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Hopf algebra structure of simple Lie groups

By

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§1. Introduction

Let G be a compact connected Lie group of rank l and p a rational prime. The group multiplication $\mu: G\times G\to G$ induces a map

$$\mu^*: H^*(G; Z_p) \longrightarrow H^*(G\times G; Z_p). \tag{1.1}$$

By the virtue of the Künneth formular, μ^* gives a Hopf algebra structure

$$\phi: H^*(G; Z_p) \longrightarrow H^*(G; Z_p)\otimes H^*(G; Z_p) \tag{1.2}$$

of $H^*(G; Z_p)$.

Since

$$\phi(x)-(x\otimes 1+1\otimes x)\in \tilde{H}^*(G; Z_p)\otimes\tilde{H}^*(G; Z_p)$$
$$\text{for}\quad \tilde{H}^*(G; Z_p)=\sum_{i>0} H^i(G; Z_p),$$

we put

$$\bar{\phi}(x)=\phi(x)-(x\otimes 1+1\otimes x).$$

An element $x\in\tilde{H}^*(G; Z_p)$ is said to be primitive if $\bar{\phi}(x)=0$.

On the other hand consider the universal G bundle

$$G \longrightarrow EG \longrightarrow BG. \tag{1.3}$$

An element $x\in\tilde{H}^*(G; Z_p)$ is called to be universally transgressive if x is transgressive with respect to (1.3).

As is well known

(1.4) if x is universally transgressive, then x is primitive (cf. Borel [5]).

As is well known

(1.5) if $H^*(\boldsymbol{G}; \boldsymbol{Z})$ is p-torsion free, then $H^*(\boldsymbol{G}; \boldsymbol{Z}_p) \cong \Lambda(x_1, x_2, \ldots, x_l)$ with $\deg x_i$ odd and $l = \operatorname{rank} \boldsymbol{G}$.

So by the Hopf Samelson theorem, each x_i can be chosen to be primitive. Moreover each x_i can be chosen to be universally transgressive.

On the other hand if $H^*(\boldsymbol{G}; \boldsymbol{Z})$ has p-torsion, it seems to be very difficult to determine the Hopf algebra structure of $H^*(\boldsymbol{G}; \boldsymbol{Z}_p)$. In fact Browder [12] showed that

(1.6) if p is an odd prime, $H^*(\boldsymbol{G}; \boldsymbol{Z}_p)$ is primitively generated if and only if $H^*(\boldsymbol{G}; \boldsymbol{Z})$ is p-torsion free.

The purpose of the present paper is to determine the Hopf algebra structure of $H^*(\boldsymbol{G}; \boldsymbol{Z}_p)$ for any p and any simple $\boldsymbol{G}$.

For a classical type $\boldsymbol{G}$, the Hopf algebra structure of $H^*(\boldsymbol{G}; \boldsymbol{Z}_p)$ is determined by Baum-Browder [4] except for $\boldsymbol{G} = \boldsymbol{Spin}(n)$ and $\boldsymbol{Ss}(4m)$. The Hopf algebra structure of $H^*(\boldsymbol{Spin}(n); \boldsymbol{Z}_2)$ and $H^*(\boldsymbol{Ss}(4m); \boldsymbol{Z}_2)$ was determined by Ishitoya-Toda and the author [13].

For exceptional type $\boldsymbol{G}$, $H^*(\boldsymbol{G}; \boldsymbol{Z})$ has p-torsion if and only if $(\boldsymbol{G}, p)$ is one of the following:

$$\begin{aligned}(\boldsymbol{G}, p) &= (\boldsymbol{G}_2, 2),\\ &= (\boldsymbol{F}_4, 2), (\boldsymbol{F}_4, 3),\\ &= (\boldsymbol{E}_6, 2), (\boldsymbol{E}_6, 3), (\boldsymbol{AdE}_6, 2), (\boldsymbol{AdE}_6, 3),\\ &= (\boldsymbol{E}_7, 2), (\boldsymbol{E}_7, 3), (\boldsymbol{AdE}_7, 2), (\boldsymbol{AdE}_7, 3),\\ &= (\boldsymbol{E}_8, 2), (\boldsymbol{E}_8, 3), (\boldsymbol{E}_8, 5),\end{aligned}$$

where $\boldsymbol{G}_2, \boldsymbol{F}_4, \boldsymbol{E}_6, \boldsymbol{E}_7, \boldsymbol{E}_8$ are compact 1-connected simple Lie groups and $\boldsymbol{AdE}_i$ is the quotient of $\boldsymbol{E}_i$ by its center for $i = 6, 7$.

Note that since the covering projection $\rho\colon \boldsymbol{E}_6 \to \boldsymbol{AdE}_6$ (resp. $\rho'\colon \boldsymbol{E}_7 \to \boldsymbol{AdE}_7$)

is a 3-fold covering (resp. a 2-fold covering), $\rho^*: H^*(\boldsymbol{AdE}_6; \boldsymbol{Z}_p)\rightarrow H^*(\boldsymbol{E}_6; \boldsymbol{Z}_p)$ is an isomorphism for any prime $p\neq 3$ (resp. $\rho'^*: H^*(\boldsymbol{AdE}_7; \boldsymbol{Z}_p)\rightarrow H^*(\boldsymbol{E}_7; \boldsymbol{Z}_p)$ is an isomorphism for any prime $p\neq 2$). The following are the cases where the Hopf algebra structure of $H^*(\boldsymbol{G}; \boldsymbol{Z}_p)$ was determined for the above $(\boldsymbol{G}, p)$:

$(\boldsymbol{G}_2, 2)$, $(\boldsymbol{F}_4, 2)$	by Borel [6]
$(\boldsymbol{AdE}_6, 2)$, $(\boldsymbol{E}_i, 2)$ for	by Toda [34], Kono-Mimura [17]
$i=6, 7, 8$	and Kono-Mimura-Shimada [21],
$(\boldsymbol{F}_4, 3)$	by Araki [2],
$(\boldsymbol{E}_6, 3)$	by Kono-Mimura [18] and Toda [34],
$(\boldsymbol{E}_7, 3)$, $(\boldsymbol{AdE}_7, 3)$, $(\boldsymbol{E}_8, 3)$	by Kono-Mimura [19].

So the Hopf algebra structure of $H^*(\boldsymbol{G}; \boldsymbol{Z}_p)$ was determined except for $(\boldsymbol{G}, p)=(\boldsymbol{AdE}_6, 3)$ and $(\boldsymbol{E}_8, 5)$. In this paper we shall determine these two cases. Cohomology operations of $H^*(\boldsymbol{AdE}_6; \boldsymbol{Z}_3)$ and $H^*(\boldsymbol{E}_8; \boldsymbol{Z}_5)$ are also determined.

The results of this paper are stated as follows, for details see Theorem 5.15 and 6.10:

$H^*(\boldsymbol{E}_8; \boldsymbol{Z}_5)\cong \boldsymbol{Z}_5[x_{12}]/(x_{12}^5)\otimes\Lambda(x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47})$ *with*

$\deg x_i=i$,

$\bar{\phi}(x_3)=\bar{\phi}(x_{11})=\bar{\phi}(x_{12})=0$

$\bar{\phi}(x_{15})=x_{12}\otimes x_3, \quad \bar{\phi}(x_{23})=x_{12}\otimes x_{11}$,

$\bar{\phi}(x_{27})=2x_{12}\otimes x_{15}+x_{12}^2\otimes x_3$,

$\bar{\phi}(x_{35})=2x_{12}\otimes x_{23}+x_{12}^2\otimes x_{11}$,

$\bar{\phi}(x_{39})=3x_{12}\otimes x_{27}+3x_{12}^2\otimes x_{15}+x_{12}^3\otimes x_3$,

$\bar{\phi}(x_{47})=3x_{12}\otimes x_{35}+3x_{12}^2\otimes x_{23}+x_{12}^3\otimes x_{11}$,

$x_{11}=\mathscr{P}^1x_3, \quad x_{12}=\beta x_{11}, \quad x_{23}=\mathscr{P}^1x_{15}, \quad x_{35}=\mathscr{P}^1x_{27}$,

$x_{47}=\mathscr{P}^1x_{39}$.

$H^*(\boldsymbol{AdE}_6; \boldsymbol{Z}_3)=\boldsymbol{Z}_3[x_2, x_8]/(x_2^9, x_8^3)\otimes\Lambda(x_1, x_3, x_7, x_9, x_{11}, x_{15})$ *with* $\deg x_i=i$,

$\bar{\phi}(x_1)=\bar{\phi}(x_2)=0,$

$\bar{\phi}(x_3)=x_2\otimes x_1, \quad \bar{\phi}(x_7)=x_2^3\otimes x_1, \quad \bar{\phi}(x_8)=x_2^3\otimes x_2,$

$\bar{\phi}(x_9)=x_2\otimes x_7-x_2^3\otimes x_3+x_8\otimes x_1+x_2^4\otimes x_1,$

$\bar{\phi}(x_{11})=x_2\otimes x_9-x_2^2\otimes x_7+x_8\otimes x_3-x_2^4\otimes x_3+x_8x_2\otimes x_1-x_2^5\otimes x_1,$

$\bar{\phi}(x_{15})=x_2^3\otimes x_9+x_8\otimes x_7+x_2^6\otimes x_3+x_8x_2^3\otimes x_1,$

$x_2=\beta x_1, \quad x_7=\mathscr{P}^1x_3, \quad x_8=\beta x_7, \quad x_{15}=\mathscr{P}^1x_{11}.$

The paper is organized as follows:

In §2 $\mathrm{Cotor}^A(\boldsymbol{Z}_5, \boldsymbol{Z}_5)$ for some Hopf algebra over $\boldsymbol{Z}_5$ is calculated. In §3 a generalization of Theorem 1.1 of [13] is given. In §4 the invariant subalgebra $H^*(B\boldsymbol{T}^8; \boldsymbol{Z}_5)^{W(\boldsymbol{E}_8)}$ under the action of the Weyl group of $\boldsymbol{E}_8$ is calculated. In §5 the Hopf algebra structure of $H^*(\boldsymbol{E}_8; \boldsymbol{Z}_5)$ is determined by the results of §2, §3, §4. In this section the Rothenberg-Steenrod spectral sequence [29] plays an important role. In the next section, §6, the Hopf algebra structure of $H^*(\boldsymbol{AdE}_6; \boldsymbol{Z}_3)$ is determined. Note that in §6 we only need the result of §3. In §7 the Hopf algebra structure of $H^*(\boldsymbol{E}_6; \boldsymbol{Z}_3)$ and $H^*(\boldsymbol{F}_4; \boldsymbol{Z}_3)$ is determined by making use of the result of §6.

Throughout the paper the augmentation ideal of a connected algebra A is denoted by $\tilde{A}$.

§2. Injective resolutions of $\boldsymbol{Z}_5$

In this section we shall construct injective resolutions of $\boldsymbol{Z}_5$ over some Hopf algebras.

Notation 2.1. (A, ϕ) is a graded Hopf algebra over $\boldsymbol{Z}_5$ such that

(1) As an algebra

$$A\cong \boldsymbol{Z}_5[x_{12}]/(x_{12}^5)\otimes\Lambda(x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}),$$

where $\deg x_i=i$,

(2) The coalgebra structure is given by

$$\bar{\phi}(x_3)=\bar{\phi}(x_{11})=\bar{\phi}(x_{12})=0,$$

$$\bar{\phi}(x_{15})=x_{12}\otimes x_3, \quad \bar{\phi}(x_{23})=x_{12}\otimes x_{11},$$

$$\bar{\phi}(x_{27}) = 2x_{12} \otimes x_{15} + x_{12}^2 \otimes x_3,$$

$$\bar{\phi}(x_{35}) = 2x_{12} \otimes x_{23} + x_{12}^2 \otimes x_{11},$$

where $\bar{\phi}(x) = \phi(x) - (x \otimes 1 + 1 \otimes x)$.

To construct an injective resulusion of $\boldsymbol{Z}_5$ over the coalgebra (A, ϕ) we use the method of [32] (see also [20]).

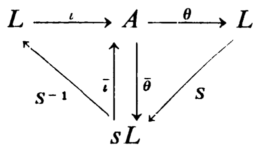
Let L be a graded submodule of $\tilde{A}$ generated by

$$\{x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{12}, x_{12}^2, x_{12}^3, x_{12}^4\}.$$

Let $s\colon L \to sL$ be the suspension. Denote the corresponding elements under the suspension by

$$\{a_4, a_{12}, b_{16}, b_{24}, c_{28}, c_{36}, d_{13}, d_{25}, d_{37}, d_{49}\}.$$

Let $\theta\colon A \to L$ be the projection and $\iota\colon L \to A$ the inclusion. Define $\bar{\theta}\colon A \to sL$ and $\bar{\iota}\colon sL \to A$ by the following commutative diagram:



Construct the free tensor algebra $T(sL)$ over sL and denote the product by ψ. Let I be the two-sided ideal of $T(sL)$ generated by $\operatorname{Im}(\psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi) \circ (1 - (\iota \circ \theta))$. Then we have

Lemma 2.2. *I is generated by*

$[a_4, a_{12}]$, $[a_i, b_j]$, $[a_i, c_k]$, $[a_i, d_l]$, $[b_{16}, b_{24}]$,

$[b_j, c_k]$, $[c_{28}, c_{36}]$,

$[d_{13}, b_{16}] + d_{25} \cdot a_4$, $[d_{13}, b_{24}] + d_{25} \cdot a_{12}$,

$[d_{13}, c_{28}] + d_{37} \cdot a_4 + 2d_{25} \cdot b_{16}$,

$[d_{13}, c_{36}] + d_{37} \cdot a_{12} + 2d_{25} \cdot b_{24}$,

$[d_{25}, b_{16}] + d_{37} \cdot a_4$, $[d_{25}, b_{24}] + d_{37} \cdot a_{12}$,

$[d_{25}, c_{28}] + d_{49} \cdot a_4 + 2d_{37} \cdot b_{16}$,

$[d_{25}, c_{36}]+d_{49}\cdot a_{12}+2d_{37}\cdot b_{24},$

$[d_{37}, b_{16}]+d_{49}\cdot a_4,\quad [d_{37}, b_{24}]+d_{49}\cdot a_{12},$

$[d_{37}, c_{28}]+d_{49}\cdot b_{16},\quad [d_{37}, c_{36}]+d_{49}\cdot b_{24},$

$[d_{49}, b_j], [d_{49}, c_k],$

where $i=4, 12, j=16, 24, k=28, 36, l=13, 25, 37, 49, x\cdot y=\psi(x, y)$ *and* $[x, y]=x\cdot y-(-1)^{\varepsilon}y\cdot x$ *with* $\varepsilon=\deg x\cdot\deg y$.

