeat the argument made for f_1 above). Hence $f_2V_{f_2}\cap k[f, g]=Nk[x, y]_N\cap k[f, g]$ and $g\in f_2V_{f_2}$, that is $f_2|a$. Now $f, a', ff'b\in N$, and applying 3.1.2 to the generating pair $(f, a'/ff'b)$ we find that again N is the only maximal ideal with $a', f\in N$ and that $f=f_2f''$ with $f''\notin N$. Since $f_2|a, f_2\nmid f'$ and $(f', a')k[x, y]=k[x, y]$ as before.

3.2.1 Corollary: *Let $f\in k[x, y]$ be a field generator. Then any irreducible component of $V(f)\subset \mathbf{A}_k^2$ is a non-singular rational curve, any two components either do not meet or meet normally in exactly one point, and no three components have a point in common.*

Proof: Choose a complementary generator $g=a/b$ such that $(a, b)k[x, y]=k[x, y]$. Then the birational map $\tilde{\varphi}$ of (1) has no fundamental point at finite distance, that is π_1 induces an isomorphism of an open subset of Z with U. Let F be the closure of $V(f)$ in $\mathbf{P}_k^2$. Then $\pi_1^{-1}(F)=\pi_2^{-1}(L)$, where L is a line in $\mathbf{P}_k^2$, and the irreducible components of $\pi_2^{-1}(L)$ clearly have the properties claimed for those of $V(f)$.

We conclude by strengthening 3.1.2.

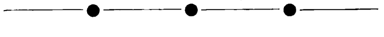
3.3 Proposition: *Let* $f\in k[x, y]$ *be a field generator and* $g=a/b$ *a complementary generator with* $a, b\in k[x, y]$ *and* $GCD(a, b)=1$. *Let* $M\subset k[x, y]$ *be a maximal ideal such that* $a, b, f\in M$. *Then* $M=(a, b, f)k[x, y]$ *and* $Mk[x, y]_M=(a+\lambda b, f)k[x, y]_M$ *for almost all* $\lambda\in k$.

Proof: Let f_1, a', f' be as in the proof of 3.2 (we assume $f_1|a$). Let $n\geqq 1$ be such that $a\in M^n-M^{n+1}$. We show by induction on n that $(f_1, b)k[x, y]_M=Mk[x, y]_M$. Since we are free to replace b by $a+\lambda b$ for almost all $\lambda\in k$, this proves the proposition in view of 3.1.2. Suppose first $n>1$ and consider the generating pair $(f, a'/f'b)$. We have $a'\in M^{n-1}-M^n$ and $f'\notin M$, so $(f_1, b)k[x, y]_M=(f_1, f'b)k[x, y]_M$ and we are done by induction. If $n=1$, consider the generating pair $(f, f'b/a')$. Let again $\tilde{\varphi}$ denote the associated birational map and let F' denote the closure in $\mathbf{P}_k^2$ of $V(ff'b)$. Then $\pi_1^{-1}(F')=\pi_2^{-1}(L_1\cup L_2)$, where L_1, L_2 are lines in $\mathbf{P}_k^2$. Now $a'\notin M$. Hence $\tilde{\varphi}$ is defined at M and we conclude that f_1 and b meet normally at M (components of $\pi_2^{-1}(L_1\cup L_2)$ meet normally).

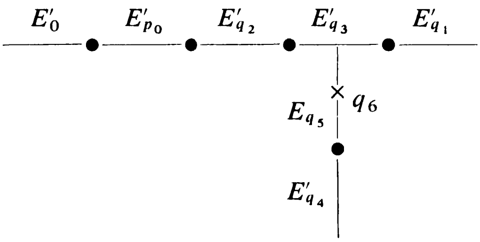
Appendix: The technique of determining a plane curve from its multiplicity tree is, of course, well known in principle. In practice, the calculations can assume imposing proportions, and it may be worth-while to illustrate the technique at work in a non-trivial example, here in the determination of the family of all bad field generators of degree 21.

