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On Hartshorne's conjecture

By

Shigefumi MORI* and Hideyasu SUMIHIRO

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§0. Introduction

After studying ample vector bundles on algebraic varieties, R. Hartshorne has posed the following problem in [5] and now it is known as the conjecture of Harshorne's.

(H–n) If X is an n-dimensional non-singular projective algebraic variety with ample tangent vector bundle defined over an algebraically closed field k, then X is (algebraically) isomorphic to $\mathbf{P}^n$ over k.

In the case $k=\mathbf{C}$ (the complex number field), it is known that this conjecture is deeply connected with the following famous conjecture of Frankel's in complex differential geometry.

(F–n) A compact Kaehler manifold X of dimension n with positive sectional curvature is biholomorphic to the complex projective space $\mathbf{P}^n(\mathbf{C})$.

From now on, we assume that the characteristic of k is 0. (H–1) and (F–1) are obvious. Using classification of algebraic surfaces, (H–2) and (F–2) are solved affirmatively by R. Hartshorne [5] and by Frankel and Andreotti [3] respectively. Recently, T. Mabuchi has succeeded in proving (H–3) under the assumption that the second Betti number of X is equal to 1 [9]. In this paper, we will prove that (H–3) holds true without the assumption on the second Betti number. The keys to our proof of (H–3) are the following.

(1) A criterion for $\mathrm{Pic}(X)=\mathbf{Z}$: Let X be a non-singular projective algebraic variety with ample anti-canonical divisor $c_1=c_1(T_X)$. Then the Picard number $\rho(X)$ of X is equal to 1 if and only if every effective divisor on X is ample (Theorem 3). Using this criterion, we prove that if the tangent vector bundle T_X of X is ample, then the Picard number $\rho(X)$ of X is equal to 1 (Theorem 4).

(2) A characterization of projective spaces: If a non-singular projective algebraic variety X has a non-zero global vector field vanishing on an ample irreducible effective divisor D on X, then X is isomorphic to a projective space $\mathbf{P}^n$ and D cor-

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responds to a hyperplane in $\mathbf{P}^n$ (Theorem 8).

(3) Bialynicki-Birula's results on G_m-actions [2] and T. Mabuchi's argument: We use T. Mabuchi's argument in simplified form on vector fields.

Finally we note that the conjecture $(H-2)$ is proved by our method without using the classification of algebraic surfaces and it seems that our method might work in higher dimensional cases.

Notations

T_X: the tangent vector bundle of a non-singular algebraic variety X, i.e., a locally free $\mathcal{O}_X$-sheaf with rank $=\dim X$.

$c_1=c_1(T_X)$: the anti-canonical divisor of X, i.e., the first Chern class of T_X.

K_X: the canonical divisor of X, i.e., $K_X=-c_1$.

$H^i(X, F)$: i-th cohomology group for a coherent $\mathcal{O}_X$-sheaf F.

$h^i(X, F)$, $h^i(X, D)=h^i(D)$: $h^i(X, F)=\dim H^i(X, F)$, $h^i(X, D)=h^i(X, \mathcal{O}_X(D))$ for a divisor D on X.

$\chi(F)$: the Euler-Poincare characteristic of a coherent $\mathcal{O}_X$-sheaf F, i.e., $\chi(F)=\sum(-1)^i h^i(X, F)$.

Pic(X): the Picard group of X.

$(D\cdot C)$: intersection number of a divisor D and a curve C in a non-singular projective algebraic variety.

Aut(X), Aut$(X)^0$: the automorphism group of an algebraic variety X and the connected component of Aut(X) containing the unit element.

X^G: G-fixed points scheme with reduced structure of an algebraic variety X on which a linear algebraic group G acts.

$V_+(\mathfrak{A})$, $D_+(F)$: the closed subscheme defined by a homogeneous ideal $\mathfrak{A}(\subset R)$ in Proj(R) (R being a graded ring) and the open subscheme defined by a homogeneous element F in Proj(R).

§1. A criterion for Pic$(X)=\mathbf{Z}$

Let X be a non-singular projective algebraic variety defined over an algebraically closed field of characteristic 0. In this section, we will give a criterion for Pic(X) to be isomorphic to $\mathbf{Z}$ when the anti-canonical divisor $c_1=c_1(T_X)$ of X is ample and using it, we will prove that the ampleness of the tangent vector bundle T_X of X implies Pic$(X)=\mathbf{Z}$.

Before stating our criterion, we shall begin with the following lemmas.

Lemma 1. *Let D be an ample divisor on X ($n=\dim X$). Then $h^0(mD-c_1)\neq 0$ for some integer m with $1\leq m\leq n+1$.*

Proof. For every integer m, we put $P(m)=\chi(mD-c_1)=\chi(mD+K_X)$. Since D is ample, $P(m)=\frac{D^n}{n!}m^n+\cdots$ is a numerical polynominal of degree n in m by the Riemann-Roch theorem and hence $P(m)=0$ has only n roots. We have $h^i(mD+$

$K_X)=h^{n-i}(-mD)=0$ for $i(1\leq i\leq n)$ and $m(\geq 1)$ by Serre duality and Kodaira vanishing theorem. Hence $P(m)=h^0(mD-c_1)$ $(m\geq 1)$ and $h^0(mD-c_1)\neq 0$ for some m $(1\leq m\leq n+1)$. q.e.d.