The proof is easy.

Put $X=T(sL)/I$. Then $\bar{\theta}$ induces a map $A\to X$ which is again denoted by $\bar{\theta}$. We define a map

$$d=-\psi\circ(\bar{\theta}\otimes\bar{\theta})\circ\phi\circ\iota\colon sL\longrightarrow T(sL)$$

and extend it naturally over $T(sL)$. Then $d(I)\subset I$ and so d induces a map $X\to X$, which is again denoted by $d\colon X\to X$ (cf. [32]). Then it is easy to see $d^2=d\circ d=0$. So (X, d) is a differential coalgebra over $\boldsymbol{Z}_5$.

Since the relation

$$d\circ\bar{\theta}+\psi(\bar{\theta}\otimes\bar{\theta})\circ\phi=0$$

holds, we now can construct the twisted tensor product $\bar{X}$ with respect to $\bar{\theta}$. That is, $\bar{X}=A\otimes X$ is an A-comodule with the differential operator

$$\bar{d}=1\otimes d+(1\otimes\psi)\circ(1\otimes\bar{\theta}\otimes 1)(\phi\otimes 1).$$

Then we have

Lemma 2.3. $\bar{d}$ *is given by*

(2.1) $\bar{d}(x_3)=a_4,\quad \bar{d}(x_{11})=a_{12},\quad \bar{d}(x_{12})=d_{13},$

$\bar{d}(x_{15})=b_{16}+x_{12}\cdot a_4,\quad \bar{d}(x_{23})=b_{24}+x_{12}\cdot a_{12},$

$\bar{d}(x_{27})=c_{28}+2x_{12}\cdot b_{16}+x_{12}^2\cdot a_4,$

$\bar{d}(x_{35})=c_{36}+2x_{12}\cdot b_{24}+x_{12}^2\cdot a_{12},$

$\bar{d}(x_{12}^2)=d_{25}+2x_{12}\cdot d_{13},$

$\bar{d}(x_{12}^3)=d_{37}+3x_{12}\cdot d_{25}+3x_{12}^2\cdot d_{13},$

$\bar{d}(x_{12}^4)=d_{49}-x_{12}\cdot d_{37}+x_{12}^2\cdot d_{25}-x_{12}^3\cdot d_{13}$,

(2.3) $\bar{d}(a_4)=\bar{d}(a_{12})=0$,

$\bar{d}(b_{16})=-d_{13}\cdot a_4,\quad \bar{d}(b_{24})=-d_{13}\cdot a_{12}$

$\bar{d}(c_{28})=-2d_{13}\cdot b_{16}-d_{25}\cdot a_4,\quad \bar{d}(c_{36})=-2d_{13}\cdot b_{24}-d_{25}\cdot a_{12}$,

$\bar{d}(d_{13})=0,\quad \bar{d}(d_{25})=-2d_{13}^2,\quad \bar{d}(d_{37})=-3[d_{13}, d_{25}]$,

$\bar{d}(d_{49})=[d_{37}, d_{13}]-d_{25}^2$.

Note that

$$d(x)=\bar{d}(x) \quad \text{for any} \quad x\in sL.$$

Now define weight $\bar{X}$ by

(2.4)

A:	x_3	x_{11}	x_{15}	x_{23}	x_{27}	x_{35}	x_{12}^j
X:	a_4	a_{12}	b_{16}	b_{24}	c_{28}	c_{36}	d_{12j+1}
weight:	1	1	1	1	1	1	j

Define filtration

$$F_r(\ )=\{x;\ \text{weight } x\geq r\}.$$

Then

$$\bar{d}(F_r(\bar{X}))\subset F_r(\bar{X}).$$

Put

$$E_0(\bar{X})=\Sigma_i F_i(\bar{X})/F_{i+1}(\bar{X}).$$

Let $\bar{d}_0$: $E_0(\bar{X})\to E_0(\bar{X})$ be the induced map. Then

$E_0(\bar{X})$

$$\cong \Lambda(x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35})\otimes \boldsymbol{Z}_5[a_4, a_{12}, b_{16}, b_{24}, c_{28}, c_{36}]\otimes C(Q(x_{12})),$$

where $C(Q(x_{12}))$ is the cobar construction of $\boldsymbol{Z}_5[x_{12}]/(x_{12}^5)$. Since

$$\bar{d}_0(x_4)=a_4,\ \bar{d}_0(x_{11})=a_{12},\ \bar{d}_0(x_{15})=b_{16},$$

$$\bar{d}_0(x_{23})=b_{24},\ \bar{d}_0(x_{27})=c_{28},\ \bar{d}_0(x_{35})=c_{36},$$

$(E_0(\bar{X}), \bar{d}_0)$ is acyclic and so is $(\bar{X}, \bar{d})$.

So by definition we have

$$(2.5) \qquad H(X; d) = \mathrm{Cotor}^A(\boldsymbol{Z}_5, \boldsymbol{Z}_5).$$

Remark 2.4. $E_0(A) = \Sigma_i F_i(A)/F_{i+1}(A)$ is primitively generated (cf. [25]).

Now we prove the following:

Lemma 2.5. *$H^{41}(X; d) \cong \boldsymbol{Z}_5$ generated by* $u_{41} = \{d_{13}c_{28} + \text{others}\}$.

Proof. Consider the spectral sequence associated with the above filtration.

$$E_0 \cong \boldsymbol{Z}_5[a_4, a_{12}, b_{16}, b_{24}, c_{28}, c_{36}] \otimes T\{d_{13}, d_{25}, d_{37}, d_{49}\}$$

The element x_{41} of degree 41 is of the form

$$\alpha_1 d_{13} \cdot c_{28} + \alpha_2 d_{13} \cdot b_{24} \cdot a_4 + \alpha_3 d_{13} \cdot b_{16} \cdot f_1 + d_{13} a_4 \cdot f_2 + \alpha_4 d_{25} \cdot b_{16} + d_{25} a_4 \cdot f_3 + \alpha_5 d_{37} \cdot a_4$$

where $\alpha_i \in \boldsymbol{Z}_5$ and f_i is a polynomial of a_4 and a_{12}.

Since $d_0(x_{41}) = 0$ we have $\alpha_4 = 0, f_3 = 0$ and $\alpha_5 = 0$.

But $d_1(c_{28} \cdot f_1) = -2d_{13} \cdot b_{16} \cdot f_1$, $d_1(b_{16} \cdot f_2) = -d_{13} \cdot a_4 \cdot f_2$ and $d_1(c_{36} \cdot a_4) = -2d_{13} \cdot b_{24} a_4$. On the other hand $d_{13} \cdot c_{28} \notin \mathrm{Im}\, d_1$. So $E_1 \cong \boldsymbol{Z}_5$ generated by $d_{13} \cdot c_{28}$ for deg$=41$. Clearly $d_{13} \cdot c_{28}$ is a permanent cycle (cf. Remark 2.6) and is not a coboundary by the reason of the filtration. So $E_1 \cong E_\infty$ for deg $=41$ and we have the result. Q.E.D.

Remark 2.6. u_{41} is represented by $d_{13}c_{28} + d_{25}b_{16} + 2d_{37}a_4$.

Proof. $d(d_{13}c_{28} + d_{25}b_{16} + 2d_{37}a_4)$

$$= 2d_{13}^2 b_{16} + d_{13}d_{25}a_4 + d_{25}d_{13}a_4 - 2d_{13}^2 b_{16} - [d_{25}, d_{13}]a_4$$

$$= 0.$$

Let (A', ϕ') be the Hopf subalgebra of (A, ϕ) generated by

$$\{x_3, x_{11}, x_{12}, x_{15}, x_{23}\}.$$

Put

$$L'=\{x_3, x_{11}, x_{15}, x_{23}, x_{12}, x_{12}^2, x_{12}^3, x_{12}^4\}$$

and

$$sL'=\{a_4, a_{12}, b_{16}, b_{24}, d_{13}, d_{25}, d_{37}, d_{49}\}.$$

Similarly we can construct (X', d') and $(\bar{X}', \bar{d}')$. Then we have

$$(\bar{X}', \bar{d}') \text{ is acyclic}. \tag{2.7}$$

And so we have

$$H(X'; d')=\mathrm{Cotor}^{A'}(\boldsymbol{Z}_5, \boldsymbol{Z}_5).$$

Moreover we have

Lemma 2.7. *$H^{29}(X'; d')\cong Z_5$ generated by $u_{29}=\{b_{16}\cdot d_{13}+2a_4\cdot d_{25}\}$.*

The proof is similar.

Remark 2.8. (1) The spectral sequence used in the proof of Lemma 2.5 is essentially May's spectral sequence [23].
(2) $H^{41}(X; d)\neq 0$ and $H^{29}(X'; d')\neq 0$ are also proved by the fact that x_{39} and x_{27} are not universally transgressive which is proved in §5 without using the above results.

§3. A transgression theorem

In this section $\boldsymbol{G}$ denotes a compact connected Lie group of rank l and p a rational prime. Let $\boldsymbol{U}$ be a closed connected subgroup of $\boldsymbol{G}$ of rank l'. As is seen in §2 of [35], the fibering

$$\boldsymbol{G} \xrightarrow{\pi} \boldsymbol{G}/\boldsymbol{U} \xrightarrow{i} \boldsymbol{BU} \tag{3.1}$$

is equivalent to the principal $\boldsymbol{G}$ bundle

$$\boldsymbol{G} \xrightarrow{\pi} E \xrightarrow{i} \boldsymbol{BU} \tag{3.2}$$

where $E=\boldsymbol{EG}\times_{\boldsymbol{U}}\boldsymbol{G}$ and $\boldsymbol{BU}=\boldsymbol{EG}\times_{\boldsymbol{U}} pt$ for the total space of the universal $\boldsymbol{G}$ bundle $\boldsymbol{EG}$.

Denote by $\boldsymbol{T}^i(\boldsymbol{G}; \boldsymbol{U})$ a graded submodule of $H^i(\boldsymbol{G}; \boldsymbol{Z}_p)$ which consists of the transgressive elements with respect to (3.1) or (3.2) and $\boldsymbol{T}^*(\boldsymbol{G}; \boldsymbol{U})=\sum_{i>0} \boldsymbol{T}^i(\boldsymbol{G}; \boldsymbol{U})$.

Thus

$$(3.3)\qquad T^*(G;U)=\delta^{-1}(i^*H^{*+1}(BU,\ pt;Z_p))$$

for the coboundary operator δ: $H^*(G;Z_p)\to H^{*+1}(E,G;Z_p)$ and the homomorphism i^*: $H^{*+1}(BU,pt;Z_p)\to H^{*+1}(E,G;Z_p)$ induced by the bundle projection i: $(E,G)\to(BU,pt)$.

Obviously we have

$$(3.3)\qquad \widetilde{\mathrm{Im}\,\pi^*}\subset T^*(G;U)\quad\text{for}\quad \pi^*: H^*(G/U;Z_p)\longrightarrow H^*(G;Z_p).$$

We use the following notations

$$T^{\mathrm{even}}(G;U)=\textstyle\sum_{i>0}T^{2i}(G;U),$$

$$T^{\mathrm{odd}}(G;U)=\textstyle\sum_{i\geq 0}T^{2i+1}(G;U).$$

Let T be a maximal torus of G. Since any maximal tori are conjugate to each other, we have

(3.4) $T^*(G;T)$ is independent of the choice of T.

Following [13], $T^*(G;T)$ is denoted by T_G^*.

In [13] the following is proved (cf. Theorem 1.1 of [13]):

Theorem 3.1. *There exist elements $a_1,\dots,a_l$ of odd degrees such that*

(1) $H^*(G;Z_p)=\Delta(a_1,\dots,a_l)\otimes\mathrm{Im}\,\pi^*$ *as an* $\mathrm{Im}\,\pi^*$*-module,*

(2) $T_G^*=\langle a_1,\dots,a_l\rangle\oplus\widetilde{\mathrm{Im}\pi^*}$,

where $\Delta(a_1,\dots,a_l)$ indicates the submodule spanned by the simple monomials $a_1^{\varepsilon_1}\cdots a_l^{\varepsilon_l}$ ($\varepsilon_i=0$ or 1) which are linearly independent, and $\langle a_1,\dots,a_l\rangle$ does a submodule spanned by $\{a_1,\dots,a_l\}$.

Let U' be also a closed connected subgroup of G such that $U'\supset U$. Consider the following commutative diagram:

$$(3.5)\qquad \begin{array}{ccccc} G & \longrightarrow & G/U & \longrightarrow & BU \\ \| & & \downarrow & & \downarrow \\ G & \longrightarrow & G/U' & \longrightarrow & BU'. \end{array}$$

Then by the naturality of the transgressions we can easily get the following:

Lemma 3.2. $\boldsymbol{T}^*(\boldsymbol{G};\boldsymbol{U}) \supset \boldsymbol{T}^*(\boldsymbol{G};\boldsymbol{U}')$.