As has been mentioned, all but two of the computer solutions of 2.3 can be ruled out as the multiplicity sequence of a field generator. Here is an example: $\mu_1=10$, $\mu_2=13$, $\mu_{1,1}=\mu_{1,2}=3$, $\mu_{1,3}=\mu_{1,4}=2$, $\mu_{2,1}=8$, $\mu_{2,2}=5$, $\mu_{3,1}=5$, $\mu_{3,2}=4$, $\mu_{3,3}=2$. We find $\varepsilon(E_{q_{1,1}})=\varepsilon(E_{q_0})-8=2$ by [2, 3.5.4]. By [2, 3.5.6] all points of Λ on $E_{q_{1,1}}$ are base points of Λ. There are two possibilities: (i) $q_{1,2}$ is not i.n. to $q_{1,1}$. Then the multiplicities of all base points of Λ on $E_{q_{1,1}}$ are among the $\mu_{3,i}$ and their sum is $\mu_{2,1}=8$, which is impossible. (ii) $q_{1,2}$ is i.n. to $q_{1,1}$, i.e., $q_{1,2}\in E'_{q_0}\cap E_{q_{1,1}}$ (here and in the sequel a prime will denote taking proper transform). Now the multiplicities of all remaining base points of Λ on $E_{q_{1,1}}$ are among the $\mu_{3,i}$ and their sum is $\mu_{2,1}-\mu_{2,2}=3$, again an impossibility.

Let us next indicate how the sequence 2.4 determines the tree of singularities of a bad field generator f of degree 21. We first consider $q_0\in E_0$ with $\mu(q_0, \Lambda)=12$. By arguments like those above we find that $q_2=q_{1,2}$ is i.n. to $q_{1,1}=q_1$ on E_{q_0}. After blowing up q_2, the support of the total transform of E_0 looks as follows:

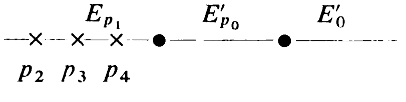


By [2, 3.5.4], $\varepsilon(E_{q_1})=9-8=1$ and $\varepsilon(E_{q_2})=9+1-4=6$. Since $\mu_{3,i}>1=\varepsilon(E_{q_1})$ for all i, Λ has no base point on $E'_{q_1}-E_{q_2}$ by [2, 3.5.3 and 3.5.4]. Since $(\Lambda, E'_{q_1})=8-4>0$, there is a base point $q_3 \in E'_{q_1} \cap E_{q_2}$, and we claim that $\mu(q_3, \Lambda)=\mu_{3,1}=4$. In fact, if $q_3,\ldots, q_c$ are the base points of Λ on E_{q_2}, then $\sum_{3\leq i\leq c} \mu(q_i, \Lambda)=\mu_{1,2}=4$. The $\mu(q_i, \Lambda)$ are among the $\mu_{3,j}$, and if $\mu(q_3, \Lambda)\neq 4$, then $c=4$ and $\mu(q_3, \Lambda)=\mu(q_4, \Lambda)=2$. If q is the base point of Λ with $\mu(q, \Lambda) =\mu_{3,1}=4$, then q is not i.n. to p_0, as follows from 1.6 (4), and hence q is i.n. to q_3 or q_4, which is impossible since $\mu(q, \Lambda)>2$. This proves our claim. Now $\varepsilon(E_{q_3})=6+1-4=3$, the base points of Λ on E_{q_3} are not on $E'_{q_1} \cup E'_{q_2}$, the sum of their multiplicities is 4, and hence there are two, q_4 and q_5 say, of multiplicity 2 each. We have $\varepsilon(E_{q_4})=1$, and we argue as above that $q_5 \in E_{q_4} \cap E'_{q_3}$ is i.n. to q_4. We find $\varepsilon(E_{q_5})=2$, and there is a unique base point $q_6 \in E_{q_5}$, $q_6 \notin E'_{q_4} \cup E'_{q_3}$. Now $\mu(q_6, \Lambda)=2=\varepsilon(E_{q_5})$, and q_6 is a terminal base point (see [2, 2.5]). The support of the total transform of E_0 at this stage has the following configuration:

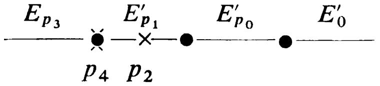


Above p_0 we first find a unique base point $p_1 \in E_{p_0}$ and then three base points, p_2, p_3, p_4 say, of multiplicity 3 each on $E_{p_1}-E'_{p_0}$. We note $\varepsilon(E_{p_1})=3$ and $(\Lambda, E_{p_1})=9$. There are three cases to consider:

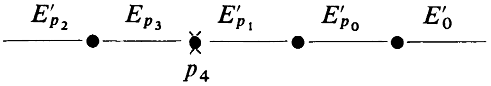
(1) p_2, p_3, p_4 are distinct. They are then all terminal base points. We obtain the diagram



(2) p_2, p_3 are distinct, p_4 is i.n. to p_3. Then p_2, p_4 are terminal. The diagram is



(3) p_3 is i.n. to p_2 and p_4 is i.n. to p_3. Then p_4 is terminal. The diagram is



Let $f(x, y)=\Sigma a_{i,j}x^i y^j$. We choose $p_0=(0, 0, 1)$ and $q_0=(0, 1, 0)$ as points at infinity of f. $F(X_0, X_1, X_2)=X_0^{21}f(X_1/X_0, X_2/X_0)$ has unique tangents at p_0 and q_0. We choose these as X_1 and X_2 respectively. Then $f(x, y)=x^9y^{12}$ +terms of lower degree in both x and y. At q_0, F has local equation $F(z, 1, y)=\Sigma a_{i,j}y^j z^{21-i-j}$. Blow up q_0. The proper transform $F^{(1)}$ of F has a local equation of the form $f^{(1)}(y, z)=F(z, 1, yz)/z^{12}=y^{12}+\sum_{i\leq 8, j\leq 11} a_{i,j}y^j z^{9-i}$, and z is a local equation for E_{q_0}. We require that $F^{(1)}$ have $q_1=(0, 0)$ as zero of multiplicity 8 with E_{q_0} as unique tangent. This is so if and only if $a_{i,j}=0$ for $i>j$, except that $a_{1,0}\neq 0$. Blow up q_1. $F^{(2)}$ has a local equation of the form $f^{(1)}(y, yz)/y^8=y^4+\Sigma a_{i,j}y^{j-i+1}z^{9-i}$. We require that $q_2=(0, 0)$ be a fourfold point of $F^{(2)}$ with unique tangent E_{q_1}, which implies the vanishing of six more of the a_{ij}. On blowing up q_2 one obtains a local equation of the form $f^{(3)}=y^4+\Sigma a_{ij}y^{j-i+1}z^{j-2i+6}$, and $F^{(3)}$ is to have $q_3=(0, 0)$ as fourfold point with unique tangent, meeting E'_{q_1} and E_{q_2} normally. Hence the leading form of $f^{(3)}$ is $(y+\alpha z)^4$ for some $\alpha\in k^*$. Consequently, three more a_{ij} vanish, and four more are determined as functions of α.

One proceeds in this fashion. There is a choice of tangent direction at q_5, amounting to a choice of $\gamma\in k^*$. Writing down all conditions at q_3, q_4, q_5, q_6 one finds: all but 29 of the a_{ij} vanish, the leading coefficient is 1, four more are determined by $\alpha\in k^*$, and three more are rational functions, defined for $\alpha\neq 0$, in $a_{8,11}$, α, γ. The remaining coefficients are $a_{i,i+3}$ $(0\leq i\leq 8)$, $a_{0,0}$ and eleven others. In each of the cases (1), (2), (3), the $a_{i,i+3}$ are determined as symmetric polynomials with integer coefficients in β_1, β_2, β_3 corresponding to a choice of $(p_2, p_3, p_4)\in(E_{p_1}-\{p\})^3\simeq \mathbf{A}_k^3$, where $\{p\}=E_{p_1}\cap E'_{p_0}$. In addition nine more equations result that are linear in the eleven coefficients mentioned above. In each case $a_{0,0}$ is arbitrary. Together with two equations left from the analysis at the q_i, these give matrix equations $M_iA=W_i$, $1\leq i\leq 3$ (one for each of the cases (1), (2), (3)), where A is the column with entries $a_{i,i+2}$ $(0\leq i\leq 5)$, $a_{3,4}$, $a_{1,1}$, $a_{0,1}$, $a_{1,2}$, $a_{2,3}$. We exhibit M_1 and W_1.

