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A sharp form of the existence theorem for hyperbolic mixed problems of second order

By

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§1. Introduction

In this paper we consider the following initial-boundary value problem (P) which we denote also by $\{P, B\}$

$$(P)\qquad \begin{cases} Pu=f(x, t), & \text{for } x\in\Omega,\ t\in R_+^1, \\ Bu|_S=g(s, t), & \text{for } s\in S,\ t\in R_+^1, \\ D_t^j u|_{t=0}=u_j(x),\ (j=0, 1), & \text{for } x\in\Omega, \end{cases}$$

in the cylinder domain $\Omega\times(0, \infty)$, where Ω is the exterior or interior of a smooth and compact hypersurface S in R^{n+1}. P is a regularly hyperbolic operator of second order and S is non characteristic to P. Moreover we assume that the only one of $\tau_1(\nu)$ and $\tau_2(\nu)$ is negative for all $(s, t)\in S\times(0, \infty)$, where $\tau_1(\xi)$ and $\tau_2(\xi)$ are the roots of $P(s, t, \xi, \tau)=0$ and ν is the inner unit normal at s. This assumption means that the number of boundary conditions is one[1]. Therefore we assume $P(s, t, 0, 1)<0$ and $P(s, x, \nu, 0)=1$. B is a first order operator:

$$B(s, t\ D_x, D_t)=\sum_{j=1}^{n+1} b_j D_{x_j}-cD_t, \qquad \left(D_t=\frac{1}{i}\frac{\partial}{\partial t}, \text{ etc.}\right),$$

and we suppose $B(s, t, \nu, 0)=1$.

We assume that all the coefficients of P and B are smooth and that they

1) This assumption is equivalent to the condition that only one root $q(\eta, \tau)$ of $P(s, t, q\nu+\eta, \tau)=0$ has positive imaginary part for $\mathrm{Im}\ \tau<0$. (cf. p. 121 in [3].) In fact, to see this, we may consider the case $\eta\equiv 0$, taking account of the hyperbolicity of P.

remain constant outside some compact sets[2].

We are concerned with the following question (Q): 'Under what condition the solution $u(t)$ of (P) has the continuity for the initial data in the same Sobolev space?' The answer to the above question is just the condition (H) below, which is equivalent to 'all the roots of $\alpha z^2-2iz-\beta=0$ are in $\{z: \mathrm{Im}\, z\geq 0\}$ and they are not real double roots.' We state it as

Theorem 1. *The above problem $\{P, B\}$ satisfies the following estimate:*

$$(1.1)\qquad \sum_{j=0}^{1}\|(D_t^j u)(t)\|_{1-j}\leq c(T)\{\sum_{j=0}^{1}\|(D_t^j u)(0)\|_{1-j}+\int_0^t\|(Pu)(s)\|_0 ds\},$$

for all $u\in C_0^\infty(\overline{\Omega\times R_+^1})$[3] *satisfying $Bu|_S=0$, if and only if the following condition (H) holds.*

$$(H)\qquad (\mathrm{I})\quad A=\begin{pmatrix} 2\,\mathrm{Re}\,\alpha & \mathrm{Im}(\alpha\bar{\beta}) \\ \mathrm{Im}\,(\alpha\bar{\beta}) & 2\mathrm{Re}\,\beta \end{pmatrix}\geq 0,\qquad \textit{when } |\mathrm{Re}\,\alpha|+|\mathrm{Re}\,\beta|\neq 0,$$

$$(\mathrm{II})\quad 1+(\mathrm{Im}\,\alpha)(\mathrm{Im}\,\beta)\geq\delta>0,\qquad \textit{when } |\mathrm{Re}\,\alpha|+|\mathrm{Re}\,\beta|=0,$$

for all $(s, t, \eta)\in S\times R_+^1\times(R^{n+1}-\{v\})$, with some positive constant δ. Here $\eta\in\{R^{n+1}-\{v\})$ means $\eta\in R^{n+1}$ and $\eta\cdot v=0$. α and β are defined by

$$\alpha=\tilde{c}(s, t)+\tilde{b}(s, t, \eta),\quad \beta=\tilde{c}(s, t)-\tilde{b}(s, t, \eta),$$

where $\tilde{c}$ and $\tilde{b}$ are determined[4] *uniquely by*

$$B(s, t, qv+\eta, \tau)=\tilde{q}+\tilde{b}d(\eta)-\tilde{c}\tilde{\tau}.$$

Here we have used the following notations[5]:

$$\tilde{q}=\frac{1}{2}\frac{\partial}{\partial q}P(s, t, qv+\eta, \tau),$$

$$\tilde{\tau}=-\frac{1}{2}\frac{\partial}{\partial\tau}\{P(s, t, qv+\eta, \tau)-\tilde{q}(s, t, qv+\eta, \tau)^2\},$$

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2) This assumption can be replaced by the assumption that all the coefficients belong to the class of smooth and bounded functions $\mathscr{B}$. Concerning this argument we can see the result of J. Kato: Remarks on the hyperbolic mixed problems.

3) $u\in C_0^\infty(\Omega\times R_+^1)$ means that u is infiniely differentiable up to the boundary of $\Omega\times R_1^+$ and its support is compact.

4) In case of wave equation in a half space, $B(y, t, q, \eta, \tau)=q+(\sum_{j=1}^{n}b_j\eta_j/|\eta|)|\eta|-c\tau$.

5) See p. 121～p. 122 in [3]. As for our notations, it is sufficient to consider the case where $\Omega=R_+^{n+1}$.

$$d(\eta)^2=P(s,\ t,\ q(\eta)\nu+\eta,\ \tau(\eta)),\quad (d(\eta)>0 \quad \text{for} \quad \eta\neq 0),$$

where $q(\eta)$ and $\tau(\eta)$ are the solutions[6] of $\frac{\partial}{\partial q}P(s,\ t,\ q\nu+\eta,\ \tau)=0$ and $\frac{\partial P}{\partial\tau}(s,\ t,\ q\nu+\eta,\ \tau)=0$. They satisfy the relation

$$P(s,\ t,\ q\nu+\eta,\ \tau)=\tilde{q}^2-\tilde{\tau}^2+d(\eta)^2,$$

which corresponds to the symbol of wave equation.

$\|\cdot\|_k$ means the norm of Sobolev space H^k in Ω, $(k=0,\ 1,\ 2,\dots)$. Denote by $\langle\!\langle\cdot\rangle\!\rangle_r$ the Sobolev r-norm on S, $(r\in R^1)$. We can say more, that is

Theorem 2. *Suppose* (H). *If* $f(x)\in\mathscr{E}_t^0(L^2(\Omega))$[7], $g\in\mathscr{E}_t^0(H^{\frac{1}{2}}(S))$ *and* $u_j\in H^{1-j}(\Omega)$, $(j=0,\ 1)$, *then there exists a unique solution* $u(t)$ *of* $\{P,\ B\}$ *in* $\mathscr{E}_t^0((H^1\Omega))\cap\mathscr{E}_t^1(L^2(\Omega))$ *satisfying the following energy estimate* (E) *with* $k=0$. *Moreover if we assume that the smooth data* $\{f,\ g,\ u_0,\ u_1\}$ *satisfy the compatibility conditions*[8] *of order* k, $(k\geq 1)$, *then the solution satisfies*

$$(E)\qquad \sum_{j=0}^{1}\|(D_t^j u)(t)\|^2_{1-j+k}+\sum_{i+j\leq k}\int_0^t e^{2\gamma_k(t-s)}\langle\!\langle(D_x^i D_t^j u)\ (s)\rangle\!\rangle^2_{\frac{1}{2}-i-j+k}ds$$

$$\leq C_k e^{2\gamma_k t}\{\sum_{j=0}^{1}\|u_j\|^2_{1-j+k}+\sum_{j\leq k}\int_0^t e^{-2\gamma_k s}(\|(D_t^j f)(s)\|^2_{k-j}+(D_t^j g)(s)\rangle\!\rangle^2_{\frac{1}{2}-j+k})ds\},$$

where C_k *and* γ_k *are positive constants. The solution has the some propagation speed as that in the case of Cauchy problem.*

Remark. If $g=0$ in the problem $\{P,\ B\}$, the above solution u satisfies (1.1) even if in the case of $k=0$. Therefore (H) is necessary to obtain (E).

The condition (H) was originally introduced in [3] as a concrete necessary and sufficient condition to obtain the estimate:

$$(*)\qquad |u|_{1,\gamma}\leq\frac{C}{\gamma}|Pu|_{0,\gamma},\qquad (\gamma>0),$$

for any smooth function u satisfying $Bu|_{\alpha\Omega}=0$, in the case where all the coefficients are constant and $\Omega=R_+^{n+1}$. Then [1] and [4] treated the estimate

6) See p. 122 in [3].

7) $f(t)\in\mathscr{E}_t^k(L^2(\Omega))$ means that u is continuous up to k-times derivatives with values in $L^2(\Omega)$.

8) All the smooth solutions must satistisfy the compatibility conditions. See p. 145 in [3]. As for notation see the last part of the introduction in [3].

in the case of variable coefficients independently. Here remark that the estimate $(*)$ follows from the estimate (1.1) by the integration in t. This process will be explained in §2. However the converse can not be proved directly. Namely we can say that the structure of (1.1) is finer than that of $(*)$. In order to obtain (1.1) we must employ a special device concerning the reverse process of Green formula, which is related to the quadratic differential form $\mathrm{Im}(Pu, Qu)_{0,\gamma}$ in §4. Q is a first order differential operator with respect to t with coefficients of pseudo-differential operators in x. The choice of Q is more difficult for the estimate (1.1), because the localization used in the proof must not depend on τ, which is a dual variable of t. In §5 and §6 we succeed in the choice of Q with the help of the detailed considerations on (H) which are prepared in §3. In order to obtain (E) we need to consider another localization in (x, t, η, τ) and the dual problem of $\{P, B\}$, even if we restrict ourselves to the Neumann problem for $\square$, which is explained in §7. Since this paper is continued from [3], the references cited here should be added to those in [3].