Let $\boldsymbol{U}_G^*$ be a graded submodule of $\tilde{H}^*(\boldsymbol{G};\boldsymbol{Z}_p)$ which consists of universally transgressive elements.

Since

$$\boldsymbol{U}_G^* = \boldsymbol{T}^*(\boldsymbol{G};\boldsymbol{G}), \tag{3.6}$$

we have

$$\boldsymbol{U}_G^* \subset \boldsymbol{T}^*(\boldsymbol{G};\boldsymbol{U}) \quad \text{for} \quad \text{any } \boldsymbol{U} \tag{3.7}$$

by Lemma 3.2.

Also by the commutative diagram

$$\begin{array}{ccccc} \boldsymbol{U}' & \longrightarrow & \boldsymbol{U}'/\boldsymbol{U} & \longrightarrow & \boldsymbol{BU} \\ {\scriptstyle j}\downarrow & & \downarrow & & \| \\ \boldsymbol{G} & \longrightarrow & \boldsymbol{G}/\boldsymbol{U} & \longrightarrow & \boldsymbol{BU} \end{array} \tag{3.8}$$

we have

Lemma 3.3. $j^*\boldsymbol{T}^*(\boldsymbol{G};\boldsymbol{U}) \subset \boldsymbol{T}^*(\boldsymbol{U}';\boldsymbol{U})$.

Now recall from [27]. Let $R=k[X_1,\dots,X_n]$ be a graded polynomial algebra over a commutative field k with $\deg X_i>0$ for any i, $1\le i\le n$ a finite sequence of homogeneous elements with positive degrees $\{f_1,\dots,f_s\}$ is called a regular sequence if f_i is *a* non zero divisor in $R/(f_1,\dots,f_{i-1})$ for any i, $1\le i\le s$.

Let $R'=k[Y_1,\dots,Y_n]$ be also a graded polynomial algebra with $\deg Y_i>0$ for any i, $1\le i\le n$. Let $\varphi\colon R'\to R$ be a homomorphism of graded algebra. Then the following is well known:

Lemma 3.4. *The following three conditions are equivalent*:
(1) *R is a finite R'-module under φ,*
(2) *$\{\varphi(Y_1),\dots,\varphi(Y_n)\}$ is a regular sequence,*
(3) *R is a free finite R'-module under φ.*

The proof is given in [27]. See also [16].

On the other hand let $\{g_1,\dots,g_s\}$ be a finite sequence of homogeneous elements with positive degrees in R'. Then the following is well known [27]:

Lemma 3.5. *If φ is faithfully flat and $\{\varphi(g_1),\dots,\varphi(g_s)\}$ is a regular sequence, then $\{g_1,\dots,g_s\}$ is also a regular sequence.*

Note that

(3.9) if R is a free R'-module under φ, φ is faithfully flat.

An important example of a regular sequence is given in this section.

Let $\boldsymbol{G}_1$ be a compact connected Lie group $\boldsymbol{G}_2$ its closed connected subgroup and $i\colon \boldsymbol{G}_2\to\boldsymbol{G}_1$ the inclusion. Then we have

Lemma 3.6. *If $H^*(\boldsymbol{G}_1;\boldsymbol{Z})$ and $H^*(\boldsymbol{G}_2;\boldsymbol{Z})$ are p-torsion free and* rank $\boldsymbol{G}_1=\operatorname{rank}\boldsymbol{G}_2=l$, *then $H^*(B\boldsymbol{G}_2;\boldsymbol{Z}_p)$ is a free $H^*(B\boldsymbol{G}_1;\boldsymbol{Z}_p)$-module under i^*.*

Proof. Since by Borel's theorem (cf. [5] see also [29]), $H^*(B\boldsymbol{G}_1;\boldsymbol{Z}_p)$ and $H^*(B\boldsymbol{G}_2;\boldsymbol{Z}_p)$ are both graded polynomial algebras in l-variables of positive degrees the result follows from Quillen's finiteness theorem (Corollary 2.4 of [28]) and Lemma 3.4. Q.E.D.

Remark 3.7. We can also prove Lemma 3.7 by the cohomology Serre spectral sequence for the fibering

$$\boldsymbol{G}_1/\boldsymbol{G}_2 \longrightarrow B\boldsymbol{G}_2 \longrightarrow B\boldsymbol{G}_1$$

and the fact that $H^{odd}(\boldsymbol{G}_1/\boldsymbol{G}_2;\boldsymbol{Z}_p)=H^{odd}(B\boldsymbol{G}_1;\boldsymbol{Z}_p)=0$.

Now consider the following commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & E_1 & \longrightarrow & B_1 \\ \| & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f} \\ F & \longrightarrow & E_2 & \longrightarrow & B_2 \end{array} \tag{3.10}$$

where $F\to E_1\to B_1$ and $F\to E_2\to B_2$ are fiberings F is arcwise connected and B_1, B_2 are 1-connected. Also we assume that F, E_1, E_2, B_1, B_2 have homotopy type of CW complexes of finite type.

Let $\{E_r^{**}(1), d_r^1\}$ (resp. $\{E_r^{**}(2), d_r^2\}$) be the cohomology Serre spectral sequence for the fibering $F\to E_1\to B_1$ (resp. $F\to E_2\to B_2$) with $\boldsymbol{Z}_p$ coefficient. Then we have

Theorem 3.8. *If $H^*(B_1;\boldsymbol{Z}_p)$ is a free $H^*(B_2;\boldsymbol{Z}_p)$-module of finite rank under f^*, then $E_r^{**}(1)$ is a free $E_r^{**}(2)$-module for $r\geq 2$.*

Proof. Let $\{x_1=1, x_2,\dots, x_n\}$ be a free basis of $H^*(B_1; \boldsymbol{Z}_p)$ over $H^*(B_2; \boldsymbol{Z}_p)$.

Let $\{E_r^{**}(3), d_r^3\}$ be a spectral sequence such that
(1) As a module $E_r^{**}(3)$ is generated by $1, X_2,\dots, X_n$ with bi-degree $(\deg x_i, 0)$,
(2) $d_r(X_i)=0$ for any i.

Define $\varphi: E_r^{**}(3)\to E_r^{**}(1)$ by $\varphi(X_i)=x_i\otimes 1$.

Since $d_r^1(x_i\otimes 1)=0$ for any $r\geq 2$, φ is a map of spectral sequence. Then we only need to prove

(3.11) $\varphi\otimes f^{\#}: E_r^{**}(3)\otimes E_r^{**}(2) \longrightarrow E_r^{**}(1)$ is isomorphic for $r\geq 2$.

Moreover we only need to prove

(3.12) $\varphi\otimes f^{\#}: E_2^{**}(3)\otimes E_2^{**}(2) \longrightarrow E_2^{**}(1)$ is isomorphic.

But (3.12) follows from the following (3.13):

(3.13)
$$\begin{array}{ccc} E_2^{**}(2) & \xrightarrow{\quad f^{\#} \quad} & E_2^{**}(1) \\ \wr\| & \circlearrowleft & \wr\| \\ H^*(B_2; \boldsymbol{Z}_p)\otimes H^*(F; \boldsymbol{Z}_p) & \xrightarrow{f^*\otimes 1} & H^*(B_1; \boldsymbol{Z}_p)\otimes H^*(F; \boldsymbol{Z}_p)\,. \end{array}$$

Q.E.D.

Corollary 3.9. *Under the assumption of Theorem* 3.8, $x\in\tilde{H}(F; \boldsymbol{Z}_p)$ *is transgressive with respect to* $F\to E_1\to B_1$ *if and only if with respect to* $F\to E_2\to B_2$.

Now we can prove the following:

Theorem 3.10. *Let* $\boldsymbol{G}$ *be a compact connected Lie group and* $\boldsymbol{U}$ *be its closed connected subgroup such that* $H^*(\boldsymbol{U}; \boldsymbol{Z})$ *is p-torsion free. Then*

$$\boldsymbol{T}_G^*\subset \boldsymbol{T}^*(\boldsymbol{G}; \boldsymbol{U})\,.$$

Moreover if rank $\boldsymbol{U}=$ rank $\boldsymbol{G}$, *then*

$$\boldsymbol{T}_G^*= \boldsymbol{T}^*(\boldsymbol{G}; \boldsymbol{U})\,.$$

Proof. Let $\boldsymbol{T}'$ be a maximal torus of $\boldsymbol{U}$ and $\boldsymbol{T}$ a maximal torus of $\boldsymbol{G}$

such that $T' \subset T$. Then by Lemma 3.2,

$$T_G^* = T^*(G;\ T) \subset T^*(G;\ T').$$

On the other hand since $H^*(BT'; Z_p)$ is a free $H^*(BU; Z_p)$-module by Lemma 3.6,

$$T^*(G; U) = T^*(G; T')$$

by Corollary 3.9. So we have

$$T_G^* \subset T^*(G; U).$$

If $\operatorname{rank} U = \operatorname{rank} G$, we may assume that $T' = T$. And so we have the second assertion of the theorem. Q.E.D.

Corollary 3.11. $T^*(G; U) = T^*(G; T)$.

From now on we assume that $H^*(U; Z)$ is p-torsion free and $\operatorname{rank} U = \operatorname{rank} G$.

Now recall from [13] (see also [35]):

Theorem 3.12. *Let $a_1, \dots, a_l$ be the elements in Theorem 3.1 and τ_0 be the transgression with respect to the fibering*

$$G \longrightarrow G/T \longrightarrow BT \tag{3.14}$$

where T is a maximal torus of G. Then $\{\tau_0(a_1), \dots, \tau_0(a_l)\}$ is a regular sequence.

Remark 3.13. $\tau_0(a_i)$ is not uniquely determined. But the property $\{\tau_0(a_1), \dots, \tau_0(a_l)\}$ is a regular sequence is independent of the choice of $\tau_0(a_i)$.

Now let τ be the transgression with respect to

$$G \longrightarrow G/U \longrightarrow BU. \tag{3.15}$$

Since $H^*(BT; Z_p)$ is a free $H^*(BU; Z_p)$ module by Lemma 3.6, $a_1, \dots, a_l$ are also transgressive with respect to (3.15). Moreover we have

Theorem 3.14. $\{\tau(a_1),\dots,\tau(a_l)\}$ *is a regular sequence.*

The proof is easy (cf. Lemma 3.5).

Remark 3.15. Let $\boldsymbol{G}_1$, $\boldsymbol{G}_2$ be closed connected subgroups of $\boldsymbol{G}$ such that

(1) $H^*(\boldsymbol{G}_1; \boldsymbol{Z})$ and $H^*(\boldsymbol{G}_2; \boldsymbol{Z})$ are p-torsion free,

(2) $\operatorname{rank}\boldsymbol{G}_1=\operatorname{rank}\boldsymbol{G}_2$,

(3) $\boldsymbol{G}_1\supset\boldsymbol{G}_2$.

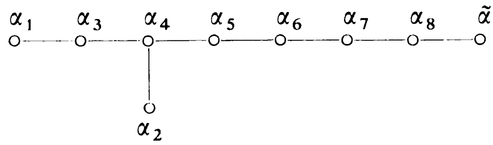
Then consider the following commutative diagram

$$\begin{array}{ccccc} \boldsymbol{G}_1/\boldsymbol{G}_2 & \xrightarrow{\iota} & \boldsymbol{G}/\boldsymbol{G}_2 & \longrightarrow & \boldsymbol{G}/\boldsymbol{G}_1 \\ & \searrow^{\iota'} & \downarrow & & \\ & & B\boldsymbol{G}_2 & & \end{array} .$$

As is well known ι'^* is surjective and so ι^* is surjective. And so $\boldsymbol{G}/\boldsymbol{G}_1$ is totally non homologous to zero mod p in $\boldsymbol{G}/\boldsymbol{G}_2$. So $H^*(\boldsymbol{G}/\boldsymbol{G}_2; \boldsymbol{Z}_p)$ is a free $H^*(\boldsymbol{G}/\boldsymbol{G}_1; \boldsymbol{Z}_p)$-module.

§4. Mod 5 invariant subalgebras of Weyl groups

Let $\boldsymbol{T}$ be a maximal torus of $\boldsymbol{E}_8$. The completed Dynkin diagram of $\boldsymbol{E}_8$ is



where α_i's $(1\leq i\leq 8)$ are the simple roots and

$$\tilde{\alpha}=2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7+2\alpha_8$$

is the highest root ($\tilde{\alpha}=\bar{\omega}_8$) (cf. [11]).

Let $\boldsymbol{U}$ be the identity component of the centralizer of the element $x_1\in\boldsymbol{T}$ such that $\alpha_1(x_1)=\frac{1}{2}$ and $\alpha_i(x_1)=0$ for $i=2, 3,\dots, 8$. Then the Weyl groups $W(\ )$ of $\boldsymbol{E}_8$ and $\boldsymbol{U}$ are generated by the following elements:

$$(4.1)\qquad \begin{cases} W(\boldsymbol{E}_8)=\langle\varphi_i;\ i=1, 2,\dots, 8\rangle \\ W(\boldsymbol{U})=\langle\varphi_i,\ \tilde{\varphi};\ i=2, 3,\dots, 8\rangle \end{cases}$$

where φ_i (resp. $\tilde{\varphi}$) denotes the reflection through the plane $\alpha_i=0$ (resp. $\tilde{\alpha}=0$) in the universal covering of $\boldsymbol{T}$. (See Borel-Siebenthal [10].)

