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Subrings of a polynomial ring of one variable

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The following problem was communicated to the writer by Dr. A. Zaks of the University of Oregon:

We consider the polynomial ring $A[X]$ of one variable X over a normal domain A. Give a criterion for a ring R to coincide with $A[X]\cap K$ with a suitable field K containing A.

In this article, we give an answer as follows:

Theorem 1. *Such an R is characterized by the property that there is a polynomial f which belongs to $XA[X]$ (i.e., the constant term of f is zero) such that R is generated by $S_i=\{g\in A[X]\mid \exists a,\ b\in A,\ a\neq 0,\ ag=bf^i\}$ $(i=1, 2, \cdots)$.*

As for the proof, if $R=A$, then f is zero, and we assume that $R\neq A$. On the other hand, let k and L be the fields of quotients of A and R, respectively. Then we may assume that $K=L$. First we prove a lemma:

Lemma. 2 *Assume that A is a valuation ring of k and that $f=c_1X^n+c_2X^{n-1}+\cdots+c_nX$ is a polynomial over A such that some of the coefficients c_i are units in A. Then a polynomial $h=e_0+e_1f+\cdots+e_sf^s$, in f with coefficients e_i in k, is in $A[X]$ if and only if all e_i are in A.*

Proof. The if part is obvious, and we want to prove the only if part. Assume that $h\in A[X]$. $e_0=h(0)$, and therefore $e_0\in A$. Then $f(e_1+\cdots+e_sf^{s-1})\in A[X]$. Since f is a primitive polynomial, we see that $e_1+\cdots+e_sf^{s-1}\in A[X]$. Thus we prove the assertion by induction on s. QED

The if part of Theorem 1 follows from the following result:

Proposition 3. *Under the assumption at the beginning, if $f\in XA[X]$, then $A[X]\cap k(f)$ is the ring generated by S_i $(i=1, 2, \cdots)$ over A.*

Proof. It is obvious that all the S_i are contained in $A[X]\cap k(f)$. Conversely, let h be an arbitrary element of $A[X]\cap k(f)$. We may assume that $f=c_1X^n+c_2X^{n-1}+\cdots+c_nX$, $c_i\in A$, $c_1\neq 0$. Then X is integral over $A[f, c_1^{-1}]$ and therefore $A[X]\cap k(f)\subseteq A[f, c_1^{-1}]$. This shows that $h=e_0+e_1f+\cdots+e_sf^s$ with e_i in $A[c_1^{-1}]\subseteq k$. Since A is normal, A is the intersection of valuation rings A_λ of k containing A. For each A_λ, the expression of h is modified: $h=e_{\lambda 0}+e_{\lambda 1}f_\lambda+\cdots+e_{\lambda s}f_\lambda{}^s$ with $f_\lambda\in S_1$ such that f_λ is a primitive polynomial over A_λ. Obviously $e_{\lambda i}f^i=e_if^i$ for each i. Now Lemma 2 shows that $e_{\lambda i}f^i$ is in $A_\lambda[X]$. Namely, e_if^i is in $A_\lambda[X]$ for any i, λ. Thus each e_if^i is in $A[X]$ and $e_if^i\in S_i$. QED

Next we prove another lemma:

Lemma 4. *Let f and g be polynomials in X over k such that (i) f and g are coprime and (ii) $\deg f>\deg g\geq 1$. Then we have $k(f/g)\cap k[X]=k$. and $k(f/g)\cap A[X]=A$.*

Proof. Assume that $h=e_0+e_1X+\cdots+e_sX^s$ $(e_i\in k,\ e_s\neq 0,\ s\geq 1)$ is in $k(f/g)$. Then we can write

$$h=\frac{b_0(f/g)^n+b_1(f/g)^{n-1}+\cdots+b_n}{(f/g)^m+c_1(f/g)^{m-1}+\cdots+c_m}\quad (b_i,c_j\in k;\ b_0\neq 0).$$

Since $s\geq 1$ and $\deg f>\deg g$, we see that $n>m$. Then we have

$$b_0f^n+b_1f^{n-1}g+\cdots+b_ng^n=h(f^m+c_1f^{m-1}g+\cdots+c_mg^m)g^{n-m}$$

and we see that f^n is devisible by g, contradicting our assupmtion. Therefore $s=0$ and $k(f/g)\cap k[X]=k$. Consequently, $k(f/g)\cap A[X]=A$. QED

Now we come to the proof of the converse part of Theorem 1. By the theorem of Lüroth, L is a simple transcendental extension of k, and $L=k(f/g)$ with $f,g\in k[X]$ (f and g coprime). We may assume that $\deg f\geq\deg g$. If $\deg f=\deg g$, then subtracting a suitable element of k from f/g and taking inverse, we may assume that $\deg f>\deg g$. Then Lemma 4 shows that $g\in k$ because of our assumption that $R\neq A$. Thus we may assume that $g=1$ and $f\in XA[X]$. This f is the required polynomial by virtue of Proposition 3. Thus we complete the proof of Theorem 1.

In closing this article, we add two remarks:

(1) In case A is a field, somewhat related results were given by A. Zaks [Israel J. Math. 9 (1971), pp. 285–289] and by P. M. Cohn [Proc. London Math. Soc. (3) 14 (1964), pp. 618–632].

(2) In general, under the notation of Theorem 1, assuming that $f\neq 0$, we see that the ring generated by all the S_i over A is generated by S_1 if and only if $(I^{-1})^n=I^{-n}$; where I is the ideal generated by the coefficients of f and $I^{-n}=\{x\in k\mid xI^n\subseteq A\}$ $(n=1,2,\cdots)$.

(The proof is easy.)

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