§2. Necessity of (H) for (1.1)

In this section we show that the condition (H) is necessary in order to obtain the estimate (1.1). Namely we want to prove

Theorem 2.1. *Assume that the problem* $\{P, B\}$ *in* §1 *satisfies the estimate* (1.1) *for* $u \in C_0^\infty(\overline{\Omega \times R_+^1})$ *satisfying* $Bu|_S = 0$. *Then the condition* (H) *must follow.*

For this purpose we need some analysis concerning the estimate (1.1), in order to arrive at the position to apply the results obtained in the chapter one in [3]. More precisely we show at first

Proposition 2.2. *Suppose the assumption in Theorem* 2.1. *Then we have the following estimate*

$$\sum_{j=0}^{1} \|(D_t^j v)(t)\|_{1-j, L^2(R_+^{n+1})} \le C \int_{-\infty}^{t} \|(P_0 v)(s)\|_{0, L^2(R_+^{n+1})} ds \tag{2.1}$$

for all $v(x, y, t) \in C_0^\infty(\overline{R_+^1 \times R^n \times R^1})$ *satisfying* $B_0 v|_{x=0} = 0$. *Here* P_0 *and* B_0 *are the operators with constant coefficients which are considered in a half space* $R_+^1 \times R^n \times R^1$ $(\ni (x, y, t))$ *and on* $R^n \times R^1$ $(\ni (y, t))$ *respectively such that*

$$P_0 = P_0(D_x, D_y, D_t) = P(s_0, t_0, D_x \nu + D_y, D_t)$$

$$B_0 = B_0(D_x, D_y, D_t) = B(s_0, t_0, D_x \nu + D_y, D_t)$$

for any $(s_0, t_0) \in S \times R^1_+$, *and* $v = v(s_0)$.

Next we prove

Proposition 2.3. *Assume* (2.1) *in Proposition* 2.2. *Then we have*

$$\gamma |u|^2_{1,\gamma} \leq \frac{C}{\gamma} |P_0 u|^2_{0,\gamma} \tag{2.2}$$

for any $u = u(x, y, t) \in C_0^\infty(\overline{R^1_+ \times R^n \times R^1})$ *satisfying* $B_0 u|_{x=0} = 0$, *where*

$$|u|^2_{k,\gamma} = \sum_{i+j+|\alpha|+l \leq k} \iiint e^{-2\gamma t} |\gamma^l D_x^i D_y^\alpha D_t^j u|^2 \, dx\, dy\, dt.$$

From Propositions 2.2 and 2.3 and by the proofs of theorems in Chapter I of [3], we have Theorem 2.1.

Now we prepare the following lemma which make the proof of Proposition 2.2 easier.

Lemma 2.4. *Let* $v(x, y, t)$ *be in* $C_0^\infty(\overline{R^1_+ \times R^n \times R^1})$ *satisfying* $B_0 v|_{x=0} = 0$, *where* B_0 *is a first order operator* $B(0, 0, D_x, D_y, D_t)$. *Here* $B(y, t, D_x, D_y, D_t) = D_x + \sum_{j=1}^n b_j(y, t) D_{y_j} - c(y, t) D_t$. *Then there exist functions* $v_\varepsilon(x, y, t)$ *for* $0 < \varepsilon < 1$, *satisfying the following conditions*:

(i) $\operatorname{supp} v_\varepsilon \subset$ *a compact set* $K \cap (\overline{R^1_+ \times R^n \times R^1})$

(ii) v_ε *converge to* v *uniformly up to their all derivatives.*

(iii) $B_\varepsilon v_\varepsilon|_{x=0} = 0$, *where* $B_\varepsilon = B(\varepsilon y, \varepsilon t, D_x, D_y, D_t)$

This lemma is proved in an elemental method[9] which we will explain in Appendix

Proof of Proposition 2.2. Let us consider a local mapping from a neighbourhood of $s_0 \in S$ to the neighbourhood of origin, such that Ω is mapped into $R^1_+ \times R^n$, the boundary S to the hyperplane $x = 0$ and s_0 to the origin. As for $t \in R^1$ we consider the simple transition: $t \to t - t_0$, then $(s_0, t_0) \to (0, 0) \in R^n \times R^1$. By virtue of these transformations we have from the assumption in Theorem 2.1,

$$\sum_{j=0}^{1} \|(D_t^j v)(t)\|_{1-j, L^2(R^{n+1}_+)} \leq C(T) \int_{-\infty}^{t} \|(Pv)(s)\|_{0, L^2(R^{n+1}_+)} \, ds, \ (t + t_0 < T), \tag{2.3}$$

for any $v(x, y, t)$ with its support in the sufficiently small neighbourhood of the origin satisfying $Bv|_{x=0} = 0$. Here we have denoted the transformed operators by the same letters. Now take v_ε in Lemma 2.4. Then we can see that $w_\varepsilon(x, y, t) = v_\varepsilon(x/\varepsilon, y/\varepsilon, t/\varepsilon)$ satisfies $Bw_\varepsilon|_{x=0} = 0$. Hence we can substitute w_ε

9) If all b_j and c are real, lemma 2.4 is evident from the geometrical viewpoint.

to v in (2.3), then by the changes of variables: $x\to\varepsilon x$ etc., we have

$$(2.4)\qquad \sum_{j=0}^{1}\|D_t^j v_\varepsilon(t)\|_{1-j,L^2(R_+^{n+1})}\le C(T)\int_{-\infty}^{t}\|(P_\varepsilon v_\varepsilon)(s)\|_{0,L^2(R_+^{n+1})}\,ds\,.$$

When ε tends to zero we have (2.1) with $C=C(T)$, $(T>t_0)$. q.e.d.

Finally we give a heuristic proof of Proposition 2.3[10]. Let us consider the simplest case of (2.1), that is corresponding to the case of the ordinary differential equation: $D_t u(t)=f(t)$,

$$(2.1)'\qquad |u|\le\int_{-\infty}^{t}|f(s)|ds$$

By Laplace transform $\tau\hat{u}(\tau)=\hat{f}(\tau)$, $(\tau=\sigma-i\gamma,\ \gamma>0)$ and Plancherel's theorem we have

Lemma 2.5. *For any smooth real function $f(t)$ defined in R^1 we have for any $\gamma>0$.*

$$(2.5)\qquad \int_{-\infty}e^{-2\gamma t}\Big(\int_{-\infty}^{t}f(s)\,ds\Big)^2 dt\le\frac{1}{\gamma^2}\int_{-\infty}^{\infty}e^{-2\gamma t}f(t)^2dt\,.$$

In fact the left-hand side equals

$$\int_{-\infty}^{\infty}e^{-2\gamma t}|u(t)|^2\,dt=\int_{-\infty}^{\infty}\Big|\frac{\hat{f}(\tau)}{\tau}\Big|^2 d\sigma\,,$$

while the right-hand side equals $\dfrac{1}{\gamma^2}\displaystyle\int_{-\infty}^{\infty}|\hat{f}(\tau)|^2d\sigma$.

Now multiply $e^{-2\gamma t}$ to (2.1) and integrate it in t using (2.5) with $f(t)=\|(P_0v)(t)\|_0$, then we have (2.2). q.e.d.

§3. Some analysis concerning the condition (H)

In this section we consider the various properties of (H), which we will use later. Put

$$(3.1)\qquad \alpha=\alpha_1+i\alpha_2,\quad \beta=\beta_1+i\beta_2,$$

where α_i and β_i, $(i=1,2)$, are real. First we see easily

Lemma 3.1. *Suppose (H). Then we have the followings:*

10) The estimates (2.1) and (2.2) were considered by many authors in treating the estemate of L^2-well-posedness. For example [2].

(*i*) $\alpha_1 \geq 0$, $\beta_1 \geq 0$ *and* $\det A = 4\alpha_1\beta_1(1+\alpha_2\beta_2)-(\alpha_1\beta_2+\alpha_2\beta_1)^2 \geq 0$,
(*ii*) $1+\alpha_2\beta_2 \geq 0$
(*iii*) $1+\alpha_2\beta_2 > 0$, *if* $\alpha_1\beta_1 = 0$.

Next we give a characterization of (H), relating to an algebraic equation:

$$f(x) = |\alpha|^2 x^2 + 2(2+\operatorname{Re}\alpha\bar{\beta})x + |\beta|^2 = 0, \tag{3.2}$$

which we consider later associating with the boundary integrals.

Lemma 3.2. *Assume* (H). *Then we have the followings*
(*i*) *All roots of* $f(x)$ *are real and non-positive,* ($-\infty$ *is involved*). *They are double only if* $1+\alpha_2\beta_2 = 0$ *and* $\alpha_1\beta_1 > 0$.
(*ii*) *Denote the two roots by* $-\varepsilon_1$ *and* $-\varepsilon'_2$, ($\varepsilon_1 \leq \varepsilon'_2$. *Later we note* $\varepsilon_2 = \dfrac{1}{\varepsilon'_2}$ *if* $\varepsilon'_2 \neq 0$). *We have*

$$\varepsilon_1 \leq \frac{\beta_1}{\alpha_1} \leq \varepsilon'_2 \qquad \text{if } \alpha_1 \neq 0. \tag{3.3}$$

Conversely (H) *follows from* (*i*) *and* (*ii*), *if* $\alpha_1 \geq 0$, $\beta_1 \geq 0$ *and* $1+\alpha_2\beta_2 \geq 0$.