Remark 4.1. According to Borel-Siebenthal [10] the local type of $\boldsymbol{U}$ is $\boldsymbol{D}_8$. Since the center of $\boldsymbol{Spin}(16)$ is $\boldsymbol{Z}_2\oplus\boldsymbol{Z}_2$, $\pi^*: H^*(\boldsymbol{U}: \boldsymbol{Z}_p)\to H^*(\boldsymbol{Spin}(16): \boldsymbol{Z}_p)$ is isomorphic for any odd prime p, where $\pi: \boldsymbol{Spin}(16)\to\boldsymbol{U}$ is the universal covering map.

Since the center of $\boldsymbol{E}_8$ is trivial we may consider that all roots are elements of $H^2(B\boldsymbol{T})=H^1(\boldsymbol{T})$, then the simple roots $\alpha_1, \alpha_2,\dots, \alpha_8$ form a basis of $H^2(B\boldsymbol{T})$ and $H^*(B\boldsymbol{T})=\boldsymbol{Z}[\alpha_1, \alpha_2,\dots, \alpha_8]$ (cf. [9]).

Following Bourbaki [11], we put

$$\alpha_1=\frac{1}{2}(\varepsilon_1+\varepsilon_8)-\frac{1}{2}(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7)\,,$$

$$\alpha_2=\varepsilon_1+\varepsilon_2\,,$$

$$\alpha_i=\varepsilon_{i-1}-\varepsilon_{i-2}\qquad\text{for}\quad 3\le i\le 8\,.$$

Then $\tilde{\alpha}=\varepsilon_7+\varepsilon_8$.

Put

$$t_1=-\varepsilon_1\,,$$

$$t_8=-\varepsilon_8$$

and

$$t_i=\varepsilon_i\qquad\text{for}\quad i=2, 3,\dots, 7\,.$$

Then

$$\alpha_1=-\frac{1}{2}\sum_{i=1}^{8}t_i\,,$$

$$\alpha_2=t_2-t_1\,,$$

$$\alpha_3=t_2+t_1\,,$$

$$\alpha_i=t_{i-1}-t_{i-2}\qquad\text{for}\quad 4\le i\le 8$$

and

$$\tilde{\alpha}=t_7-t_8.$$

Since $t_j=\sum_{i=1}^{8} a_{ij}\alpha_i$ for $a_{ij}\in \boldsymbol{Z}\left[\frac{1}{2}\right]$, the elements $t_1, t_2,\dots, t_8$ also form a basis of $H^2(B\boldsymbol{T};\boldsymbol{Z}_5)$.

Then the actions of $W(\boldsymbol{E}_8)$ and $W(\boldsymbol{U})$ over $H^2(B\boldsymbol{T};\boldsymbol{Z}_5)$ are given by the following table (4.2)

	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	$\tilde{\varphi}$
t_1	t_1+c_1	t_2	$-t_2$						
t_2	t_2+c_1	t_1	$-t_1$	t_3					
t_3	t_3+c_1			t_2	t_4				
t_4	t_4+c_1				t_3	t_5			
t_5	t_5+c_1					t_4	t_6		
t_6	t_6+c_1						t_5	t_7	
t_7	t_7+c_1							t_6	t_8
t_8	t_8+c_1								t_7

(4.2)

where the blanks indicate the trivial action and

$$c_1=t_1+t_2+\cdots+t_8.$$

Denote by

$$c_i=\sigma_i(t_1, t_2,\dots, t_8)$$

the i-th elementary symmetric function on the variables t_i's ($c_0=1$).

Also denote by

$$p_i=\sigma_i(t_1^2, t_2^2,\dots, t_8^2)$$

the i-th elementary symmetric functions on the variables t_i^2's ($p_0=1$). $W(\boldsymbol{E}_8)$ and $W(\boldsymbol{U})$ act on $H^*(B\boldsymbol{T};\boldsymbol{Z}_5)$ as a ring homomorphism and the invariant sub-algebras are denoted by $H^*(B\boldsymbol{T};\boldsymbol{Z}_5)^{W(\ )}$.

Then we have the following:

Lemma 4.2. $H^*(B\boldsymbol{T}^8;\boldsymbol{Z}_5)^{W(\boldsymbol{U})}=\boldsymbol{Z}_5[p_1, p_2,\dots, p_7, c_8]$.

The proof is easy.

Put $\varphi=\varphi_1$. Note that since $\boldsymbol{U}\subset\boldsymbol{E}_8$, $W(\boldsymbol{U})\subset W(\boldsymbol{E}_8)$ and so if $f\in H^*(B\boldsymbol{T};\boldsymbol{Z}_5)^{W(\boldsymbol{E}_8)}$, $f\in H^*(B\boldsymbol{T};\boldsymbol{Z}_5)^{W(\boldsymbol{U})}$. But φ does not act on $H^*(B\boldsymbol{T}^8;\boldsymbol{Z}_5)^{W(\boldsymbol{U})}$. Also note that

Lemma 4.3. *$i^*\colon H^*(B\boldsymbol{U};\boldsymbol{Z}_5)\to H^*(B\boldsymbol{T};\boldsymbol{Z}_5)$ is injective and* $\operatorname{Im} i^*=H^*(B\boldsymbol{T};\boldsymbol{Z}_5)^{W(\boldsymbol{U})}$, *where $i\colon \boldsymbol{T}\hookrightarrow\boldsymbol{U}$* (cf. §3).

The graded subalgebra $R_0^*=\boldsymbol{Z}_5[c_1, c_2, c_3,\dots, c_8]$ is invariant under the action of φ. $\varphi(c_i)$ is given by

$$\sum_{i=1}^{8}\varphi(c_i)=\prod_{i=1}^{8}(1+t_i+c_1)=\sum_{i=1}^{8}(1+c_1)^{8-i}c_i. \tag{4.3}$$

Let $I_1=(c_1^2)$, $I_2=(c_1^2, c_2)$ and $I_3=(c_1^2, c_2, c_3)$. Since $\varphi(c_1)=-c_1$ and $\varphi(c_2)=c_2$. The ideals I_1 and I_2 are φ invariant. Since $\varphi(c_3)\equiv c_3 \bmod I_2$, I_3 is also φ invariant. $\varphi(c_4)\equiv c_4$, $\varphi(c_5)\equiv c_5-c_4c_1$, $\varphi(c_6)\equiv c_6-2c_5c_1$, $\varphi(c_7)\equiv c_7+2c_6c_1$ and $\varphi(c_8)\equiv c_8+c_7c_1 \bmod I_1$.

The identity

$$\prod_{i=1}^{8}(1-t_i)\cdot\prod_{i=1}^{8}(1+t_i)=\prod_{i=1}^{8}(1-t_i^2) \tag{4.4}$$

gives the relations between c_i's and p_i's

$$\sum_{i=1}^{8}(-1)^i c_i\cdot\sum_{i=1}^{8}c_i=\sum_{i=1}^{8}(-1)^i p_i. \tag{4.5}$$

More explicitly

$$\begin{aligned}
p_1&=c_1^2-2c_2,\\
p_2&=c_2^2-2c_3c_1+2c_4,\\
p_3&=c_3^2-2c_4c_2+2c_5c_1-2c_6,\\
p_4&=c_4^2-2c_5c_3+2c_6c_2-2c_7c_1+2c_8,\\
p_5&=c_5^2-2c_6c_4+2c_7c_3-2c_8c_2,\\
p_6&=c_6^2-2c_7c_5+2c_8c_4,\\
p_7&=c_7^2-2c_8c_6.
\end{aligned} \tag{4.5'}$$

Let $R^*=H^*(BT;\,\boldsymbol{Z}_5)^{W(E_8)}$ and $i\colon R^*\to R_0^*$ be the inclusion. Note that

Lemma 4.4. *If* $f\in R^*$, then $f\in H^*(BT;\,\boldsymbol{Z}_5)^{W(E_8)}$ *if and only if* $\varphi(f)=f$.

Now we prove the following:

Lemma 4.5. *Let* $f_{28}\in R^{28}$ *be a homogeneous element of degree* 28. *If* $f_{28}=p_7+$(other terms), then $\varphi(f_{28})\neq f_{28}$.

Proof. Since $\varphi(c_4)\equiv c_4 \bmod I_3=(c_1^2,\,c_2,\,c_3)$,

and

$$\varphi(c_6-c_5c_1)\equiv c_6-2c_5c_1+c_5c_1\equiv c_6-c_5c_1 \bmod I_3\,,$$

the ideal

$$I=(c_1^2,\,c_2,\,c_3,\,c_4,\,c_6-c_5c_1)$$

is invariant under the action of φ. Put $R'=R_0^*/I$ and $\pi'\colon R_0^*\to R'$ (the projection). Then

$$R'=\boldsymbol{Z}_5[c_5,\,c_7,\,c_8]\otimes\Lambda(c_1)$$

and

(4.6)
$$\pi'(p_1)=\pi'(p_2)=0\,,$$
$$\pi'(p_3)=2(c_5c_1-c_6)=0\,,$$
$$\pi'(p_7)=c_7^2-2c_8c_5c_1\,.$$

Since I is φ invariant φ induces a ring homomorphism

$$\varphi'\colon R'\longrightarrow R'\,.$$

Note that

(4.7)
$$\varphi'(c_1)=-c_1\,,$$
$$\varphi'(c_5)=c_5\,,$$
$$\varphi'(c_7)=c_7\,,$$
$$\varphi'(c_8)=c_8-c_7c_1\,.$$

Then f_{28} is of the form

$$\alpha p_7 + f'_{28} \quad \text{for} \quad \alpha \in \mathbf{Z}_5 \quad \text{and} \quad f'_{28} \in \operatorname{Ker} \pi' .$$

Put

$$\bar{f}_{28} = \pi'(f_{28}) = \pi'(\alpha p_7) = \alpha c_7^2 - 2\alpha c_8 c_5 c_1 .$$

On the other hand

$$\varphi'(\bar{f}_{28}) = \alpha c_7^2 + 2\alpha c_8 c_5 c_1 .$$

If $\varphi(f_{28}) = f_{28}$, then $\varphi'(\bar{f}_{28}) = \bar{f}_{28}$ and so $\alpha = 0$. Q.E.D.

Also we have

Lemma 4.6. *Let* $f_{40} \in R^{40}$. *If* $f_{40} = p_5^2 + (\text{other terms})$, then $\varphi(f_{40}) \neq f_{40}$.

Proof. Since

$$\varphi(c_4^2) \equiv c_4^2 \bmod I_3 ,$$

$$\varphi(c_6 - c_5 c_1) \equiv c_6 - c_5 c_1 \bmod I_3 ,$$

$$\varphi(c_7 + c_6 c_1) \equiv c_7 + 2c_6 c_1 - c_6 c_1 = c_7 + c_6 c_1 \bmod I_3 ,$$

and

$$\varphi(c_8 - 2c_7 c_1) \equiv c_8 + c_7 c_1 + 2c_7 c_1 = c_8 - 2c_7 c_1 \bmod I_3 ,$$

the ideal

$$J = (c_1^2, c_2, c_3, c_4^2, c_6 - c_5 c_1, c_7 + c_6 c_1, c_8 - 2c_7 c_1)$$

$$= (c_1^2, c_2, c_3, c_4^2, c_6 - c_5 c_1, c_7, c_8)$$

is invariant under the action of φ. Put $R'' = R_0^*/J$ and $\pi'': R_0^* \to R''$ (the projection). Then

$$R'' = \mathbf{Z}_5[c_5] \otimes \Lambda(c_1, c_4)$$

and

(4.8)
$$\pi''(p_1)=0\,,$$
$$\pi''(p_2)=2c_4\,,$$
$$\pi''(p_3)=2(c_5c_1-c_6)=0\,,$$
$$\pi''(p_4)=0\,,$$
$$\pi''(p_5)=c_5^2-2c_5c_4c_1\,,$$
$$\pi''(p_6)=0\,,$$
$$\pi''(p_7)=0\,.$$

Since J is φ invariant φ induces a ring homomorphism
$$\varphi'': R'' \longrightarrow R''\,.$$

Note that

(4.9)
$$\varphi''(c_1)=-c_1\,,$$
$$\varphi''(c_4)=c_4\,,$$
$$\varphi''(c_5)=c_5-c_4c_1\,.$$

Then f_{40} is of the form
$$\beta p_5^2+f'_{40} \quad \text{for} \quad \beta\in \boldsymbol{Z}_5 \quad \text{and} \quad f'_{40}\in \operatorname{Ker}\pi''$$

Put $\bar{f}_{40}=\pi''(f_{40})=\beta c_5^4-4\beta c_5^3c_4c_1$.

On the other hand
$$\varphi''(\bar{f}_{40})=\beta c_5^4-4\beta c_5^3c_4c_1+4\beta c_5c_4c_1=\beta c_5^4\,.$$

If $\varphi(f_{40})=f_{40}$, then $\varphi''(\bar{f}_{40})=\bar{f}_{40}$ and so $\beta=0$. Q.E.D.

Remark 4.7. The homogeneous space $\boldsymbol{E}_8/\boldsymbol{U}$ is an irreducible Riemannian symmetric space denoted by ***EVIII***. The subgroup $\boldsymbol{U}$ is ***Semi-Spin***(16) (cf. [16]).

§5. $\boldsymbol{H}^*(\boldsymbol{E}_8;\boldsymbol{Z}_5)$

The purpose of this section is to determine the Hopf algebra structure and the cohomology operations of $H^*(\boldsymbol{E}_8; \boldsymbol{Z}_5)$.

First recall from [7] (see also [35] and §3).