For a divisor D on X, we write $D>0$ if D is ample and $D\geq 0$ if D is numerically effective, i.e., $(D\cdot C)\geq 0$ for every effective curve C in X.

Lemma 2. *Assume that the anti-canonical divisor $c_1=c_1(T_X)$ is ample. Then we get the following*:

(1) *linear equivalence = numerical equivalence for divisors on X.*

(2) *For a divisor $D\geq 0$ on X, there is a positive integer m such that $h^0(mD)\geq 1$.*

Proof. (2) Let D be a numerically effective divisor on X. Let $P(x)$ be the polynomial such that $P(m)=\chi(mD)$ for every integer m. Since c_1 is ample, $P(0)=\chi(\mathcal{O}_X)=1$ and $P(m)=\frac{D^n}{n!}m^n+\cdots+1$. $h^i(mD)=h^{n-i}(-mD+K_X)=h^{n-i}(-(mD+c_1))=0$ for all $i>0$ because $mD+c_1$ is ample for every $m(\geq 0)$. Hence $P(m)=h^0(mD)\geq 0$ for $m(\geq 0)$ and $h^0(mD)\geq 1$ for some integer $m(\geq 1)$. (1) Let D be a divisor which is numerically equivalent to 0. Then we see easily that $h^0(\mathcal{O}_X(D))=1$ and $h^0(\mathcal{O}_X(-D))=1$ because $c_1=c_1(T_X)$ is ample. Hence D is linearly equivalent to 0. q.e.d.

Let $A^1(X)=N(X)\otimes_{\mathbf{Z}}\mathbf{R}$ where $N(X)$ is the Neron-Severi group of X and let ρ be the Picard number of X, i.e., $\rho=\dim_{\mathbf{R}}A^1(X)$ ([7]). Now we shall give a theorem which implies $\rho=1$ under some condition.

Theorem 3. *Let X be a non-singular projective algebraic variety defined over an algebraically closed field of characteristic* 0 *and let the anti-canonical divisor $c_1=c_1(T_X)$ be ample. Then the following are equivalent.*

(1) $\rho=1$

(2) *Every effective divisor on X is ample.*

Proof. We have only to prove (2)→(1). Assuming that there is an ample divisor D on X so that $D\notin\mathbf{R}c_1$ in $A^1(X)$, we will get a contradiction. By virtue of Lemma 1 and our assumption, we have $(n+1)D-c_1\geq 0$ and $(n+1)D-c_1>0$ because $D\notin\mathbf{R}c_1$. Let $(n+1)D=c_1+D_1$ in $A^1(X)$, D_1 being an ample divisor on X. Then $D_1\notin\mathbf{R}c_1$. Applying the same process to D_1, we get $(n+1)D_1=c_1+D_2$, D_2 being an ample divisor on X. Repeating this process, we obtain

$$\begin{aligned}(n+1)D &= c_1+D_1\\ (n+1)D_1 &= c_1+D_2\\ &\vdots\\ (n+1)D_{m-1} &= c_1+D_m\\ &\vdots\end{aligned}$$

Hence, $D=\frac{1-(1/(n+1))^{m+1}}{n}c_1+(1/(n+1))^{m+1}D_m$ where D_m is an ample divisor on X. Taking $m\to\infty$, $D\geq\frac{c_1}{n}$, i.e., $nD-c_1\geq 0$. By virtue of Lemma 2 and our assumption, $nD-c_1>0$ because $D\notin\mathbf{R}c_1$. Hence, $nD-c_1>0$ for any ample divisor D which is not contained in $\mathbf{R}c_1$. Applying the above argument to this situation again, we get $(n-1)D-c_1>0$. Repeating this argument, we finally get that $D-c_1>0$ if D is an ample divisor and $D\notin\mathbf{R}c_1$. Now we have $D=c_1+D_1$, $D_1=c_1+D_2,\dots$ (D_m is an ample divisor for every m.). Then $D=mc_1+D_m$. Since c_1 is ample, $D_m=D-mc_1$ is not ample for a sufficiently large m, which is a contradiction.

q.e.d.

Now we will prove the following theorem.

Theorem 4. *Let X be a non-singular projective algebraic variety with ample tangent vector bundle T_X defined over an algebraically closed field of characteristic* 0. *Then* $\mathrm{Pic}(X)=\mathbf{Z}$.

Before giving the proof, we shall show three lemmas and fix some notation. The following lemma is well-known and hence we omit the proof.