Proof. The discriminant of $f(x)$ is

$$(2+\operatorname{Re}\alpha\bar{\beta})^2 - |\alpha\bar{\beta}|^2 = 4(1+\alpha_2\beta_2) + \det A \geq 0. \tag{3.4}$$

Hence (i) holds from Lemma 3.1, (i) and (iii). (ii) follows from

$$\begin{aligned} f\left(-\frac{\beta_1}{\alpha_1}\right) &= \frac{1}{\alpha_1^2}\{|\alpha|^2\beta_1^2 - 2(2+\operatorname{Re}\alpha\bar{\beta})\alpha_1\beta_1 + |\beta|^2\alpha_1^2\} \\ &= -\frac{1}{\alpha_1^2}\det A \leq 0. \end{aligned} \tag{3.5}$$

The converse is evident. q.e.d.

Now we state another lemma concerning the positiveness of the Hermite matrix, which we will use later associating to the interior integrals. Let A_2 be

$$A_2 = \begin{pmatrix} \alpha_1 & \mu \\ \bar{\mu} & \beta_1 \end{pmatrix}, \qquad \text{where } \alpha_1, \beta_1 \in \mathrm{R} \text{ and } \mu \in C.$$

Lemma 3.3. A_2 *is positive definite if and only if the following Hermite* *matrix A_3 is positive*;

$$A_3=\begin{pmatrix} \alpha_1+\beta_1 & i\mu & -i\bar{\mu} \\ -i\bar{\mu} & \alpha_1 & 0 \\ i\mu & 0 & \beta_1 \end{pmatrix}.$$

Proof. This is evident from $\det A_3=(\alpha_1+\beta_1)\det A_2$.

Example. Let α_1 and β_1 be those in (H) and let $\mu=\frac{1}{2}\operatorname{Im}(\alpha\bar{\beta})$. Then A in (H) is $2A_2$.

In the proof of Theorems 1 and 2, it plays an important role to show the positiveness of the matrix of type A_3 in Lemma 3.3. At that time we may consider the simple A_2.

To arrive at the above position, we must employ a special devise concerning the integration by parts, which we explain in the next section.

§4. Preliminaries

In this section we explain the outline of the proof of Theorem 1 simply. (See also §6 in [3].) First we notice that, as for the estimate (1.1), $\{P, B\}$ in $\Omega\times(0, \infty)$ is reduced to the same problem in $R_+^{n+1}\times(0, \infty)$ by the localization: $u=\sum_j^{\text{finite}}\psi_j u=\sum_j u_j$, where $\Sigma\psi_j(x)=1$ in Ω and by the transformation of the coordinates in the boundary patch. We want to prove the estimate (1.1) by the integration by parts of

$$\mathscr{G}(0, t); P, Q; u)=2i\operatorname{Im}\int_0^\infty\int_{R^n}\int_0^t e^{-2\gamma t}\, Pu\,\overline{Qu}\,dx\,dy\,dt, \tag{4.1}$$

where Q is a first order operator which satisfies $\tilde{G}_x(P, Q)\geq 0$ and $\tilde{G}_t(P, Q)>0$. Here $\tilde{G}_x(P, Q)$ and $\tilde{G}_t(P, Q)$ are explained below in §4.1 for simplicity in the case of $P=\square$. We show the actual integration by parts in the last part of §6, to obtain the estimate (1.1).

§4.1. Green formula associated with the bouudary condition

As in §6.3, in [3] to $\mathscr{G}((0, t); P, Q; u)$ there corresponds the following symbolic calculus:

$$\begin{aligned} G(P, Q)&=P(\xi, \eta, \tau)Q(\zeta, \eta, \bar{\tau})-Q(\xi, \eta, \tau)P(\zeta, \eta, \bar{\tau}) \\ &=(\xi-\zeta)G_x(P, Q)-(\tau-\bar{\tau})G_t(P, Q). \end{aligned} \tag{4.2}$$

Here we consider the case of $P=\square$. $G_x(\square, Q)$ and $G_t(\square, Q)$ are quadratic forms in (ξ, z_1, z_2), where z_1 and z_2 are $z_1=\tau-|\eta|$ and $z_2=\tau+|\eta|$ respectively. Then $\sum_{i=1}^{n} b_j\eta_j - c\tau = -\frac{1}{2}(\alpha z_1+\beta z_2)$, and the boundary condition is $D_x u|_{x=0}=\frac{1}{2}(\alpha z_1+\beta z_2)(D)u|_{x=0}+g$. Hence we substitute $\frac{1}{2}(\alpha z_1+\beta z_2)$ into ξ in $G_x(\square, Q)$ and $\frac{1}{2}(\bar\alpha \bar z_1+\bar\beta \bar z_2)$ into ζ in $G_x(\square, Q)$, then $G_x(\square, Q)$ becomes an Hermite form $G'_x(\square, Q)$ in (z_1, z_2). Denote the anti-symmetric part of $G'_x(\square, Q)$ by $i\operatorname{Im} G(\square, Q)_{1,2}(z_1\bar z_2 - z_2\bar z_1)$ and notice that

$$(\xi-\zeta)\{z_1\bar z_2-z_2\bar z_1\}=-(\tau-\bar\tau)\ \{\xi\bar z_1-\xi\bar z_2-z_1\zeta+z_2\zeta\}$$

follows from $z_1\bar z_2-z_2\bar z_1=-(\tau-\bar\tau)(z_1-z_2)=-(\tau-\bar\tau)(\bar z_1-\bar z_2)$. Then we have

$$G(\square, Q)=(\xi-\zeta)\tilde G_x(\square, Q)-(\tau-\bar\tau)\tilde G_t(\square, Q), \tag{4.3}$$

where $\tilde G_x(\square, Q)$ is a symmetric part of $G'_x(\square, Q)$, and $\tilde G_t(\square, Q)$ is an Hermite form:

$$\tilde G_t(\square, Q)=G_t(\square, Q)+i\operatorname{Im} G(\square, Q)_{1,2}\{\xi\bar z_1-\xi\bar z_2-z_1\zeta+z_3\zeta\}, \tag{4.4}$$

to which corresponds an Hermite matrix.

Here we choose Q as a polynomial in ξ and τ, with coefficients depending on (x, y, t, η), in order to prove the estimate of type (1.1) and (E). Therefore we can use only the following type of localization.

§4.2. The localization for the estimate (1.1) and (E)

Let us consider the partition of unity of type

$$\sum_{j=1}^{\text{finite}} \varphi_j(x, y, t, \eta)=1 \quad \text{on} \quad \overline{R^1_+}\times R^n\times\overline{R^1_+}\times(R^n-0),$$

where φ_j are sufficiently smooth and homogeneous in η. Corresponding to the above partition of unity we have the localizations of the function u: $\varphi_j(D)u=\varphi_j u=\bar F_y\varphi_j F_y u$ such that $u=u(x, y, t)=\Sigma(\varphi_j u)(x, y, t)$. We can take φ_j so that the oscillations of α and β are arbitrary small on the support of one φ_j by making the number of $\{\varphi_j\}$ larger if necessary. By this property we can choose $Q=Q_j$ for each φ_j such that $\tilde G_x(P, Q)\geq 0$ and $\tilde G_t(P, Q)>0$ on the support φ_j, that is shown in the next section.

§5. The choice of Q in the case of wave equation

In this section we show the proof of Theorem 1 in the case of $P=\square$

and $\Omega=R_+^{n+1}$. For the purpose we prove $\tilde{G}_x(\square, Q)\geq 0$ and $\tilde{G}_t(\square, Q)>0$ for the following Q:

(I) $Q=(\alpha_1 z_1+\beta_1 z_2)+\varepsilon(z_1+cz_2-d\xi)$, if the support φ_j contains any point satisfying $\alpha_1\beta_1=0$.

(II) $Q=(\alpha_1 z_1+\beta_1 z_2)-\varepsilon(2\xi-c_1 z_1-c_2 z_2)$, in other cases, where ε is a sufficiently small positive number, and c_1, c_2 and c are positive functions in (y, x, η). Here we choose these as follows: c_1 and c_2 are determined in the domain where $\alpha_1\beta_1 \neq 0$, by the identity, (See (3.5).),

$$2(\alpha_1 z+\beta_1)(c_1 z+c_2)=|\alpha|^2 z^2+2(2+\operatorname{Re}\alpha\bar{\beta})z+|\beta|^2+(\det A)/\alpha_1^2 \tag{5.1}$$

$$=f(z)+(\det A)/\alpha_1^2\,.$$

c is an arbitrary positive function defined in a neighbourhood of the points where $\alpha_1\beta_1=0$, such that

$$\varepsilon_1<c<\frac{1}{\varepsilon_2}\,, \tag{5.2}$$

which is assured by $1+\alpha_2\beta_2>0$. (See Lemma 3.1, (iii).) We put

$$d=4(d_1 X+d_2 Y), \tag{5.3}$$

where

$$(5.3)_1 \qquad \begin{cases} X=(\beta_1-\alpha_1\varepsilon_1)/\rho(1-\varepsilon_1\varepsilon_2), \\ Y=(\alpha_1-\beta_1\varepsilon_2)/\rho(1-\varepsilon_1\varepsilon_2), \end{cases}$$

and

$$(5.3)_2 \qquad \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}=\frac{1}{(1-\varepsilon_1\varepsilon_2)}\begin{pmatrix} 1 & -\varepsilon_2 \\ -\varepsilon_1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ c \end{pmatrix}.$$

Here ε_1, ε_2 and ρ are defined by

$$f(z)=|\alpha|^2 z^2+2(2+\operatorname{Re}\alpha\bar{\beta})z+|\beta|^2=\rho(z+\varepsilon_1)(1+\varepsilon_2 z). \tag{5.4}$$

From now on we exhibit the positiveness of $\tilde{G}_t(\square, Q)$ and non-negativeness of $\tilde{G}_x(\square, Q)$, together with the reason why we choose Q as above.

As for $\tilde{G}_x$ we notice the following simple lemma:

Lemma 5.1. *The symmetric quadratic form* $az_1\bar{z}_1+b(z_1\bar{z}_2+z_2\bar{z}_1)+cz_2\bar{z}_2$ *is non-negative if and only if* $ax^2+2bx+c\geq 0$ *for all real* x.