Theorem 5.1. *There exist* $x_3, x_{11}, x_{12}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47} \in H^*(\boldsymbol{E}_8; \boldsymbol{Z}_5)$ *such that*

(1) *As an algebra*

$$H^*(\boldsymbol{E}_8; \boldsymbol{Z}_5) \cong \boldsymbol{Z}_5[x_{12}]/(x_{12}^5) \otimes \Lambda(x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47})$$

where $\deg x_i = i$,

(2) $\boldsymbol{T}^*_{\boldsymbol{E}_8} = \widetilde{\boldsymbol{Z}_5[x_{12}]}/(x_{12}^5) \oplus \langle x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47} \rangle$

and $\operatorname{Im} \pi^* = \boldsymbol{Z}_5[x_{12}]/(x_{12}^5)$, *where* $\pi: \boldsymbol{E}_8 \to \boldsymbol{E}_8/\boldsymbol{T}^8$,

(3) $x_{11} = \mathscr{P}^1 x_3$ *and* $x_{12} = \beta x_{11}$.

The following is easily proved

Lemma 5.2. *As an algebra*

$$H^*(B\boldsymbol{E}_8; \boldsymbol{Z}_5) = \boldsymbol{Z}_5[y_4, y_{12}, y_{13}] \qquad \textit{for} \quad * \leq 14,$$

where $\sigma(y_{j+1}) = x_j$ *for* $j = 3, 11, 12$ *under the cohomology suspension* σ. *Moreover* $y_{12} = \mathscr{P}^1 y_4$ *and* $y_{13} = \beta y_{12}$.

The following is also easily proved:

Lemma 5.3. $\bar{\phi}(x_{15}) \neq 0$ *if and only if* $y_4 \cdot y_{13} = 0$.

Let $\mu: \boldsymbol{E}_8 \to \boldsymbol{U}(240)$ be the representation defined in [26]. We use the notation of [26]. Since the coefficient of $(c_2')^2$ in $\rho^* \mu^*(c_4)$ is $9 \cdot 2^3 - 45 - 3^3 - ((-1)^2 - 2^2 + 9) \cdot \binom{4}{2} = -36 \not\equiv 0 \bmod 5$, $\mu^*(c_4) \not\equiv 0 \bmod 5$. Since $H^8(B\boldsymbol{E}_8; \boldsymbol{Z}_5) \cong \boldsymbol{Z}_5$ generated by y_4^2 and so

(5.1) $\mu^*(c_4) = \alpha y_4^2$ for $\alpha \neq 0$ (c_4 is the mod 5 reduction of c_4).

On the other hand $\beta \mathscr{P}^1 c_4 = 0$ and so

$$\beta \mathscr{P}^1 y_4^2 = 0. \tag{5.2}$$

But $\beta \mathscr{P}^1 y_4^2 = \beta y_4 y_{12} = y_4 y_{13}$.

So we have

(5.3) $$y_4 y_{13} = 0 .$$

So we have

Lemma 5.4. $\bar{\phi}(x_{15}) \neq 0$.

Now we use the following (cf. [13]):

Theorem 5.5. *For* $x \in \tilde{H}^*(\boldsymbol{G}; \boldsymbol{Z}_p)$ *($\boldsymbol{G}$; a compact connected Lie group), the following three conditions are equivalent*:
(1) $x \in \boldsymbol{T}_{\boldsymbol{G}}^*$,
(2) $\phi(x) - x \otimes 1 \in \operatorname{Im} \pi^* \otimes H^*(\boldsymbol{G}; \boldsymbol{Z}_p)$,
(3) $\phi(x) - x \otimes 1 \in \operatorname{Im} \pi^* \otimes \boldsymbol{T}_{\boldsymbol{G}}^*$.

So we may assume that

(5.4) $$\bar{\phi}(x_{15}) = x_{12} \otimes x_3 .$$

Lemma 5.6. $\mathscr{P}^1 x_i = 0$ *for* $i = 11, 12$.

Proof. $\mathscr{P}^1 x_i \in \boldsymbol{T}_{\boldsymbol{E}_8}^{i+8} = 0$ for $i = 11, 12$. Q.E.D.

So $\bar{\phi}(\mathscr{P}^1 x_{15}) = \mathscr{P}^1(x_{12} \otimes x_3) = x_{12} \otimes x_{11}$ and so $\mathscr{P}^1 x_{15} \neq 0$. But $\mathscr{P}^1 x_{15} \in \boldsymbol{T}_{\boldsymbol{E}_8}^{23}$ and so

(5.5) $$\mathscr{P}^1 x_{15} = x_{23} \quad \text{and} \quad \bar{\phi}(x_{23}) = x_{12} \otimes x_{11} .$$

Let $\boldsymbol{U}$ be the closed subgroup of $\boldsymbol{E}_8$ defined in §4. Since $H^*(\boldsymbol{U}; \boldsymbol{Z})$ is 5-torsion free (cf. Remark 4.1) and $\operatorname{rank} \boldsymbol{U} = \operatorname{rank} \boldsymbol{E}_8$, we can apply Theorem 3.10 and Theorem 3.14. Let τ be the transgression with respect to the fibering

(5.6) $$\boldsymbol{E}_8 \longrightarrow \boldsymbol{E}_8 / \boldsymbol{U} \longrightarrow B\boldsymbol{U} .$$

Note that

(5.7) $$\{\tau(x_3), \tau(x_{11}), \tau(x_{15}), \tau(x_{23}), \tau(x_{27}), \tau(x_{35}), \tau(x_{39}), \tau(x_{47})\}$$

is a regular sequence in

(5.8) $$H^*(B\boldsymbol{U}; \boldsymbol{Z}_5) \cong \boldsymbol{Z}_5[p_1, p_2, \ldots, p_7, c_8] ,$$

where $\deg p_i = 4i$ and $\deg c_8 = 16$ (p_i, c_8 are in §4).

Then we have

Lemma 5.7. (1) $\tau(x_{27}) \equiv p_7 + (\text{other terms})$

and (2) $\tau(x_{39}) \equiv p_5^2 + (\text{other terms})$ *up to non-zero constant.*

Proof of (1). If $\tau(x_{27})$ does not contain the term p_7, $\tau(x_{27}) \in J = (p_1, p_2, p_3, p_4, p_5, p_6, c_8)$. On the other hand $\tau(x_i) \in J$, for $i = 3, 11, 15, 23, 35, 39, 47$, by the dimensional reason. This contradicts the fact that (5.7) is a regular sequence. Similarly we can prove (2). Q. E. D.

Now consider the following commutative diagram:

(5.9)
$$\begin{array}{ccccc} \boldsymbol{E}_8 & \longrightarrow & \boldsymbol{EE}_8 & \longrightarrow & \boldsymbol{BE}_8 \\ \| & & \uparrow & & \uparrow j \\ \boldsymbol{E}_8 & \longrightarrow & \boldsymbol{E}_8/\boldsymbol{U} & \longrightarrow & \boldsymbol{BU} \\ \| & & \uparrow & & \uparrow i \\ \boldsymbol{E}_8 & \longrightarrow & \boldsymbol{E}_8/\boldsymbol{T}^8 & \longrightarrow & \boldsymbol{BT}^8. \end{array}$$

Note that

(5.10) $\operatorname{Im}\{i^*j^*: H^*(\boldsymbol{BE}_8; \boldsymbol{Z}_5) \longrightarrow H^*(\boldsymbol{BT}^8; \boldsymbol{Z}_5)\} \subset H^*(\boldsymbol{BT}^8; \boldsymbol{Z}_5)^{W(\boldsymbol{E}_8)}$

(cf. [33]).

If x_{27} is universally transgressive, by the naturality of the transgressions there exists an element $x \in H^*(\boldsymbol{BT}; \boldsymbol{Z}_5)^{W(\boldsymbol{U})}$ such that

(5.11) $x = p_7 + (\text{other terms})$ and $x \in H^*(\boldsymbol{BT}; \boldsymbol{Z}_5)^{W(\boldsymbol{E}_8)}$.

But (5.11) contradicts Lemma 4.5 and so we have

Lemma 5.8. x_{27} *is not universally transgressive.*

For a compact connected Lie group $\boldsymbol{G}$, Milnor-Rothenberg-Steenrod constructed a spectral sequence of algebra $\{E_r^*, d_r\}_{r \geq 1}$ such that

(1) E_1^* is naturally isomorphic to the cobar construction of $H^*(\boldsymbol{G}; \boldsymbol{Z}_p)$,

(2) $E_2^* \cong \operatorname{Cotor}^{H^*(\boldsymbol{G}; \boldsymbol{Z}_p)}(\boldsymbol{Z}_p, \boldsymbol{Z}_p)$,

(3) $E_\infty^* = Gr(H^*(\boldsymbol{BG}; \boldsymbol{Z}_p))$.

(For details see [29]. See also [11] and [24].)

If $\bar{\phi}(x_{27}) = 0$ then

(5.12) $\operatorname{Cotor}^{H^*(\boldsymbol{E}_8; \boldsymbol{Z}_5)}(\boldsymbol{Z}_5, \boldsymbol{Z}_5) \cong \operatorname{Cotor}^{A'}(\boldsymbol{Z}_5, \boldsymbol{Z}_5) \otimes \boldsymbol{Z}_5[y_{28}]$ for $\deg \leq 35$.

But since $u_{29} \in E_2^2$ (cf. Lemma 2.7) and $y_{28} \in E_2^1$, y_{28} is a permanent cycle. So applying the cohomology suspension (cf. Remark 5.17) we have

(5.13) If $\bar{\phi}(x_{27})=0$, then x_{27} is universally transgressive.

So we have

(5.14) $$\bar{\phi}(x_{27})\neq 0.$$

By Theorem 5.5, $\bar{\phi}(x_{27})$ is of the form

(5.15) $$\alpha_1 x_{12}^2\otimes x_3+\alpha_2 x_{12}\otimes x_{15} \quad \text{for} \quad \alpha_1, \alpha_2\in \mathbf{Z}_5.$$

Using the coassociativity we have $\alpha_2=2\alpha_1$ and so we have

Lemma 5.9. $\bar{\phi}(x_{27})=2x_{12}\otimes x_{15}+x_{12}^2\otimes x_3$.

Since $\mathscr{P}^1 x_{23}\in \boldsymbol{T}_{E_8}^{31}=0$, $\mathscr{P}^1 x_{23}=0$. $\bar{\phi}(\mathscr{P}^1 x_{27})=\mathscr{P}^1(2x_{12}\otimes x_{15}+x_{12}^2\otimes x_3)=2x_{12}\otimes x_{23}+x_{12}^2\otimes x_{11}\neq 0$. So we have

(5.16) $$\mathscr{P}^1 x_{27}=x_{35} \quad \text{and} \quad \bar{\phi}(x_{35})=2x_{12}\otimes x_{23}+x_{12}^2\otimes x_{11}.$$

Corollary 5.10. *The subalgebra generated by* $\{x_3, x_{11}, x_{12}, x_{15}, x_{23}, x_{27}, x_{35}\}$ *is isomorphic to* (A, ϕ) *as a Hopf algebra.*

Then by the argument similar to the above we have

Lemma 5.11. $\bar{\phi}(x_{39})\neq 0$.

Moreover by Theorem 5.6, $\bar{\phi}(x_{39})$ is of the form

(5.17) $$\alpha_1 x_{12}^3\otimes x_3+\alpha_2 x_{12}^2\otimes x_{15}+\alpha_3 x_{12}\otimes x_{27} \quad \text{for} \quad \alpha_1, \alpha_2, \alpha_3\in \mathbf{Z}_5.$$

Using the coassociativity we have $\alpha_2=3\alpha_1$ and $\alpha_3=3\alpha_1$. So we have

Lemma 5.12. $\bar{\phi}(x_{39})=x_{12}^3\otimes x_3+3x_{12}^2\otimes x_{15}+3x_{12}\otimes x_{27}$.

Since $\mathscr{P}^1 x_{35}\in T_{E_8}^{43}=0$, $\mathscr{P}^1 x_{35}=0$. $\bar{\phi}(\mathscr{P}^1 x_{35})=\mathscr{P}^1(x_{12}^3\otimes x_3+3x_{12}^2\otimes x_{15}+3x_{12}\otimes x_{27})=x_{12}^3\otimes x_{11}+3x_{12}^2\otimes x_{23}+3x_{12}\otimes x_{35}\neq 0$. So we have

(5.18) $$\mathscr{P}^1 x_{39}=x_{47} \quad \text{and} \quad \bar{\phi}(x_{47})=3x_{12}\otimes x_{35}+3x_{12}^2\otimes x_{23}+x_{12}^3\otimes x_{11}.$$

Now we compute β-operation.

(5.19) $\bar\phi(\beta x_{15})=0$ and so $\beta x_{15}=0$,

$\bar\phi(\beta x_{23})=x_{12}\otimes x_{12}$ and so $\beta x_{23}=\frac{1}{2}x_{12}^2=3x_{12}^2$,

$\bar\phi(\beta x_{27})=0$ and so $\beta x_{27}=0$,

$\bar\phi(\beta x_{35})=x_{12}\otimes{}^2_{12}+x^2_{12}\otimes{}_{12}$ and so $\beta x_{35}=\frac{1}{3}x^3_{12}=2x^3_{12}$,

$\bar\phi(\beta x_{39})=0$ and so $\beta x_{39}=0$,

and

$$\bar\phi(\beta x_{47})=x^3_{12}\otimes x_{12}-x^2_{12}\otimes x^2_{12}+x_{12}\otimes x^3_{12}$$

and so $\beta x_{47}=\frac{1}{4}x^4_{12}=-x^4_{12}$.

Let $\boldsymbol{P}^i_{\boldsymbol{E}_8}=\{x\in H^i(\boldsymbol{E}_8;\boldsymbol{Z}_5);\ \bar\phi(x)=0\}$. Note that $\boldsymbol{P}^i_{\boldsymbol{E}_8}\subset\boldsymbol{T}^i_{\boldsymbol{E}_8}$ by [13]. Easily we have

Lemma 5.13. $\boldsymbol{P}^i_{\boldsymbol{E}_8}\neq 0$ *if and only if* $i=3, 11, 12$.