$$M_1=\begin{bmatrix} \left.\begin{matrix} 1 & \beta_i & \beta_i^2 & \cdots & \beta_i^5 \\ 0 & 1 & 2\beta_i & \cdots & 5\beta_i^4 \end{matrix}\right\} i=1, 2, 3 & & & & \\ & -\alpha^2 & \alpha & -1 & \\ & 3\alpha^2\gamma & -2\alpha\gamma & \gamma & \\ & & & & \beta_i^3 \quad 0 \quad 1 \quad \beta_i \quad \beta_i^2\} i=1, 2, 3 \end{bmatrix}$$

$$W_1=\begin{bmatrix} \left.\begin{matrix} -a_{6,8}\beta_i^6 \ -4\alpha\beta_i^7 \\ -6a_{6,8}\beta_i^5-28\alpha\beta_i^6 \end{matrix}\right\} i=1,2,\ 3 \\ \gamma^2-a_{7,10}\alpha^3 \\ \gamma(a_{6,8}\gamma+4a_{7,10}\alpha^3-4a_{8,11}\alpha\gamma) \\ -a_{4,5}\beta_i^4-6\alpha^2\beta_i^5\} i=1,\ 2,\ 3 \end{bmatrix}$$

The factors of $\det M_1$ are α, γ and $\prod_{i<j}(\beta_i-\beta_j)^5$, those of $\det M_2$ are α, γ and $(\beta_h-\beta_i)^{10}$ (in case $\beta_h\neq\beta_i=\beta_j$) and $\det M_3=\alpha\gamma$. For general $(\alpha, \gamma, \beta_1, \beta_2, \beta_3)\in \mathbf{A}_k^5$ (with $\det M_1\neq 0$) there is a unique corresponding pencil $\Lambda(f)$, the inverse image $\Lambda(f)$ being determined up to permutation of the β_i. Elementary row operations and extraction of five factors $\beta_j-\beta_i$ from appropriate rows allow one to transform $M_1A=W_1$ into $M_1^*A=W_1^*$, where the factors of $\det M_1^*$ are $\alpha, \gamma, (\beta_h-\beta_i)^5$ and $(\beta_h-\beta_j)^5$, and M_1^*, W_1^* specialize to M_2, W_2 when $\beta_i=\beta_j$. By further operations of the same type one obtains $M_1^{**}A=W_1^{**}$, where $\det M_1^{**}=\alpha\gamma$ and M_1^{**}, W_1^{**} specialize to M_3, W_3 when $\beta_i=\beta_j=\beta_h$. It follows easily that there is a one-to-one correspondence between the points of $V_1=(\mathbf{A}_k^1-\{0\})^2\times\mathbf{A}_k^{(3)}$ ($\mathbf{A}_k^{(3)}$=symmetric threefold product of $\mathbf{A}_k^1$) and pencils $\Lambda(f)$ satisfying our initial choice of p_0, q_0 and tangents at these points. These choices amount to picking a point in $V_2=(\mathbf{P}_k^1\times\mathbf{P}_k^1-\Delta)\times\mathbf{A}_k^2$ (Δ stands for diagonal). Taking into account the free choice of $a_{0,0}$, we find that bad field generators of degree 21 are parametrized by $V_1\times V_2\times\mathbf{A}_k^1$.

DEPARTMENT OF MATHEMATICS
McGILL UNIVERSITY
MONTREAL, P.Q.
CANADA

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