This lemma is trivial in the proof but is useful in process of choosing Q below.

First we recall §4.1 and collect the followings:

$$(5.5) \qquad \tilde{G}_x(P, z_i)=\frac{1}{2}(\alpha_1 z_1+\beta_1 z_2)\bar{z}_i+z_i\cdot\frac{1}{2}(\alpha_1\bar{z}_1+\beta_1\bar{z}_2), \qquad (i=1, 2),$$

$$(5.6) \qquad \tilde{G}_x(P, \xi)=\frac{1}{4}(\alpha z_1+\beta z_2)(\overline{\alpha z_1+\beta z_2})+\frac{1}{2}(z_1\bar{z}_2+z_2\bar{z}_1)$$

$$-\frac{i}{4}\mathrm{Im}\,(\alpha\bar{\beta})(z_1\bar{z}_2-z_2\bar{z}_1)$$

$$=\frac{1}{4}\{|\alpha|^2 z_1\bar{z}_1+(2+\mathrm{Re}\,\alpha\bar{\beta})(z_1\bar{z}_2+z_2\bar{z}_1)+|\beta|^2 z_2\bar{z}_2\},$$

$$(5.7) \qquad \tilde{G}_t(P, z_i)=\xi\zeta+z_i\bar{z}_i+\frac{\sqrt{-1}}{2}(\delta_{2i}\alpha_2-\delta_{1i}\beta_2)\{\xi\bar{z}_1-\xi\bar{z}_2-z_1\zeta+z_2\zeta\},$$

$$(i=1, 2), \qquad (\delta_{ij}=0, \text{ if } i\neq j, \ =1, \ \text{ if } i=j),$$

$$(5.8) \qquad \tilde{G}_t(P, \xi)=\frac{1}{2}(\xi\bar{z}_1+\xi\bar{z}_2+z_1\zeta+z_2\zeta)+\frac{i}{4}\mathrm{Im}\,(\alpha\bar{\beta})(\xi\bar{z}_1-\xi\bar{z}_2-z_1\zeta+z_2\zeta).$$

These relations are shown by (4.2), (4.3) and (4.4) combined with (7.4) in [3].

§5.1. The case (II)

Now we show how to choose Q in the case of (II) in several steps. The guiding principle is the above Lemma 5.1 concerning $\tilde{G}_x(P, Q)$.

First step. First consider $Q_0=\alpha_1 z_1+\beta_1 z_2$. Then we have from (5.5) and (5.7)

$$(5.9) \qquad \tilde{G}_x(P, Q_0)=(\alpha_1 z_1+\beta_1 z_2)(\alpha_1\bar{z}_1+\beta_1\bar{z}_2)\geq 0,$$

$$(5.10) \qquad \tilde{G}_t(P, Q_0)=(\alpha_1+\beta_1)\xi\zeta+\alpha_1 z_1\bar{z}_1+\beta_1 z_2\bar{z}_2$$

$$+\frac{i}{2}\mathrm{Im}\,(\alpha\bar{\beta})\{\xi\bar{z}_1-\xi\bar{z}_2-z_1\zeta+z_2\zeta\}$$

$$=(\xi, z_1, z_2)\begin{pmatrix}\alpha_1+\beta_1 & i\sigma & -i\sigma\\ -i\sigma & \alpha_1 & 0\\ i\sigma & 0 & \beta_1\end{pmatrix}\begin{pmatrix}\zeta\\ \bar{z}_1\\ \bar{z}_2\end{pmatrix}, \quad \text{where } \sigma=\frac{\mathrm{Im}\,(\alpha\bar{\beta})}{2},$$

which is positive definite if and only if $\frac{1}{4}\det A=\alpha_1\beta_1-\sigma^2>0$, from Lemma 3.3. (the case of uniform Lopatinski condition.) However we must consider the case up to $\det A\geq 0$.

Second step. By the way we consider $Q_1=a_1z_1+a_2z_2$ similarly, where a_1 and a_2 are real. We have from (5.5) and (5.7)

$$\tilde{G}_x(P, Q_1)=\frac{1}{2}(\alpha_1z_1+\beta_1z_2)(a_1\bar{z}_1+a_2\bar{z}_2)+\frac{1}{2}(a_1z_1+a_2z_2)(\alpha_1\bar{z}_1+\beta_1\bar{z}_1) \tag{5.11}$$

$$\tilde{G}_t(P, Q_1)=(\xi, z_1, z_2)\begin{pmatrix} a_1+a_2 & i\sigma' & -i\sigma' \\ -i\sigma' & a_1 & 0 \\ i\sigma' & 0 & a_2 \end{pmatrix}\begin{pmatrix} \zeta \\ \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}, \tag{5.12}$$

where $\sigma'=\frac{1}{2}(a_2\alpha_2-a_1\beta_2)$.

Hence $\tilde{G}_x(P, Q_1)$ is not positive semi-definite unless $\frac{a_1}{\alpha_1}=\frac{a_2}{\beta_1}$. $\tilde{G}_t(P, Q_1)$ is positive if and only if $a_1>0, a_2>0$ and $a_1a_2-\sigma'^2>0$. Put $\frac{a_2}{a_1}=a$. Then $\frac{4}{a_1^2}(a_1a_2-\sigma'^2)\geq 0$ equals

$$\alpha_2^2a^2-2(2+\alpha_2\beta_2)a+\beta_2^2\leq 0. \tag{5.13}$$

The positive solution a of (5.13) exists if and only if $1+\alpha_2\beta_2\geq 0$. (Equalities hold simultaneously.) In the case II, $1+\alpha_2\beta_2$ may be zero, then Q_1 does not play the desired role by itself. Thus we must take account of ξ.

Third step. Let us consider $G(P, \xi)$. In view of Lemma 5.1 we associate, to $\tilde{G}_x(P, \xi)$ in (5.6), the polynomial $f(x)=|\alpha|^2x^2+2(2+\mathrm{Re}\,\alpha\bar{\beta})x+|\beta|^2$, which was already appeared in Lemma 3.2. As for $\tilde{G}_t(P, \xi)$ we have from (5.8)

$$\tilde{G}_t(P, 2\xi)=(\xi, z_1, z_2)\begin{pmatrix} 0 & 1+i\sigma & 1-i\sigma \\ 1-i\sigma & 0 & 0 \\ 1+i\sigma & 0 & 0 \end{pmatrix}\begin{pmatrix} \zeta \\ z_1 \\ z_2 \end{pmatrix}, \tag{5.8$'$}$$

which is in a special subclass of the type A_3 in Lemma 3.3. This matrix does not contribute the positiveness by itself, but by $Q_3=Q_0-2\varepsilon\xi$ with small positive ε, $G_t(P, Q_3)$ becomes positive even if $\tilde{G}_t(P, Q_0)\geq 0$ as is shown below. In view of Lemma 3.3, that proof is reduced to show

$$\begin{pmatrix} \alpha_1 & \sigma-\varepsilon(\sigma-i) \\ \sigma-\varepsilon(\sigma+i) & \beta_1 \end{pmatrix}>0,$$

which equals $\alpha_1\beta_1-|\sigma-\varepsilon(\sigma-i)|^2>0$ since $\alpha_1>0$ and $\beta_1>0$. In Gaussian plane we can see easily that

$$\sigma^2>|\sigma-\varepsilon(\sigma-i)|^2$$

holds for small positive ε if σ is real and not zero. Therefore we can find a positive ε such that $\alpha_1\beta_1-|\sigma-\varepsilon(\sigma-i)|^2>0$ holds in the domain (of $R^n\times R^1\times S^{n-1}\ni(y,t,\eta)$) where $\alpha_1\beta_1>\delta_0>0$ is satisfied with given δ_0.

Fourth step. On the other hand $\tilde{G}_x(P,-2\xi)$ is not non-negative. We want to show that $\tilde{G}_x(P,-2\xi+c_1z_1+c_2z_2)$ becomes non-negative for the positive functions c_i introduced before in (5.1). First recall Lemma 3.2 and (3.5), then we see that $f(x)+\frac{1}{\alpha_1^2}\det A=0$ has a smooth root $-\frac{\beta_1}{\alpha_1}$. Therefore another root of $f(x)+\frac{1}{\alpha_1^2}\det A$ is negative smooth function even if $f(x)$ has double roots. (Remark that the roots of $f(x)=0$ are not smooth in general if $1+\alpha_2\beta_2=0$.)

Thus we have (5.1) with positive smooth functions c_1 and c_2. Now from (5.1), (5.6) and (5.11), the polynomial $-\frac{1}{2}f(x)+(\alpha_1x+\beta_1)(c_1x+c_2)=(\det A)/2\alpha_1^2$ corresponds to $\tilde{G}_x(P,-2\xi+c_1z_1+c_2z_2)$. Hence $\tilde{G}_x(P,-2\xi+c_1z_1+c_2z_2)\geq 0$ in view of Lemma 5.1.

On the other hand let us prove $\tilde{G}_t(P,c_1z_1+c_2z_2)\geq 0$. From (5.12) with $a_i=c_i$ $(i=1,2)$, $\tilde{G}_t(P,c_1z_1+c_2z_2)\geq 0$ equals $4c_1c_2-(c_2\alpha_2-c_1\beta_2)^2\geq 0$. Hence let us remark

$$4x-(\alpha_2x-\beta_2)^2=-\alpha_2^2x^2+2(2+\alpha_2\beta_2)x-\beta_2^2=-f(-x)+(\alpha_1x-\beta_1)^2\,. \tag{5.14}$$

Put $x=\frac{c_2}{c_1}$, then from (5.1) $-f\left(-\frac{c_2}{c_1}\right)=(\det A)/\alpha_1^2$ holds. Hence

$$4c_1c_2-(c_2\alpha_2-c_1\beta_2)^2=\frac{c_1^2}{\alpha_1^2}\left\{(\det A)+\alpha_1^4\left(\frac{c_2}{c_1}-\frac{\beta_1}{\alpha_1}\right)^2\right\}\geq 0\,,$$

which equals zero from (5.1) and Lemma 3.2 if and only if $1+\alpha_2\beta_2=0$.