So we have $\mathscr{P}^i x_{11}=\mathscr{P}^i x_{12}=0$ for $i>0$ and $\mathscr{P}^i x_3=0$ for $i>1$.

Lemma 5.14. $\mathscr{P}^i x_j=0$ *for* $i>1$.

Proof. $\bar\phi(\mathscr{P}^i x_{15})=\mathscr{P}^i(x_{12}\otimes x_3)=0$ and so $\mathscr{P}^i x_{15}=0$ for $i>1$. $\bar\phi(\mathscr{P}^i x_{23})=\mathscr{P}^i(x_{12}\otimes x_{11})=0$ and so $\mathscr{P}^i x_{23}=0$ for $i>1$. Similarly we have $\bar\phi(\mathscr{P}^i x_j)=0$, $i>1$ and so $\mathscr{P}^i x_j=0$ for $i>1$.

Q. E. D.

Thus the following Theorem 5.15 is proved:

Theorem 5.15. *In* Theorem 5.1.

$\bar\phi(x_{15})=x_{12}\otimes x_3$, $\bar\phi(x_{23})=x_{12}\otimes x_{11}$,

$\bar\phi(x_{27})=2x_{12}\otimes x_{15}+x^2_{12}\otimes x_3$,

$\bar\phi(x_{35})=2x_{12}\otimes x_{23}+x^2_{12}\otimes x_{11}$,

$\bar\phi(x_{39})=3x_{12}\otimes x_{27}+3x^2_{12}\otimes x_{15}+x^3_{12}\otimes x_3$,

$$\bar{\phi}(x_{47})=3x_{12}\otimes x_{35}+3x_{12}^2\otimes x_{23}+x_{12}^3\otimes x_{11},$$

$$\bar{\phi}(x_3)=\bar{\phi}(x_{11})=\bar{\phi}(x_{12})=0,$$

$$\beta x_{11}=x_{12},\quad \beta x_{23}=\frac{1}{2}x_{12}^2,\quad \beta x_{35}=\frac{1}{3}x_{12}^3,\quad \beta x_{47}=\frac{1}{4}x_{12}^4,$$

$$\beta x_i=0 \quad \textit{for} \quad i=3,\ 12,\ 15,\ 27,\ 39,$$

$$\mathscr{P}^1 x_3=x_{11},\quad \mathscr{P}^1 x_{15}=x_{23},\quad \mathscr{P}^1 x_{27}=x_{35},\quad \mathscr{P}^1 x_{39}=x_{47},$$

$$\mathscr{P}^1 x_i=0 \quad \textit{for} \quad i=11,\ 12,\ 23,\ 35,\ 47$$

and

$$\mathscr{P}^j x_i=0 \quad \textit{for any} \quad j>1.$$

Let ε: $\boldsymbol{E}_8\to K(\boldsymbol{Z}, 3)$ be the generator of $H^3(\boldsymbol{E}_8;\ \boldsymbol{Z})\cong\boldsymbol{Z}$. Consider the fibering

(5.20) $$K(\boldsymbol{Z},2)\longrightarrow \tilde{\boldsymbol{E}}_8 \xrightarrow{q} \boldsymbol{E}_8,$$

which is classified by ε. $\tilde{\boldsymbol{E}}_8$ is called the 3-connective fibre space over $\boldsymbol{E}_8$.

Also consider the fibering

(5.21) $$K(\boldsymbol{Z},3)\longrightarrow B\tilde{\boldsymbol{E}}_8 \xrightarrow{q'} B\boldsymbol{E}_8,$$

which is classified by ε': $B\boldsymbol{E}_8\to K(\boldsymbol{Z}, 4)$ corresponding to the generator of $H^4(B\boldsymbol{E}_8;\ \boldsymbol{Z})\cong\boldsymbol{Z}$. Note that $B\tilde{\boldsymbol{E}}_8$ is the classifying space of $\tilde{\boldsymbol{E}}_8\simeq\Omega B\tilde{\boldsymbol{E}}_8$.

Making use of Kudo's transgression theorem to the fibering (5.20) we have

(5.22) $$H^*(\tilde{\boldsymbol{E}}_8;\ \boldsymbol{Z}_5)\cong\boldsymbol{Z}_5[y_{50}]\otimes\Lambda(\tilde{x}_{15},\ \tilde{x}_{23},\ \tilde{x}_{27},\ \tilde{x}_{35},\ \tilde{x}_{39},\ \tilde{x}_{47},\ y_{51},\ y_{59}),$$

where $\deg y_i=i$ and $\tilde{x}_i=q^*(x_i)$.

Also easily we have

(5.23) $$H^*(B\boldsymbol{E}_8;\ \tilde{\boldsymbol{Z}}_5)\cong\boldsymbol{Z}_5[y_{16},\ y_{24},\ y_{28},\ y_{36},\ y_{40},\ y_{48}] \quad \text{for} \quad *\leq 50.$$

Consider the Serre spectral sequence for the fibering

$$B\tilde{\boldsymbol{E}}_8\longrightarrow B\boldsymbol{E}_8 \xrightarrow{\varepsilon'} K(\boldsymbol{Z},4).$$

(5.24) $$E_2^* \cong H^*(K(Z, 4); Z_5) \otimes H^*(B\tilde{E}_8; Z_5)$$

$$\cong Z_5[u_4, u_{12}] \otimes \Lambda(u_{13}) \otimes H^*(B\tilde{E}_8; Z_5) \qquad * \leq 50,$$

where $\deg u_i = i$, $u_{12} = \mathscr{P}^1 u_4$ and $u_{13} = \beta u_4$.

Clearly y_{16} is transgressive with $\tau(y_{16}) = u_4 \cdot u_{13}$. Applying $\mathscr{P}^1$ we have $y_{24} = \mathscr{P}^1 y_{16}$ and $\tau(y_{24}) = u_{12} \cdot u_{13}$ and so $\tilde{x}_{23} = \mathscr{P}^1 \tilde{x}_{15}$.

Moreover we have

(5.25) y_{28} is a permanent cycle or $d_{13}(1 \otimes y_{28}) = u_{13} \otimes y_{16}$.

By (5.25) we can also get

(5.26) y_{28} is a permanent cycle or $\bar{\Phi}(x_{27}) \neq 0$.

Similarly we have

(5.27) y_{40} is a permanent cycle or $d_{13}(1 \otimes y_{40}) = u_{13} \otimes y_{28}$.

Moreover $u_4 \otimes y_{24}$ is a permanent cycle and corresponds to the Massey product $\langle y_4, y_{13}, y_{12} \rangle$.

Remark 5.16. Using the Milnor-Rothenberg-Steenrod spectral sequence, the cohomology suspension

$$\sigma: H^{*+1}(BG; Z_p) \longrightarrow \tilde{H}^*(G; Z_p)$$

is represented by the following composition

$$H^{*+1}(BG; Z_p) \twoheadrightarrow E_\infty^1 \hookrightarrow E_2^1 \hookrightarrow E_1^1 \cong \tilde{H}^{*+1}(\Sigma G; Z_p) \cong \tilde{H}^*(G; Z_p).$$

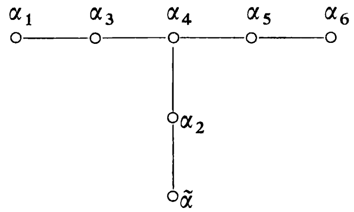
On the other hand

$$\operatorname{Im}\{E_2^1 \hookrightarrow E_1^1 \cong \tilde{H}^*(G; Z_p)\} = P_G^* \qquad \text{(cf. Browder [12]).}$$

§6. $H^*(AdE_6; Z_3)$

Let E_6 be the compact 1-connected simple Lie group of type E_6. As is well known the center of E_6 is a cyclic group of order 3 and denoted by Z_3. The quotient of E_6 by the center, E_6/Z_3 is denoted by AdE_6 and the covering projection $E_6 \to AdE_6$ is denoted by ρ.

Let T^6 be a maximal torus of E_6. The completed Dynkin diagram is



where α_i $(1 \leq i \leq 6)$ are the simple roots and

$$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$$

is the highest root.

Let $\tilde{V}$ be the identity component of the centralizer of the element $x \in T^6$ such that $\alpha_2(x) = \frac{1}{2}$ and $\alpha_i(x) = 0$ for $i \neq 2$. According to Borel-Siebenthal [10] the local type of $\tilde{V}$ is $A_5 \times A_1$. Moreover $\tilde{V}$ is $SU(6)\cdot SU(2)$ for $SU(6) \cap SU(2) = Z_2$ (cf. [17]). Note that since $\tilde{V} \supset T^6$ and so $\tilde{V} \supset Z_3$.

Remark 6.1. The homogeneous space $E_6/\tilde{V}$ is an irreducible Riemannian symmetric space and denoted by ***EII***.

Since $\pi' : SU(6) \times SU(2) \to \tilde{V}$ is a double covering,

(6.1) $\pi'^* : H^*(\tilde{V}; Z_p) \longrightarrow H^*(SU(6) \times SU(2); Z_p)$ is an isomorphism for any odd prime p. In particular $H^*(\tilde{V}; Z)$ is p-torsion free for any odd prime p.

Recall from [3]

Theorem 6.2. (1) $H^*(E_6; Z_3) \cong Z_3[\tilde{x}_8]/(\tilde{x}_8^3) \otimes \Lambda(\tilde{x}_3, \tilde{x}_7, \tilde{x}_9, \tilde{x}_{11}, \tilde{x}_{15}, \tilde{x}_{17})$, *where* $\tilde{x}_i \in T^i_{E_6}$, $\tilde{x}_7 = \mathscr{P}^1 \tilde{x}_3$, $\tilde{x}_8 = \beta \tilde{x}_7$ *and* $\tilde{x}_{15} = \mathscr{P}^1 \tilde{x}_{11}$,
(2) $H^*(AdE_6; Z_3) \cong Z_3[x_2, x_8]/(x_2^9, x_8^3) \otimes \Lambda(x_1, x_3, x_7, x_9, x_{11}, x_{15})$,
where $x_i \in T^i_{AdE_6}$, $x_2 = \beta x_1$, $x_7 = \mathscr{P}^1 x_3$, $x_8 = \beta x_7$ *and* $x_{15} = \mathscr{P}^1 x_{11}$,
(3) $\operatorname{Ker} \rho^*$ *is the ideal generated by* x_1 *and* x_2.

Since by (2.2) of [13]

$$\rho^*(T^*_{AdE_6}) \subset T^*_{E_6}, \tag{6.2}$$

and

(6.3) $$T^*_{E_6}=\widetilde{Z_3[\tilde{x}_8]/(\tilde{x}_8^3)}\oplus<\tilde{x}_3, \tilde{x}_7, \tilde{x}_9, \tilde{x}_{11}, \tilde{x}_{15}, \tilde{x}_{17}> \quad \text{and}$$

$$T^*_{AdE_6}=\widetilde{Z_3[x_2, x_8]/(x_2^9, x_8^3)}\oplus<x_1, x_3, x_7, x_9, x_{11}, x_{15}>,$$

we have

Lemma 6.3. $\rho^*(x_i)=\tilde{x}_i$ *for* $i=3, 7, 8, 9, 11, 15$.

Put $V=\tilde{V}/Z_3$ and consider the following commutative diagram:

(6.4) $$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{j}} & E_6 \\ \downarrow{\rho} & & \downarrow{\rho} \\ V & \xrightarrow{j} & AdE_6, \end{array}$$

where $\rho: \tilde{V}\to V$ is the restriction of $\rho: E_6\to AdE_6$.

Note that the double covering

(6.5) $$\pi'': V \longrightarrow PU(6)\times SO(3)$$

induces an isomorphism

(6.6) $$(\pi'')^*: H^*(PU(6)\times SO(3); Z_3) \longrightarrow H^*(V; Z_3).$$

Also the following Lemma is well known:

Lemma 6.4. (1) $H^*(\tilde{V}; Z_3)\cong\Lambda(\tilde{y}_3, \tilde{y}'_3, \tilde{y}_5, \tilde{y}_7, \tilde{y}_9, \tilde{y}_{11})$,
where $\tilde{y}_i\in T^i_{\tilde{V}}$, $\tilde{y}'_3\in T^3_{\tilde{V}}$ *and* $U^*_{\tilde{V}}=T^*_{\tilde{V}}=<\tilde{y}_3, \tilde{y}'_3, \tilde{y}_5, \tilde{y}_7, \tilde{y}_9, \tilde{y}_{11}>$,
(2) $H^*(V; Z_3)\cong Z_3[y_2]/(y_2^3)\otimes\Lambda(y_1, y_3, y'_3, y_7, y_9, y_{11})$,
where $y_i\in T^i_V$, $y'_3\in T^3_V$ *and* $T^*_V=\widetilde{Z_3[y_2]/(y_2^3)}\oplus<y_1, y_3, y'_3, y_7, y_9, y_{11}>$
(3) $\tilde{y}_7=\mathscr{P}^1\tilde{y}_3$, $y_2=\beta y_1$ *and* $y_7=\mathscr{P}^1 y_3$.

By the argument similar to the above we have

(6.6) $$\rho^*(y_i)=\tilde{y}_i \quad \text{for} \quad i=3, 7, 9, 11 \quad \text{and} \quad \rho^*(y'_3)=\tilde{y}'_3.$$

Now we apply Theorem 3.10 and Theorem 3.14 to the fibering

(6.7) $$E_6 \longrightarrow E_6/\tilde{V} \longrightarrow B\tilde{V}.$$

Note that

(6.8) $$H^*(B\tilde{V};\, \boldsymbol{Z}_3)=\boldsymbol{Z}_3[u_4,\, u_6,\, u_8,\, u_{10},\, u_{12},\, u'_4],$$

where $\sigma(u_{j+1})=\tilde{y}_j$ and $\sigma(u'_4)=\tilde{y}'_3$ under the cohomology suspension σ.