Combining the above steps we have $\tilde{G}_x(P,Q)\geq 0$ and $\tilde{G}_t(P,Q)>0$ in the case (II). Now we turn to the case (I) where $\alpha_1\beta_1$ is so small that $1+\alpha_2\beta_2>0$ from Lemma 3.1.

§5.2. The case (I)

From $1+\alpha_2\beta_2>0$ we can choose a positive function c satisfying

$$-\alpha_2^2c^2+2(2+\alpha_2\beta_2)c-\beta_2^2>0\,. \tag{5.15}$$

Recall the second step in §5.1, then we see $\tilde{G}_t(P,z_1+cz_2)>0$. At that time

$f(x)$ has real distinct roots which are non-positive. Then as in (5.4) we can denote $f(x)=\rho(x+\varepsilon_1)(1+\varepsilon_2 x)$ by smooth functions ρ, ε_1 and ε_2, where ρ is a positive functions, and ε_1 is non-negative such that

$$\varepsilon_1<\frac{1}{\varepsilon_2}(=\varepsilon_2').$$

Notice that $-\varepsilon_1$ and $-\dfrac{1}{\varepsilon_2}$ are two roots of $f(x)=0$, then we have from (5.14) that (5.15) holds if c satisfies (5.2).

First step. Now we state

Proposition 5.2. *Assume the condition* (*H*). *Then we can choose smooth functions X and Y such that for*

$$(5.16) \qquad Q_4=z_1+\varepsilon_1 z_2-4X\xi, \quad Q_5=z_2+\varepsilon_2 z_1-4Y\xi,$$

$\tilde{G}_x(P, Q_j)\geq 0$, $(j=4, 5)$ *if* $\alpha_1\beta_1$ *is sufficiently small.*

Proof. From (3.3) and (3.5) we have always

$$(5.17) \qquad \alpha_1-\beta_1\varepsilon_2\geq 0 \quad \text{and} \quad \beta_1-\alpha_1\varepsilon_1\geq 0,$$

because $\alpha_1\geq 0$ and $\beta_1\geq 0$. To $\tilde{G}_x(P, Q_4)$ we associate

$$h(x)=(\alpha_1 x+\beta_1)(x+\varepsilon_1)-Xf(x)$$

from (5.4), (5.5), (5.6) and Lemma 5.1. Now we put

$$(5.3)_1' \qquad X=(\beta_1-\alpha_1\varepsilon_1)/f'(-\varepsilon_1)=(\beta_1-\alpha_1\varepsilon_1)/\rho(1-\varepsilon_1\varepsilon_2)\geq 0.$$

Then $h(-\varepsilon_1)=h'(-\varepsilon_1)=0$. Since $h\left(-\dfrac{\beta_1}{\alpha_1}\right)=-Xf\left(-\dfrac{\beta_1}{\alpha_1}\right)\geq 0$ from (3.5) if $\alpha_1\neq 0$, we have $h(x)\geq 0$ for all real x.[11] This means $\tilde{G}_x(P, Q_4)\geq 0$ by virtue of Lemma 5.1. Similarly putting $Y=(\beta_1-\alpha_1\varepsilon_2)/\rho(1-\varepsilon_1\varepsilon_2)$ as in $(5.3)_1$, we have $\tilde{G}_x(P, Q_5)\geq 0$. q.e.d.

Second step. We denote by d_1 and d_2 the solution of $d_1+\varepsilon_2 d_2=1$ and $\varepsilon_1 d_1+d_2=c$, which were given in $(5.3)_2$. Remark that they are positive from (5.2). Since

$$z_1+cz_2-d\xi=d_1 Q_4+d_2 Q_5$$

11) If $\alpha_1>0$ and $\varepsilon_1=\beta_1/\alpha_1$, then $h=(\alpha_1 z+\beta_1)^2/\alpha_1\geq 0$. If $\alpha_1=0$ and $\beta_1\neq 0$, then we have $\alpha_2=0$ from Lemma 3.1 and h is at most of degree one, thus $h=0$.

holds, where $d=4(d_1X+d_2Y)$, we see $\tilde{G}_x(P, z_1+cz_2-d\xi)\geq 0$ by Proposition 5.2. Hence $\tilde{G}_x(P, Q)\geq 0$.

Now let us prove $\tilde{G}_t(P, Q)>0$. We notice that

$$(5.18) \qquad 0\leq d\leq C\max\{\alpha_1, \beta_1\}\leq C(\alpha_1+\beta_1)$$

holds from $(5.3)_1$, with some constant C. Recall Lemma 3.3, then from (5.10), (5.8)′ and the positiveness of $\tilde{G}_t(P, z_1+cz_2)$ we see that $\tilde{G}_t(P, Q)>0$ is equivalent to

$$(5.19) \qquad M=\begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}+\varepsilon\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}-\varepsilon d\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}>0,$$

where p is a positive constant. This is true if we confine ourselves to the points satisfying $\alpha_1\beta_1=0$. $M>0$ follows from (5.18) for small positive ε. Thus we see $\tilde{G}_t(P, Q)>0$ in a neighbourhood of $\alpha_1\beta_1=0$.

§6. Symbolic calculus in the general case

In this section we extend the results in the precedent section to the problem for regularly hyperbolic operator P and in the quarter space $R^1_+\times R^n\times R^1_+$, $(\ni(x, y, t))$.

Let P be

$$(6.1) \qquad P=-aD_t^2+2(\sum_{i=1}^{n} a_iD_{y_i}+a_0D_x)D_t$$

$$+(\sum_{i,j=1}^{n} a_{ij}D_{y_i}D_{y_j}+2\sum_{j=1}^{n} a_{0_j}D_{y_j}D_x+D_x^2).$$

As in §2 of [3], we assume the followings:

$$(6.2)_1 \qquad a>0$$

$$(6.2)_2 \qquad D(\eta)=(a+a_0^2)\{(\sum_{i=1}^{n} a_i\eta_i)^2+a\sum_{i,j=1}^{n} a_{ij}\eta_i\eta_j\}$$

$$-(a_0\sum_{i=1}^{n} a_i\eta_i+a\sum_{j=1}^{n} a_{ij}\eta_i)^2>0, \qquad \text{for } |\eta|\neq 0.$$

We can express P as

$$(6.3) \qquad P=\tilde{\xi}^2-\tilde{\tau}^2+d(\eta)^2,$$

where $d(\eta)=\{a^{-1}(a+a_0^2)^{-1}D(\eta)\}^{1/2}$ and

$$(6.4)\quad \begin{cases} \tilde{\xi}=\xi+a_0\tau+\sum_{j=1}^{n} a_{0_j}\eta_j,\ \tilde{\zeta}=\zeta+a_0\bar{\tau}+\sum_{j=1}^{n} a_{0_j}\eta_j \\ \tilde{\tau}=(a+a_0^2)^{1/2}\{\tau-(a+a_0^2)^{-1}(\sum_{i=1}^{n} a_i\eta_i-a_0\sum_{j=1}^{n} a_{0_j}\eta_j)\}\equiv\tilde{\tau}(\eta,\tau)\,. \end{cases}$$

We define $\tilde{z}_1$ and $\tilde{z}_2$ by

$$(6.5)\quad \begin{cases} \tilde{z}_1=\tilde{\tau}-d(\eta)\,, \\ \tilde{z}_2=\tilde{\tau}+d(\eta)\,. \end{cases}$$

Then α and β satisfies

$$(6.6)\quad c\tau-\sum_{j=1}^{n} b_j\eta_j=\frac{1}{2}(\alpha\tilde{z}_1+\beta\tilde{z}_2)\,.$$

Here we explain the symbolic calculus for the regularly hyperbolic operator P, as extensions of those in §4.1.

$$(6.7)\quad G(P,\tilde{Q})\approx(\xi-\zeta)\tilde{G}_x(P,\tilde{Q})-(\tilde{\tau}-\bar{\tilde{\tau}})\tilde{G}_t(P,\tilde{Q})\,,$$

Here $\tilde{G}_x(P,\tilde{Q})$ is $\tilde{G}_x(\Box, Q)$ replaced ξ, z_1 and z_2 by $\tilde{\xi}$, $\tilde{z}_1$ and $\tilde{z}_2$, where $\tilde{Q}(\xi, z_1, z_2)=Q(\tilde{\xi}, \tilde{z}_1, \tilde{z}_2)$, and

$$(6.8)\quad \tilde{G}_t(P,\tilde{Q})=\{G_t(\Box,Q)-\delta G_x(\Box,Q)\}\,|_{\xi=\tilde{\xi},z_1=\tilde{z}_1,z_2=\tilde{z}_2}$$
$$+i\,(\mathrm{Im}\,G(\Box,Q)_{12})\{(\tilde{\xi}\bar{\tilde{z}}_1-\tilde{\xi}\bar{\tilde{z}}_2-\tilde{z}_1\zeta+\tilde{z}_2\zeta)+\delta(\tilde{z}_1\bar{\tilde{z}}_2-\tilde{z}_2\bar{\tilde{z}}_1)\},$$

where $\delta=a_0/(a+a_0^2)^{1/2}$, $|\delta|<1$. These are verified from the argument in §4.1, if we remark

$$(6.9)\quad \begin{aligned} &(1)\quad \tilde{\tau}-\bar{\tilde{\tau}}=(a+a_0^2)^{1/2}(\tau-\bar{\tau})\,,\\ &(2)\quad \tilde{\xi}-\tilde{\zeta}=(\xi-\zeta)+\delta(\tilde{\tau}-\bar{\tilde{\tau}})\,, \end{aligned}$$

and the following relation

$$(6.10)\quad (\xi-\zeta)\{\tilde{z}_1\bar{\tilde{z}}_2-\tilde{z}_2\bar{\tilde{z}}_1\}=-(\tilde{\tau}-\bar{\tilde{\tau}})\,\{(\tilde{\xi}\tilde{z}_1-\tilde{\xi}\bar{\tilde{z}}_2-\tilde{z}_1\tilde{\zeta}+\tilde{z}_2\tilde{\zeta})+\delta(\tilde{z}_1\bar{\tilde{z}}_2-\tilde{z}_2\bar{\tilde{z}}_1)\}\,,$$

which is obtained by (6.9) (2) and

$$\tilde{z}_1\bar{\tilde{z}}_2-\tilde{z}_2\bar{\tilde{z}}_1=-(\tilde{\tau}-\bar{\tilde{\tau}})\tilde{z}_1+(\tilde{\tau}-\bar{\tilde{\tau}})\tilde{z}_2=-(\tilde{\tau}-\bar{\tilde{\tau}})\bar{\tilde{z}}_1+(\tilde{\tau}-\bar{\tilde{\tau}})\bar{\tilde{z}}_2\,.$$

Now we take $\tilde{Q}=Q(\tilde{\xi}, \tilde{z}_1, \tilde{z}_2)$ in §5, namely

(I) $\tilde{Q}=\alpha_1\tilde{z}_1+\beta_1\tilde{z}_2+\varepsilon(\tilde{z}_1+c\tilde{z}_2-d\tilde{\xi})$, in the neighbourhood of $\alpha_1\beta_1=0$, and
(II) $\tilde{Q}=\alpha_1\tilde{z}_1+\beta_1\tilde{z}_2-\varepsilon(2\tilde{\xi}-c_1\tilde{z}_1-c_2\tilde{z}_2)$, in the domain $\alpha_1\beta_1\neq 0$. Then we can prove $\tilde{G}_x(P, \tilde{Q})\geq 0$ and $\tilde{G}_t(P, \tilde{Q})>0$ in the following way. Remark that $\tilde{G}_x(\square, Q)\geq 0$ means $\tilde{G}_x(P, \tilde{Q})\geq 0$. In §5 we have proved that $\tilde{G}_t(P, \hat{Q})>0$ if $\delta\equiv 0$ in (6.8). We see that $\tilde{G}_t(P, \tilde{Q})>0$ holds for $\delta\in(-1, 1)$, if we notice the following facts (i) and (ii).

(i) From (6.8), (6.10) and (5.5)~(5.8), (1, 1)-cofactor of the matrix corresponding to $\tilde{G}_t(P, \tilde{Q})$

$$=\begin{pmatrix}\alpha_1 & i\delta\sigma\\ -i\delta\sigma & \beta_1\end{pmatrix}+\varepsilon\begin{pmatrix}1 & i\delta\sigma'\\ -i\delta\sigma' & c\end{pmatrix}-\frac{1}{2}\varepsilon d\delta\begin{pmatrix}0 & -1+i\sigma\\ -1-i\sigma & 0\end{pmatrix}\quad \text{in case (I)},$$

$$=\begin{pmatrix}\alpha_1 & i\delta\sigma\\ -i\delta\sigma & \beta_1\end{pmatrix}-\varepsilon\delta\begin{pmatrix}0 & -1+i\sigma\\ -1-i\sigma & 0\end{pmatrix}+\varepsilon\begin{pmatrix}c_1 & i\delta\sigma''\\ -i\delta\sigma'' & c_2\end{pmatrix}\quad \text{in case (II)},$$

where $\sigma=\frac{1}{2}\mathrm{Im}(\alpha\bar{\beta})$, $\sigma'=\frac{1}{2}(\alpha_2c-\beta_2)$ and $\sigma''=\frac{1}{2}(\alpha_2c_1-\beta_2c_2)$. In each case (1, 1)-cofactor of $\tilde{G}_t(P, \tilde{Q})$ is positive, because even if $\delta=1$ it is proved to be positive in just the same way as the proof of $\tilde{G}_t(\square, Q)>0$ in the precedent section.

(ii) The determinant of the matrix corresponding to $\tilde{G}_t(P, \tilde{Q})$ is zero if $\delta=\pm 1$. In fact, as in the appendix A.3 of [3] the sum of the second and the third line vector of each matrix corresponding to

$$G_t(\square, z_i)\mp G_x(\square, z_i),\ (i=1, 2),\quad \text{and}\quad G_t(\square, \xi)\mp G_x(\square, \xi)$$

equals its first line vector. In fact

$$G_t(\square, z_i)=\begin{pmatrix}1 & 0 & 0\\ 0 & \delta_{i1} & 0\\ 0 & 0 & \delta_{i2}\end{pmatrix},\ G_x(\square, z_i)=\begin{pmatrix}0 & \delta_{i1} & \delta_{i2}\\ \delta_{i1} & 0 & 0\\ \delta_{i2} & 0 & 0\end{pmatrix},\qquad (i=1, 2),$$

$$G_t(\square, \xi)=\begin{pmatrix}0 & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & 0 & 0\\ \frac{1}{2} & 0 & 0\end{pmatrix},\ G_x(\square, \xi)=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & \frac{1}{2}\\ 0 & \frac{1}{2} & 0\end{pmatrix}.$$

Moreover we see the same property as above in the matrix corresponding to the second term of $\tilde{G}_t(P, \tilde{Q})$: $(\xi\bar{\tilde{z}}_1-\bar{\xi}\tilde{z}_2-\tilde{z}_1\bar{\zeta}+\tilde{z}_2\bar{\zeta})\pm(\tilde{z}_1\bar{\tilde{z}}_2-\tilde{z}_2\bar{\tilde{z}}_1)$, namely

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Corresponding to (6.7), we consider in the next section the integration by parts of (4.1), which lead us to the proof of Theorems.

§7. Proofs of Theorems 1 and 2

Now we prove Theorems 1 and 2. Theorem 1 follows directly from the integration by parts of $G((0, t), P, Q; \varphi_j u)$ in (4.1), using Q determined in the precedent section. On the other hand in order to prove the estimate in Theorem 2 we need to obtain the boundary estimate, and for this purpose we use an existence theorem with zero initial data of the dual problem as in [3]. All these arguments are similar to those in §9 of [3], so we state only the outline.

First to $G(P, Q)$ we associate the differential form

$$(7.1) \qquad G(P, Q; u) = i\,\mathrm{Im}\, P(D_x, D_y, D_t - i\gamma)e^{-\gamma t}u\,\overline{Q(D_x, D_y, D_t - i\gamma)e^{-\gamma t}u},$$

where we denote by $P(D_x, D_y, D_t)$ the regularly hyperbolic operator in (6.1). Then to the quadratic form $\tilde{G}_x(P, Q)$ and $\tilde{G}_t(P, Q)$ we associate the quadratic differential form, $\tilde{G}_x(P, Q; u)$ and $\tilde{G}_t(P, Q; u)$ respectively. The function u is assumed to belong to $C_0^\infty(\overline{R_+^{n+1} \times R^1})$. Corresponding to (6.7) we have

$$(7.2) \qquad G(P, Q; u) = D_x\tilde{G}_x(P, Q; u) - (D_t - 2i\gamma)(a + a_n^2)^{1/2}\tilde{G}_t(P, Q; u)$$
$$+ \mathscr{B}(P, Q, B; u) + R(P, Q; u),$$

where $\mathscr{B}(P, Q, B; u)$ is a quadratic differential form on the boundary such that $\mathscr{B}(P, Q, B; u) = 0$ if $Bu|_{x=0} = 0$, and $R(P, Q; u)$ satisfies

$$(7.3) \qquad |R(P, Q; u)| \leq C\Big(\sum_{i+j+k+|\alpha| \leq 1} |e^{-\gamma t}\gamma^i D_x^j D_t^k D_y^\alpha u|^2\Big).$$

§7.1. The proof of Theorem 1

First we remark $\tilde{G}_t(P, \tilde{Q}) > 0$ on the support of φ_j then we have the following relation:

$$(7.4) \qquad c \sum_{l+i+k+|\alpha|=1} |e^{-\gamma t}\gamma^l D_x^i D_t^k \widehat{D_y^\alpha \varphi_j u}|^2 \leq \tilde{G}_t(P, \tilde{Q}; \widehat{\varphi_j u})$$

for a positive constant c, where $\wedge$ means Fourien transform in y.

Now integrate (7.2) with $u = \varphi_j u$ in $(y, t) \in R^n \times R^1$, then we have from (7.1)

$\sim$(7.4) and $\tilde{G}_x(P, Q; \varphi_j u) \geq 0$,

$$\text{(7.5)} \qquad \left(\frac{\partial}{\partial t}+2\gamma\right)\iint \tilde{G}_t(P, \tilde{Q}; \varphi_j u)\mathrm{d}y\ \mathrm{d}x \leq \frac{C}{\gamma}\left(\iint |e^{-\gamma t}Pu|^2 \mathrm{d}y\ \mathrm{d}x\right)^{1/2} \times\left(\iint \tilde{G}_t(P, \tilde{Q}; \varphi_j u)\mathrm{d}y\ \mathrm{d}x\right)^{1/2}$$

Let us divide (7.5) by $\left(\iint \tilde{G}_t(P, \tilde{Q}; \varphi_j u)\mathrm{d}y\ \mathrm{d}t\right)^{1/2}$ then it follows

$$\text{(7.6)} \qquad 2\left(\frac{\partial}{\partial t}+\gamma\right)\left(\iint \tilde{G}_t(P, \tilde{Q}; \varphi_j u)\mathrm{d}y\ \mathrm{d}t\right)^{1/2} \leq \frac{C}{\gamma}\left(\iint |e^{-\gamma t}Pu|^2\ \mathrm{d}y\ \mathrm{d}x\right)^{1/2}.$$

Integrate (7.6) in $(0, t)$ and sum up in j, then we have Theorem 1 if we take care of (7.4).