By Theorem 3.14

(6.9) $\{\tau(\tilde{x}_3),\, \tau(\tilde{x}_7),\, \tau(\tilde{x}_9),\, \tau(\tilde{x}_{11}),\, \tau(\tilde{x}_{15}),\, \tau(\tilde{x}_{17})\}$ is a regular sequence.

$\tau(\tilde{x}_3)=\alpha u_4+\beta u'_4$ for $\alpha,\, \beta\in \boldsymbol{Z}_3$. If $\alpha=0$, then $\tau(\tilde{x}_3)=\beta u'_4$. Applying $\mathscr{P}^1$ we have $\tau(x_7)=\mathscr{P}^1\beta u'_4=\beta\mathscr{P}^1 u'_4\in(u'_4)$ the ideal generated by u'_4. But this contradicts (6.9). So $\alpha\neq 0$.

Next we show

Lemma 6.5. $\tau(\tilde{x}_9)\equiv u_{10}$ *and* $\tau(\tilde{x}_{11})\equiv u_{12}$ mod *decomposables.*

Proof. If $\tau(\tilde{x}_9)$ is decomposable, $\tau(\tilde{x}_9)\in(u'_4,\, u_4,\, u_6,\, u_8)$. On the other hand $\tau(\tilde{x}_i)\in(u'_4,\, u_4,\, u_6,\, u_8,\, u_{12})$ for $i=3,\, 7,\, 11,\, 15,\, 17$. This contradicts (6.9).

$$\tau(\tilde{x}_{11})\equiv u_{12} \text{ mod } \textit{decomposables}$$

is proved similarly. Q. E. D.

By the naturalities of the transgressions we have

Corollary 6.6. $j^*(\tilde{x}_3)=\tilde{y}_3+\alpha'\tilde{y}'_3$ *for* $\alpha'\in\boldsymbol{Z}_3$

and $j^*(\tilde{x}_i)=\tilde{y}_i$ *for* $i=7,\, 9,\, 11$.

Moreover we have

Corollary 6.7. $j^*(x_3)=y_3+\alpha' y'_3$ *for* $\alpha'\in\boldsymbol{Z}_3$

and $j^*(x_i)=y_i$ *for* $i=7,\, 9,\, 11$.

On the other hand, since

(6.10) $$H^*(\boldsymbol{E}_6/\tilde{\boldsymbol{V}};\, \boldsymbol{Z}_3)=H^*(\boldsymbol{AdE}_6/\boldsymbol{V};\, \boldsymbol{Z}_3)=0 \quad \text{for} \quad *\leq 2,$$

we have

(6.11) $$j^*(x_1)=y_1 \quad \text{and} \quad j^*(x_2)=y_2\,.$$

Let G be a compact connected Lie group. Denote by P_G^i a submodule of $H^i(G; Z_p)$ which consists of primitive elements. That is

$$P_G^i = \{x \in H^i(G; Z_p);\ \bar{\phi}(x) = 0\}.$$

Note that $P_G^i \subset T_G^i$ for any i (cf. [5]).

Now recall from [4]:

Lemma 6.8. $P_{PU(6)}^i = 0$ for $i = 3, 9, 11$.

The proof is easy.

Since $j^*(x_j) \equiv y_j \bmod (y'_3)$ for $j = 3, 9, 11$, we have

Lemma 6.9. $\bar{\phi}(x_i) \neq 0$ *for* $i = 3, 9, 11$.

Now we can determine $\bar{\phi}(x_i)$ for $i = 3, 7, 8, 9, 11$.

By Theorem 5.5 we may assume that

$$\bar{\phi}(x_3) = x_2 \otimes x_1. \tag{6.12}$$

Applying $\mathscr{P}^1$ we have

$$\bar{\phi}(x_7) = \mathscr{P}^1(x_2 \otimes x_1) = x_2^3 \otimes x_1, \tag{6.13}$$

and

$$\bar{\phi}(x_8) = \beta\bar{\phi}(x_7) = \beta(x_2^3 \otimes x_1) = x_2^3 \otimes x_2. \tag{6.14}$$

Also by Theorem 5.5, $\bar{\phi}(x_9)$ is of the form

$$\alpha_1 x_2 \otimes x_7 + \alpha_2 x_2^3 \otimes x_3 + \alpha_3 x_2^4 \otimes x_1 + \alpha_4 x_8 \otimes x_1. \tag{6.15}$$

Since $\bar{\phi}(x_1) = \bar{\phi}(x_2) = 0$, we have

$$(\bar{\phi} \otimes 1)\bar{\phi}(x_9) = (\alpha_3 + \alpha_4) x_2^3 \otimes x_2 \otimes x_1 + \alpha_3 x_2 \otimes x_2^3 \otimes x_1 \tag{6.16}$$

Also we have

$$(1 \otimes \bar{\phi})\bar{\phi}(x_9) = \alpha_1 x_2 \otimes x_2^3 \otimes x_1 + \alpha_2 x_2^3 \otimes x_2 \otimes x_1. \tag{6.17}$$

Since $(\bar{\phi} \otimes 1)\bar{\phi}(x_9) = (1 \otimes \bar{\phi})\bar{\phi}(x_9)$, we have

(6.18) $$\alpha_3=\alpha_1 \quad \text{and} \quad \alpha_4=\alpha_2-\alpha_1 .$$

Recall that

(6.19) $$\beta x_3=-x_2^2 \text{ since } \phi(\beta x_3)=x_2\otimes x_2 \quad \text{and} \quad \beta x_3\in \boldsymbol{T}^4_{AdE_6} .$$

Applying β to $\bar{\phi}(x_9)$ we have

(6.20) $$\bar{\phi}(\beta x_9)=\alpha_1 x_2\otimes x_8-\alpha_2 x_2^3\otimes x_2^2+\alpha_1 x_2^4\otimes x_2+\alpha_1 x_8\otimes x_2 .$$

On the other hand, since

$$\bar{\phi}(x_2^5)=-x_2^4\otimes x_2+x_2^3\otimes x_2^2+x_2^2\otimes x_2^3-x_2\otimes x_2^4 ,$$

$$\bar{\phi}(x_2x_8)=x_2\otimes x_8+x_2^4\otimes x_2+x_8\otimes x_2+x_2^3\otimes x_2$$

and

$$\beta x_9\in \boldsymbol{T}^{10}_{AdE_6},$$

we have

(6.21) $$\beta x_9=x_2x_8$$

and

$$\bar{\phi}(x_9)=x_2\otimes x_7-x_2^3\otimes x_3+x_2^4\otimes x_1+x_8\otimes x_1 .$$

Next similarly $\bar{\phi}(x_{11})$ is of the form

(6.22) $$\beta_1 x_2\otimes x_9+\beta_2 x_2^2\otimes x_7+\beta_3 x_2^4\otimes x_3+\beta_4 x_8\otimes x_3+\beta_5 x_2^5\otimes x_1+\beta_6 x_2x_8\otimes x_1$$

for $\beta_i\in \boldsymbol{Z}_3$.

Using the relation $(1\otimes\bar{\phi})\bar{\phi}(x_{11})=(\bar{\phi}\otimes 1)\bar{\phi}(x_{11})$ we have

(6.23) $$\beta_2=\beta_3=\beta_5=-\beta_1 \quad \text{and} \quad \beta_4=\beta_6=\beta_1 .$$

Applying β-operation we have

(6.24) $$\beta x_{11}=-x_8x_2^2-x_2^6$$

and

$$\bar{\phi}(x_{11})=x_2\otimes x_9-x_2^2\otimes x_7+x_8\otimes x_3-x_2^4\otimes x_3+x_8x_2\otimes x_1-x_2^5\otimes x_1 .$$

Now we can compute $\mathscr{P}^1 x_i$ for $i=7, 8, 9$.

(6.25) $\mathscr{P}^1 x_7 = \mathscr{P}^1 \mathscr{P}^1 x_3 = -\mathscr{P}^2 x_3 = 0$ by the Adem relation,

$\bar{\phi}(\mathscr{P}^1 x_8) = \mathscr{P}^1(x_2^3 \otimes x_2) = x_2^3 \otimes x_2^3$ and so $\mathscr{P}^1 x_8 = -x_2^6$,

$\mathscr{P}^1 x_9 \in \boldsymbol{T}^{13}_{AdE_6} = 0$ and so $\mathscr{P}^1 x_9 = 0$.

Applying $\mathscr{P}^1$ we have

(6.26) $\bar{\phi}(x_{15}) = \bar{\phi}(\mathscr{P}^1 x_{11}) = \mathscr{P}^1 \bar{\phi}(x_{11}) = x_2^3 \otimes x_9 + x_8 \otimes x_7 + x_2^6 \otimes x_3 + x_8 x_2^3 \otimes x_1$

Applying β-operation we have

(6.27) $\bar{\phi}(\beta x_{15}) = \beta \bar{\phi}(x_{15}) = x_2^3 \otimes x_8 x_2 + x_8 \otimes x_8 - x_2^6 \otimes x_2^2 + x_8 x_2^3 \otimes x_2$

and so

$$\beta x_{15} = -x_8^2 \quad \text{since} \quad \beta x_{15} \in \boldsymbol{T}^{16}_{AdE_6}.$$

Since $\mathscr{P}^1 x_{15} \in \boldsymbol{T}^{19}_{AdE_6} = 0$, $\mathscr{P}^1 x_{15} = 0$.

Note that

(6.28) $$\boldsymbol{P}^i_{AdE_6} = 0 \qquad \text{for} \quad i \neq 1, 2, 6.$$

Since $\bar{\phi}(\mathscr{P}^j x_i) = 0$ for $j \geq 2$, we have

(6.29) $$\mathscr{P}^j x_i = 0 \qquad \text{for} \quad j \geq 2.$$

Thus the following theorem is proved:

Theorem 6.10. *In* (2) *of* Theorem 6.2,

$$\bar{\phi}(x_1) = \bar{\phi}(x_2) = 0, \quad \bar{\phi}(x_3) = x_2 \otimes x_1,$$

$$\bar{\phi}(x_7) = x_2^3 \otimes x_1, \quad \bar{\phi}(x_8) = x_2^3 \otimes x_2,$$

$$\bar{\phi}(x_9) = x_2 \otimes x_7 - x_2^3 \otimes x_3 + x_8 \otimes x_1 + x_2^4 \otimes x_1,$$

$$\bar{\phi}(x_{11}) = x_2 \otimes x_9 - x_2^2 \otimes x_7 + x_8 \otimes x_3 - x_2^4 \otimes x_3 + x_8 x_2 \otimes x_1 - x_2^5 \otimes x_1,$$

$$\bar{\phi}(x_{15}) = x_2^3 \otimes x_9 + x_8 \otimes x_7 + x_2^6 \otimes x_3 + x_8 x_2^3 \otimes x_1,$$

$$\beta x_1 = x_2, \quad \beta x_3 = -x_2^2, \quad \beta x_7 = x_8, \quad \beta x_9 = x_8 x_2,$$

$$\beta x_{11} = -x_8 x_2^2 - x_2^6, \quad \beta x_{15} = -x_8^2, \quad \beta x_i = 0 \quad \textit{for} \quad i = 2, 8\,,$$

$$\mathscr{P}^1 x_2 = x_2^3, \quad \mathscr{P}^1 x_3 = x_7, \quad \mathscr{P}^1 x_8 = -x_2^6, \quad \mathscr{P}^1 x_{11} = x_{15}\,,$$

$$\mathscr{P}^1 x_i = 0 \quad \textit{for} \quad i = 1, 7, 9, 15\,,$$

$$\mathscr{P}^j x_i = 0 \quad \textit{for} \quad j \geq 2\,.$$

Applying ρ^* we have

Corollary 6.11. *In* (1) *of* Theorem 6.2

$$\bar{\phi}(\tilde{x}_3) = \bar{\phi}(\tilde{x}_7) = \bar{\phi}(\tilde{x}_8) = \bar{\phi}(\tilde{x}_9) = 0\,,$$

$$\bar{\phi}(\tilde{x}_{11}) = \tilde{x}_8 \otimes \tilde{x}_3, \quad \bar{\phi}(\tilde{x}_{15}) = \tilde{x}_8 \otimes \tilde{x}_7\,,$$

Remark 6.12. Corollary 6.11 is proved by Araki [2] using Kudo's transgression theorem.

Now consider the inclusion $j' : \boldsymbol{SU}(6) \to \tilde{\boldsymbol{V}} \to \boldsymbol{E}_6$. Since the center of $\tilde{\boldsymbol{V}}$ is of order 6, we have $\boldsymbol{Z}_3 \hookrightarrow \boldsymbol{SU}(6)$. Consider the following commutative diagram:

(6.30)
$$\begin{array}{ccccc} \boldsymbol{SU}(6) & \xrightarrow{j'} & \boldsymbol{E}_6 & \longrightarrow & \boldsymbol{E}_6/\boldsymbol{SU}(6) \\ \downarrow & \circlearrowright & \downarrow & \circlearrowright & \| \\ \boldsymbol{SU}(6)/\boldsymbol{Z}_3 & \xrightarrow[j']{} & \boldsymbol{AdE}_6 & \xrightarrow[r]{} & \boldsymbol{AdE}_6/(\boldsymbol{SU}(6)/\boldsymbol{Z}_3)\,. \end{array}$$

Note that

(6.31) $H^*(\boldsymbol{SU}(6)/\boldsymbol{Z}_3\,; \boldsymbol{Z}_3) \cong H^*(\boldsymbol{PU}(6)\,; \boldsymbol{Z}_3) \cong \boldsymbol{Z}_3[y_2]/(y_2^3) \otimes \Lambda(y_1, y_3, y_7, y_9, y_{11})$.

By Corollary 6.7 and (6.1) we have

(6.32) $$j'^*(x_i) = y_i \quad \text{for} \quad i = 1, 2, 3, 7, 9, 11\,.$$

In particular j'^* is surjective and so $\boldsymbol{AdE}_6/(\boldsymbol{SU}(6)/\boldsymbol{Z}_3) = \boldsymbol{E}_6/\boldsymbol{SU}(6)$ is totally non homologous to zero mod 3 in $\boldsymbol{AdE}_6$. So we have

Corollary 6.13. $H^*(\boldsymbol{E}_6/\boldsymbol{SU}(6); \boldsymbol{Z}_3) = \boldsymbol{Z}_3[e_6, e_8]/(e_6^3, e_8^3) \otimes \Lambda(e_{15})$, *where* deg $e_i = i$, $r^*(e_6) = x_2^3$, $r^*(e_8) = x_8$ *and* $r^*(e_{15}) = x_{15}$. *Moreover* $\beta e_{15} = e_8^2$.

§7. $H^*(E_6; Z_3)$

In this section we shall determine the Hopf algebra structure of $H^*(E_6; Z_3)$. By Corollary 6.12 we only need to determine $\bar{\phi}(\tilde{x}_{17})$.

First recall from [2]. Let $k: F_4 \to E_6$ be the inclusion defined in [2]. Then the following is known:

Theorem 7.1. $k^*: H^*(E_6; Z_3) \longrightarrow H^*(F_4; Z_3)$ *is surjective and* $\operatorname{Ker} k^* = (\tilde{x}_9, \tilde{x}_{17})$.

Remark 7.2. This gives the Hopf algebra structure of $H^*(F_4; Z_3)$. In fact if $k^*(\tilde{x}_i)$ is also denoted by $\tilde{x}_i$ for $i=3, 7, 8, 11, 15$, then we have

$$H^*(F_4; Z_3) \cong Z_3[\tilde{x}_8]/(\tilde{x}_8^3) \otimes \Lambda(\tilde{x}_3, \tilde{x}_7, \tilde{x}_{11}, \tilde{x}_{15}),$$

$$\bar{\phi}(\tilde{x}_3) = \bar{\phi}(\tilde{x}_7) = \bar{\phi}(\tilde{x}_8) = 0,$$

$$\bar{\phi}(\tilde{x}_{11}) = \tilde{x}_8 \otimes \tilde{x}_3,$$

$$\bar{\phi}(\tilde{x}_{15}) = \tilde{x}_8 \otimes \tilde{x}_7.$$

Also recall from [19]:

Lemma 7.3. (1) $\operatorname{Cotor}^{H^*(F_4; Z_3)}(Z_3, Z_3)$ *is generated by elements* $\tilde{u}_i$'s *for* $i=4, 8, 9, 20, 21, 25, 26, 36, 48$, *with* $\deg \tilde{u}_i = i$,

(2) *The Rothenberg-Steenrod spectral sequence* (*or the Eilenberg-Moore spectral sequence*) (cf. [29] *or* [30]),

$$E_2 = \operatorname{Cotor}^{H^*(F_4; Z_3)}(Z_3, Z_3) \Longrightarrow E_\infty = Gr(H^*(BF_4; Z_3))$$

collapses,

(3) *Moreover* $\tilde{u}_8, \tilde{u}_9, \tilde{u}_{20}$ *are represented by* $\mathscr{P}^1 u_4$, $\beta\mathscr{P}^1 u_4$, $\mathscr{P}^3\mathscr{P}^1 u_4$ *for a generator* u_4 *of* $H^4(BF_4; Z_3) \cong Z_3$.

Now we assume that

$$\bar{\phi}(x_{17}) = 0. \tag{7.1}$$

Then we have

Lemma 7.4. (1) *Under the assumption* (7.1),

$$\mathrm{Cotor}^{H^*(E_6;Z_3)}(Z_3, Z_3) \cong \mathrm{Cotor}^{H^*(F_4;Z_3)}(Z_3, Z_3) \otimes Z_3[\tilde{u}_{10}, \tilde{u}_{18}]$$

with $\deg \tilde{u}_i = i$,

(2) *In particular an element* $\tilde{u}_{19}$ *of* $\mathrm{Cotor}^{H^*(E_6;Z_3)}(Z_3, Z_3)$ *of degree* 19 *is of the form* $\alpha\tilde{u}_{10}\cdot\tilde{u}_9$ *for* $\alpha \in Z_3$.

So by the argument similar to the proof of Lemma 5.8 we have

(7.2) Under the assumption (7.1), $\tilde{u}_{18}$ is a permanent cycle.

Since clearly $\tilde{x}_9$ is universally transgressive, $\tilde{u}_{10}$ is a permanent cycle represented by $u_{10} = \tau(\tilde{x}_9)$. So we have

Lemma 7.5. *Under the assumption* (7.1),

$H^*(BE_6; Z_3)$ *for* $* \leq 24$, *is generated by the following elements*:

$$u_4 = \tau(\tilde{x}_3),\ u_8 = \mathscr{P}^1 u_4,\ u_9 = \beta\mathscr{P}^1 u_4,\ u_{20} = \mathscr{P}^3\mathscr{P}^1 u_4,$$

$$u_{10} = \tau(\tilde{x}_9) \quad and \quad u_{18} = \tau(\tilde{x}_{17}).$$

Proof. Clearly $\tilde{u}_i$, $i = 4, 8, 9, 10, 18$ are permanent cycles. Since $\mathscr{P}^3\mathscr{P}^1 k^* u_4$ is not decomposable, so is $\mathscr{P}^3\mathscr{P}^1 u_4$. So the result follows. Q.E.D.

Now the following is well known:

(7.3) $H^*(BSU(6); Z_3) \cong Z_3[c_2, c_3, c_4, c_5, c_6]$, where c_i is the mod 3 reduction of the i-th universal Chern class.

Consider the inclusion

$$j' : BSU(6) \longrightarrow BE_6.$$

Then by the naturality of the transgression and by (6.32) we have

(7.4) $\qquad j'^*(u_{10}) = c_5 + \alpha c_2 \cdot c_3$ for $\alpha \in Z_3$ up to non-zero multiple.

Moreover we have

(7.5) $\qquad j'^*(u_4), j'^*(u_8) \in (c_2, c_4)$ by the dimensional reason.

Since $\mathscr{P}^3 u_{10} \in H^{22}(BE_6; Z_3)$, so

(7.6) $$\mathscr{P}^3 u_{10} \in (u_4, u_8) \text{ by the dimensional reason.}$$

On the other hand

(7.7) $$j'^*(\mathscr{P}^3 u_{10}) = \mathscr{P}^3(c_5 + \alpha c_2 \cdot c_3) = c_6 c_5 + (\text{other terms}) \text{ up to non-zero multiple.}$$

And so

(7.8) $$j'^*(\mathscr{P}^3 u_{10}) \notin (c_2, c_4).$$

That is a contradiction and so we have

Lemma 7.6. $\bar{\phi}(\tilde{x}_{17}) \neq 0$.

Then applying Theorem 5.5 we have

Theorem 7.7. *In* (1) *of* Theorem 6.2,

$$\bar{\phi}(\tilde{x}_3) = \bar{\phi}(\tilde{x}_7) = \bar{\phi}(\tilde{x}_8) = \bar{\phi}(\tilde{x}_9) = 0,$$

$$\bar{\phi}(\tilde{x}_{11}) = \tilde{x}_8 \otimes \tilde{x}_3, \quad \bar{\phi}(\tilde{x}_{15}) = \tilde{x}_8 \otimes \tilde{x}_7, \quad \bar{\phi}(\tilde{x}_{17}) = \tilde{x}_8 \otimes \tilde{x}_9,$$

$$\beta \tilde{x}_7 = \tilde{x}_8, \quad \beta \tilde{x}_{15} = -\tilde{x}_8^2, \quad \beta \tilde{x}_i = 0 \quad \textit{for} \quad i \neq 7, 15,$$

$$\mathscr{P}^1 \tilde{x}_3 = \tilde{x}_7, \quad \mathscr{P}^1 \tilde{x}_{11} = \tilde{x}_{15}, \quad \mathscr{P}^1 \tilde{x}_i = 0 \quad \textit{for} \quad i \neq 3, 11,$$

$$\mathscr{P}^j \tilde{x}_i = 0 \quad \textit{for} \quad j \geq 2.$$

The proof is easy (cf. Theorem 6.10).

Remark 7.8. The similar proof is given in [18]. But the proof is tedious since we need the algebra structure of $H^*(BE_6; \boldsymbol{Z}_3)$ for $* \leq 30$ under the assumption of (7.1).

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References

[1] J. F. Adams: Lectures on Lie groups, Benjamin, New York 1969.
[2] S. Araki: On the non-commutativity of Pontrjagin rings mod 3 of the compact exceptional groups, Nagoya Math. J., **17** (1960), 225–260.
[3] S. Araki: Differential Hopf algebras and the cohomology mod 3 of the compact excep-

tional groups E_7 and E_8. Ann. Math., **73** (1961), 404–436.
[4] P. Baum-W. Browder: The cohomology of quotient of classical groups, Topology, **3** (1965), 305–336.
[5] A. Borel: Sur l'homologie des espaces fibrés principaux et des espaces homogènes des groupes de Lie compacts, Ann. Math., **57** (1953), 115–207.
[6] A. Borel: Sur l'homologie et la cohomologie des groupes de Lie compacts connexes, Amer. J. Math., **76** (1954), 273–342.
[7] A. Borel: Sous groupes commutatifs et torsion des groupes de Lie compacts connexes, Tôhoku Math. J., **13** (1961), 216–240.
[8] A. Borel: Topology of Lie groups and characteristic classes, Bull. AMS, **61** (1955), 397–432.
[9] A. Borel-F. Hirzebruch: Characteristic classes and homogeneous spaces I, Amer. J. Math., **80** (1958), 459–538.
[10] A. Borel-J. de Siebenthal: Les sous groupes fermés connexes de rang maximum des groupes de Lie clos, Comment. Math. Helv., **23** (1949), 200–221.
[11] N. Bourbaki: Groupes et algèbre de Lie IV–VI, 1968.
[12] W. Browder: Homology rings of groups, Amer. J. Math., **90** (1968), 318–333.
[13] K. Ishitoya-A. Kono-H. Toda: Hopf algebra structure of mod 2 cohomology of simple Lie groups, Publ. RIMS. Kyoto Unive., 12 (1976), 141-167.
[14] A. Iwai-N. Shimada: A remark on resolutions for Hopf algebras, Publ. RIMS of Kyoto Univ., **1** (1966), 187–198.
[15] J. Kojima: On the Pontrjagin product mod 2 of Spinor groups, Mem. Fac. Sci. Kyushu Univ., Ser. A., **11** (1957), 1–14.
[16] A. Kono: On the 2-rank of compact connected Lie groups, J. Math. Kyoto Univ., **17** (1977), 1–18.
[17] A. Kono-M. Mimura: Cohomology mod 2 of the classifying space of the compact connected Lie group of type E_6, J. Pure and Applied Alg. **6** (1975), 61–81.
[18] A. Kono-M. Mimura: Cohomology mod 3 of the classifying space of the compact, 1-connected Lie group of type E_6, (preprint).
[19] A. Kono-M. Mimura: Cohomology operations and the Hopf algebra structure of the compact, exceptional Lie groups E_7 and E_8, (to appear).
[20] A. Kono-M. Mimura-N. Shimada: Cohomology of classifying spaces of certain H-spaces, J. Math. Kyoto Univ., **15** (1975), 607–617.
[21] A. Kono-M. Mimura-N. Shimada: On the cohomology mod 2 of the classifying space of the 1-connected exceptional Lie group E_7, J. Pure and Applied Alg., 8 (1976), 267-283.
[22] T. Kudo: A transgression theorem, Mem. Fac. Sci. Kyusyu Univ., **A9** (1956), 79–81.
[23] P. May: The cohomology of restricted Lie algebras and of Hopf algebras, Bull. A. M. S., **71** (1965), 372–377.
[24] J. Milnor: Construction of universal bundles II, Ann. Math., **63** (1956), 430–436.
[25] J. Milnor-J. Moore: On the structure of Hopf algebras, Ann. Math., **81** (1965), 211–264.
[26] M. Mimura-H. Toda: A representation and the quasi p-regularity of the compact Lie group, Japanese J. Math., **1** (1975), 101–109.
[27] D. Quillen: The mod 2 cohomology ring of extra special 2-groups and the Spinor groups, Math. Ann. **194** (1971), 197–212.
[28] D. Quillen: The spectrum of an equivariant cohomology rings I, Ann. Math., **94** (1971), 549–572.
[29] M. Rothenberg-N. E. Stenrod: The cohomology of the classifying spaces of H-spaces, (mimeographed notes).

[30] M. Rothenberg-N. E. Steenrod: The cohomology of the classifying spaces of H-spaces, Bull. AMS, **71** (1965), 872–875.

[31] J. P. Serre: Homologie singulière des espaces fibrés, Ann. Math., **54** (1951), 425–505.

[32] N. Shimada-A. Iwai: On the cohomology of some Hopf algebra, Nagoya Math. J., **30** (1971), 103–111.

[33] H. Toda: Cohomology mod 3 of the classifying space BF_4 of the exceptional group F_4, J. Math. Kyoto Univ., **13** (1972), 97–115.

[34] H. Toda: Cohomology of the classifying space of exceptional Lie groups, Manifolds, Tokyo 1973.

[35] H. Toda: On the cohomology rings of some homogeneous spaces, J. Math. Kyoto Univ., **15** (1975), 185–199.