§7.2. The proof of Theorem 2

We state the proof of Theorem 2 in the following several steps (See also §9 in [3].) At first let Ω be R_+^{n+1}. Later we mention to the case where Ω be a general domain.

1). First we show that the condition (H) is necessary for obtaining the estimate in Theorem 2. Let us take u in $C_0^\infty(\overline{R_+^{n+1}} \times R^1)$ such that $u \equiv 0$ in $t<0$, in the estimate (E) in Theorem 2. Then integrate it with respect to t, then we have

$$\text{(7.7)} \qquad \gamma_1(|u|_{1,\gamma_1}^2 + \sum_{j=0}^{1} < \Lambda_{y,\gamma_1}^{-\frac{1}{2}} D_x^j u >_{1-j,\gamma_1}^2) \leq \frac{C}{\gamma_1}\{|Pu|_{0,\gamma_1}^2 + < \Lambda_{y,\gamma_1}^{\frac{1}{2}} Bu >_{0,\gamma_1}^2\}, \quad \text{where } \sigma(\Lambda_{y,\gamma_1}) = (|\eta|^2+\gamma_1^2)^{1/2},$$

for some positive constant C. Thus in view of Theorem 2 in [3] we can see, after the same argument in §2, that the condition (H) is necessary for the estimate (E) in Theorem 2.

2). Next we show that the estimate (7.7) follows from (H). This is proved in the same way as the estimate in Theorem 1, if we add the following localization.

Since $B(y, t, \xi_+(0, 1), 0, 1) \neq 0$ for any (y, t), (See Theorem 1 in [3]), we can find C^∞ functions ψ_1 and ψ_2 satisfying

(1) $\psi_1+\psi_2=1$ on $R_+^{n+2} \times \Sigma \ni (x, y, t) \times (\eta, \tau)$, where $\Sigma = \{|\eta|^2+|\tau|^2=1, \operatorname{Im}\tau \leq 0\}$,

(2) ψ_1 vanishes in a neighbourhood of $(0, y, t, 0, 1) \in R_+^{n+2} \times \Sigma$, while the support of ψ_2 is contained in another neighbourhood of $(0, y, t, 0, 1)$, such that

$\mathscr{L}=B(y, t, \xi_+(\eta, \tau), \eta, \tau)$ does not vanish on the support of ψ_2.

To the homogeneous extensions of ψ_1 and ψ_2 we correspond the pseudo-differential operators with a parameter γ. (See §6 in [3]). Put $\psi_1(D)u=u_1$ and $\psi_2(D)u=u_2$ for any u in $C_0^\infty(\overline{R_+^{n+1}\times R^1})$. Assume that the operators P and B are extended to $t<0$, satisfying the condition (H).

From the support of ψ_1 we have

$$
(7.8)\qquad <\Lambda_{y,\gamma}^{-\frac{1}{2}}u_1>_{1,\gamma}^2 \leq \text{Const. } |u|_{1,\gamma}^2,
$$

if we remark that $<\Lambda_{y,\gamma}^{-\frac{1}{2}}u_1>_{1,\gamma} \leq \text{Const. } <\Lambda_{\gamma}^{-\frac{1}{2}}u_1>_{1,\gamma}$ holds for u_1 and that $<\Lambda_{\gamma}^{-\frac{1}{2}}u_1>_{1,\gamma}^2 \leq |u_1|_{1,\gamma}^2$ always holds, where $\sigma(\Lambda_{y,\gamma})=(|\eta|^2+\gamma^2)^{1/2}$ and $\Lambda_\gamma = e^{\gamma t}\bar{F}(|\eta|^2+|\tau|^2)^{1/2}Fe^{-\gamma t}$. Thus we have

$$
(7.9)\qquad \sum_{j=1}^{1} <\Lambda_{y,\gamma}^{-\frac{1}{2}}D_x^j u_1>_{1-j,\gamma}^2 \leq C(<\Lambda_{y,\gamma}^{-\frac{1}{2}}Bu_1>_{0,\gamma}^2+|u|_{1,\gamma}^2)
$$

Now we consider the quadratic differential form (7.2) replaced u by u_1, and integrate it in $R_+^{n+1}\times R^1$ using (7.9). Then we have for large γ

$$
(7.8)\qquad \gamma|u_1|_{1,\gamma}^2+\sum_{j=0}^{1} <\Lambda_{y,\gamma}^{-\frac{1}{2}}D_x^j u_1>_{1-j,\gamma}^2
$$
$$
\leq C\left\{\frac{1}{\gamma}|Pu_1|_{0,\gamma}^2+\frac{1}{\gamma}<\Lambda_{y,\gamma}^{\frac{1}{2}}Bu_1>_{0\gamma}^2+|u|_{1,\gamma}^2\right\}.
$$

As for u_2, by virtue of uniformity of $|\mathscr{L}|\neq 0$

$$
\gamma|u_2|_{1,\gamma}^2+\sum_{j=0}^{1} <D_x^j u_2>_{1-j,\gamma}^2
$$
$$
\leq C\left\{\frac{1}{\gamma}|Pu_2|_{0,\gamma}^2+<Bu_2>_{0,\gamma}^2+|u|_{1,\gamma}^2\right\},
$$

in the same way as Kreiss-Sakamoto. (See Appendix A.3 of [3] in our cases) Thus we have (7.7).

3). Here we show an existence theorem in the weak sense. Namely we have an existence theorem of the solution of

$$
(P)_0\qquad \begin{cases} Pu=f(x, y, t), & x>0,\ y\in R^n,\ t\in R^1, \\ Bu=g(y, t), & x=0,\ y\in R^n,\ t\in R^1. \end{cases}
$$

Assume $f\in\mathscr{H}_{k,\gamma}(R_+^{n+2})$ and $\Lambda_{y,\gamma}^{1/2}g\in\mathscr{H}_{k,\gamma}(R^{n+1})$ for $\gamma>\gamma_k$. Then there exists a solution u of $(P)_0$ belonging to $\mathscr{H}_{k+1,\gamma}(R_+^{n+2})$. This proof depends on the fact that the dual problem below also satisfies the condition (H), (See §9.1 of [3].)

For convenience we notice that $u \in \mathscr{H}_{k+1,\gamma}(R_+^{n+2})$ is proved by the interpolation theorem, in view of the following facts:
$u = \Lambda_\gamma^{-(k+1)} e^{2\gamma t} \Lambda_\gamma'^{-(k+1)} P^* w$ and

(i) $e^{2\gamma t} \Lambda_\gamma'^{-(k+1)} w \in \mathscr{H}_{1,\gamma}(R_+^{n+2})$, where Λ_γ' is the adjoint of Λ_γ,

(ii) $D_x^2 e^{2\gamma t} \Lambda_\gamma'^{-(k+1)} w \in \mathscr{H}_{0,\gamma}(R_+^{n+2})$, because of (i) and $\Lambda_\gamma'^{-(k+1)} P^* w \in \mathscr{H}_{0,-\gamma}(R_+^{n+2})$.

(iii) $Pu = f \in \mathscr{H}_{k,\gamma}(R_+^{n+2})$.

4). Now we have the following energy inequality for the solution of $(P)_0$: For any interval $[s_0, t_0]$ we have

$$\gamma |u|_{1,\gamma,(s_0,t_0)}^2 + \gamma \sum_{j=0}^{1} < \Lambda_{\gamma,\gamma}^{-1/2} D_x^j u >_{1-j,\gamma,(s_0,t_0)}^2 + \sum_{j=0}^{1} \|(D_t^j u)(t_0)\|_{1-j,\gamma}^2 \tag{7.9}$$

$$\leq \frac{C}{\gamma} \{ |f|_{0,\gamma,(s_0,t_0)}^2 + < \Lambda_{\gamma,\gamma}^{1/2} > g >_{0,\gamma,(s_0,t_0)}^2 \},$$

if the supports of f and g in t are in $[s_0, \infty)$.

Proof. If $f \equiv 0$ and $g \equiv 0$ in $(-\infty, t_1]$, then tending γ to ∞ in (7.7), we have $u \equiv 0$ in $(-\infty, t_1)$. Remark that no singularity appears in the right hand of (7.7), even if we replace f and g in $[t_1, \infty)$ by zeros. Hence we have (7.9) except the third term in the left hand-side if we put $t_1 = t_0$. The estimate of that term is obtained by the integration by parts

$$(Pu, c\, Qu)_{0,\gamma,(s_0,t_1)} - (c\, Qu, Pu)_{0,\gamma,(s_0,t_1)},$$

where Q is the operator defined in the previous section and c is a small positive constant. q.e.d.

5). Here we state some properties of the dual problem for $\{P, B\}$. Let u and v be in $C_0^\infty(\overline{R_+^{n+2}})$ and let P^* be the formal adjoint operator of P. Then the first order boundary operator B' is uniquely determined such that

$$(Pu, v) - (u\, P^* v) = i\{ < Bu, v > + < u, B'v > \}, \tag{7.10}$$

where (,) and < , > are L^2 norms in R_+^{n+2} and R^{n+1} respectively. Now by $\tilde{P}^*$ and $\tilde{B}'$ we denote the operator P^* and B' replaced D_t and t by $-D_t$ and $-t$. Then $\{\tilde{P}^*, \tilde{B}'\}$ satisfies the condition (H) if and only if $\{P, B\}$ does so as in Lemma 9.1 in [3]. Therefore we have an existence theorem concerning the problem

$$(P^*)_0 \begin{cases} P^*v = \psi, & x>0,\ y\in R^n,\ t\in R^1,\ \psi\in C_0^\infty(\overline{R_+^{n+2}}), \\ B'v=\varphi, & x=0,\ y\in R^n,\ t\in R^1,\ \varphi\in C_0^\infty(R^{n+1}). \end{cases}$$

And from (7.9) we have the estimate

$$(7.9)' \qquad \gamma|v|_{1,-\gamma,(s_0,t_0)}^2+\gamma\sum_{j=0}^{1}<\Lambda_{y,\gamma}^{-1/2}D_x^jv>_{1-j,-\gamma,(s_0,t_0)}^2$$

$$+\sum_{j=0}^{1}\|(D_t^jv)(s_0)\|_{1-j,-\gamma}^2\le\frac{C}{\gamma}\{|\psi|_{0,-\gamma,(s_0,t_0)}^2+<\Lambda_{y,\gamma}^{1/2}\varphi>_{0,-\gamma,(s_0,t_0)}^2\}.$$

6). Now by means of the results in the previous steps we can prove Theorem 2. In order to obtain the estimate for the second term of the energy inequality (E), we consider the following identity for smooth functions u and v:

$$(7.11) \qquad (Pu,\ Dv)_{(0,t)}-(Du,\ P^*v)_{(0,t)}$$

$$=i\{<Bu,\ Dv>_{(0,t)}+<Du,\ B'v>_{(0,t)}\}+R(u,\ v)+I[u,\ v],$$

where D means D_t or D_{y_j}, $(j=1,\dots,n)$. Here we have

$$(7.12) \qquad |R(u,\ v)|\le C|u|_{1,\gamma,(0,t)}|v|_{1,-\gamma,(0,t)},$$

$$|I[u,\ v]|\le C[u(0)]_{1,\gamma}[v(0)]_{1,-\gamma}, \qquad \text{where } [u(0)]_{k,\gamma}=\sum_{j=0}^{1}\|(D_t^ju)(0)\|_{k-j,\gamma}$$

Let us take v as the solution of the problem

$$\begin{cases} P^*v=0 \\ B'v=\varphi(y,\ t), \end{cases}$$

with the boundary data $\varphi(y, t)$ in $\mathscr{D}(R^n\times(0, t))$. The solution $v(s)$ vanishes in $t\le s$ and satisfies (7.9)' with $s_0=0$ and $t_0=t$. Therefore form (7.11) and (7.12) we have

$$|<Du,\ \varphi>_{(0,t)}|\le C\gamma^{-1/2}<\Lambda_{y,\gamma}^{1/2}\varphi>_{0,-\gamma,(0,t)}\{F+|u|_{1,\gamma,(0,t)}\},$$

$$\text{where } F=\left\{\frac{1}{\gamma}|Pu|_{0,\gamma,(0,t)}^2+\frac{1}{\gamma}<\Lambda_{y,\gamma}^{1/2}\ Bu>_{0,\gamma,(0,t)}^2+[u(0)]_{1,\gamma}^2\right\}^{1/2}.$$

Thus we have

$$\gamma<\Lambda_{y,\gamma}^{-1/2}\ Du>_{0,\gamma,(0,t)}^2\le C(F^2+|u|_{1,\gamma,(0,t)}^2)$$

Hence we have

$$(7.13)\qquad \gamma\sum_{j=0}^{1}<\Lambda_{y,\gamma}^{-1/2}D_x^j u>_{1-j,\gamma,(0,t)}^2 \le C(F^2+|u|_{1,\gamma,(0,t)}^2).$$

Moreover from the integration by parts of

$$(Pu,\ Qu)_{0,\gamma,(0,t)}-(Qu,\ Pu)_{0,\gamma,(0,t)}$$

we have

$$(7.14)\qquad \gamma|u|_{1,\gamma,(0,t)}^2+[u(t)]_{1,\gamma}^2 \le C\{F^2+\gamma\sum_{j=0}^{1}<\Lambda_{y,\gamma}^{-1/2}D_x^j u>_{1-j,\gamma,(0,t)}^2\}.$$

By (7.13) and (7.14) we have the energy estimate (E) in Theorem 2.

In the similar way, (cf. (9.11) in [3]), for any smooth function u with compact support, we have the estimate

$$(E)'\qquad \gamma|u|_{k,\gamma,(0,t)}^2+\gamma\sum_{j=0}^{1}<\Lambda_{y,\gamma}^{-1/2}D_x^j u>_{k-j,\gamma,(0,t)}+[u(t)]_{k,\gamma}^2$$
$$\le C\left\{\frac{1}{\gamma}|Pu|_{k-1,\gamma,(0,t)}^2+\frac{1}{\gamma}<\Lambda_{y,\gamma}^{1/2}Bu>_{k-1,\gamma,(0,t)}^2+[u(0)]_{k,\gamma}^2\right\}.$$

7). The existence theorem follows from the above energy estimate in the same way as §9.5 in [3], if we consider two existence theorems concerning a suitable Cauchy problem and the boundary value problem (P_0). We can show the existence theorem in the general cylindrical domain and the finiteness of the propagation speed since the condition (H) is invariant under the space-like transformation, as in §10 of [3].

Appendix

§A.1. Proof of Lemma 2.4.

For simplicity let us rewrite $(x, y_1, y_2,\dots, y_n, t)$ by $(x_0, x_1,\dots, x_n, x_{n+1})$, which we denote also by (x_0, x'). $((x_0, x')\in\overline{R_+^{n+2}},\ x_0\ge 0,\ x'\in R^{n+1}.)$ We denote the boundary operator B as

$$B=\mathrm{Re}\,B+i\,\mathrm{Im}\,B,$$

where
$$\begin{cases}\mathrm{Re}\,B=D_{x_0}+\sum_{j=1}^{n+1}\mathrm{Re}\,b_n(x')D_{x_j}, & (b_{n+1}(x')=-c(x'))\\ \mathrm{Im}\,B=\sum_{j=1}^{n+1}\mathrm{Im}\,b_j(x')D_{x_j}.\end{cases}$$

Associating to the boundary operator $\mathrm{Re}\,B$, we consider the C^∞ mapping $\Phi=(\Phi_0, \Phi_1,\dots, \Phi_n, \Phi_{n+1})$ from R^{n+2} to R^{n+2} such that
(1) Φ is diffeomorphic in a neighbourhood ω of origin to another neighbourhood ω' of origin. Moreover Φ maps $\omega\cap\overline{R_+^{n+2}}$ into $\overline{R_+^{n+2}}$. (So later we use Φ as the mapping restricted in $\overline{R_+^{n+2}}$.)
(2) $\Phi(0, x')=(0, x')$ and the Jacobi matrix at the origin is the identity.
(3) $\mathrm{Re}\,B_0u=\mathrm{Re}\,B(u(\Phi(x)))$ at any point $(0, x')\in\omega$, for any smooth function $u \in C_0^\infty(\overline{R_+^{n+2}})$.

Now we define $\tilde{u}_\varepsilon(x)=u\left(\frac{1}{\varepsilon}\Phi(\varepsilon x)\right)$. Then we can see that $\tilde{u}_\varepsilon(x)$ tends to $u(x)$ uniformly up to all derivatives if $u(x)$ is smooth and bounded function with compact support. Here remark that if B has real coefficients then $\tilde{u}_\varepsilon(x)$ satisfies the desired properties for given function $u(x)$. In fact, since (3) means

$$\sum_{j=0}^{n+1} b_j(0)(D_{x_j}u)(0, x')=\sum_{j=0}^{n+1} b_j(x')\sum_{i=0}^{n+1}\left(\frac{\partial\Phi_i}{\partial x_j}\right)(x')(D_{x_i}u)(0, x'),$$

where $b_0(x')=1$, it follows

$$(3)' \qquad \sum_{j=0}^{n+1}\frac{\partial\Phi_i}{\partial x_j}(x')b_j(x')=b_i(0) \qquad \text{for any } x'\in\omega\,|_{x_0=0}$$

If we assume $\mathrm{Re}\,B_0u|_{x_0=0}=0$ then we have $\mathrm{Re}\,B_\varepsilon\tilde{u}_\varepsilon|_{x_0=0}=0$ from (3)$'$ replaced x' by $\varepsilon x'$. So Lemma 2.4 is true, if B has real coefficients.

Next we proceed to the general case. $B_0u|_{x_0=0}=0$ means

$$(4) \qquad \mathrm{Re}\,B_0u=-i\,\mathrm{Im}\,B_0u \quad \text{on} \quad x_0=0.$$

Here we remark that from (2) we have for any ε in $[0, 1)$.

$$(5) \qquad \mathrm{Im}\,B_\varepsilon u=\mathrm{Im}\,B_\varepsilon\tilde{u}_\varepsilon \quad \text{on} \quad x_0=0$$

Therefore since we have $\mathrm{Re}\,B_0u=\mathrm{Re}\,B_\varepsilon\tilde{u}_\varepsilon$, it follows

$$(6) \qquad B_\varepsilon\tilde{u}_\varepsilon=i(\mathrm{Im}\,B_\varepsilon-\mathrm{Im}\,B_0)u \quad \text{on} \quad x_0=0.$$

Denote the right-hand side of (6) by $a_\varepsilon(x')$ and extend it to $x_0>0$ by constant, which we denote by $a_\varepsilon(x_0, x')$. Then $a_\varepsilon(x_0, x')$ tends to zero in $\mathscr{B}(\overline{R_+^{n+2}})$. We define u_ε by

$$u_\varepsilon=\tilde{u}_\varepsilon+a_\varepsilon(x_0, x')\varphi\left(\frac{1}{\varepsilon}\Phi(\varepsilon x)\right),$$

where $\varphi(x)=\varphi(x_0, x')$ is a real function in $C_0^\infty(\overline{R_+^{n+2}})$ satisfying $\varphi(0, x')=0$ and $B_0\varphi=1$ at $(0, x')\in\bar{\omega}$. Then u_ε satisfies the desired conditions. q.e.d.

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