Lemma 5. *Let D be an irreducible divisor on X. Then D is ample if and only if $\mathcal{O}_X(D)\otimes\mathcal{O}_D$ is ample and $(D\cdot C)>0$ for every curve C in X.*

Let $A_1(X)=(Z_1(X)/\text{Num. equiv.})\underset{\mathbf{Z}}{\otimes}\mathbf{R}$ where $Z_1(X)$ is the group generated by cycles of codimension $(n-1)$, i.e., curves in X. Then $A_1(X)$ is the dual space of $A^1(X)$ by the intersection pairing: $A^1(X)\otimes A_1(X)\in(D, C)\to(D\cdot C)\in R$ and $\dim A_1(X)=\rho$, ρ being the Picard number of X. We define a norm $\|\ \|$ in $A_1(X)$ by $\|C\|=\sqrt{\sum x_i^2}$ for $C=\sum_{i=1}^{\rho}x_iC_i$, where $\{C_1,\dots, C_\rho\}$ is a fixed basis of $A_1(X)$.

S. Kleiman gave a useful criterion for a divisor D on X to be ample, i.e., D is ample if and only if there exists a positive number ε such that $(D\cdot C)\geq\varepsilon\|C\|$ for every effective curve C in X([7]). C. Barton extended the criterion to vector bundles on X ([1]).

Lemma 6 (*Barton*). *Let E be a vector bundle on X. The following are equivalent to each other.*

(1) *E is ample.*

(2) *There exists a positive number ε such that $d(f^*(E))\geq\varepsilon\|f_*(C)\|$ for every finite morphism $f: C\to X$, C being a non-singular projective curve, where $d(f^*(E))$ denotes the minimum of degrees of quotient line bundles of $f^*(E)$ on C.*

The following lemma is obvious and we omit the proof.

Lemma 7. *Let A be a commutative noetherian ring, I a prime ideal in A and let D be a derivation on A. Then, $D(I^{(m)})\subseteq I^{(m-1)}$ where $I^{(m)}$ are the m-th symbolic powers of I ($m=1, 2,\dots$) and the induced homomorphism $I^{(m)}/I^{(m+1)}\to I^{(m-1)}/I^{(m)}$ is A/I-linear.*

Proof of Theorem 4. Since $c_1=c_1(T_X)$ is ample, we have only to prove that every irreducible divisor D on X is ample by virtue of Theorem 3. Using Lemma

5, we will check the following two facts (i) and (ii).

(i) $\mathcal{O}_X(D)\otimes\mathcal{O}_D$ is ample: We prove that there exists a positive number ε such that $(D\cdot C)\geq\varepsilon m(C)$ for every irreducible (reduced) curve C in D, $m(C)=\max_{P\in C}\operatorname{mult}_P(C)$ ([5], Seshadri's criterion for ampleness of divisors). Let I_C, I_D be the sheaves of defining ideals of C, D in X respectively and let m be a natural number such that $I_C^{(m)}\supset I_D$, $I_C^{(m+1)}\not\supset I_D$, where $I_C^{(l)}$ are the l-th symbolic powers of I_C $(l=1, 2,\ldots)$. Then, $m\leq\operatorname{mult}_P(D)$ for a general point P in C. The natural homomorphism $I_D\otimes\mathcal{O}_C=I_D/I_CI_D\to I_C^{(m)}/I_C^{(m+1)}$ induces a non-zero map at the generic point of C and the induced homomorphism α: $\mathcal{H}om_{\mathcal{O}_C}(I_C^{(m)}/I_C^{(m+1)}, \mathcal{O}_C)\to\mathcal{H}om_{\mathcal{O}_C}(I_D/I_CI_D, \mathcal{O}_C)=\mathcal{O}_X(D)\otimes\mathcal{O}_C$ is also non-zero at the generic point of C. By virtue of Lemma 7, we have an $\mathcal{O}_C$-homomorphism β: $S^m(T_X)\otimes\mathcal{O}_C\ni D_1\otimes\cdots\otimes D_m\to[g\to D_1(\cdots(D_m(g))\cdots)]\in\mathcal{H}om_{\mathcal{O}_C}(I_C^{(m)}/I_C^{(m+1)}, \mathcal{O}_C)$ and $\alpha\cdot\beta$: $S^m(T_X)\otimes\mathcal{O}_C\to\mathcal{O}_X(D)\otimes\mathcal{O}_C$ is non-zero at the generic point of C. Let f: $C'\to C$ be a desingularization of C. Since $f^*(S^m(T_X))\to f^*(\mathcal{O}_X(D))$ is a non-zero homomorphism and $S^m(T_X)$ is ample, there is a positive number ε' such that $(D\cdot C)\geq\varepsilon'\|C\|$ by virtue of Lemma 6. ε' may depend on the integer m. However, considering the Samuel function on D, we see that these integers m are bounded. Since $m(C)\leq\lambda\|C\|$ $(\lambda>0)$ for every curve C in X, we get a positive number ε such that $(D\cdot C)\geq\varepsilon m(C)$ for every irreducible curve C in D.

(ii) $(D\cdot C)>0$ for every curve C in X: Since $\mathcal{O}_X(D)\otimes\mathcal{O}_D$ is ample, $\mathcal{O}_X(lD)$ is generated by global sections for a sufficiently large integer l. Let C be an irreducible curve in X. Let $lD\sim\sum_i r_iD_i$ ($\sim$ denotes linear equivalence), D_i being irreducible divisor so that $D_i\cap C\neq\phi$ for some i. If $D_i\supset C$, then $(D_i\cdot C)>0$ by virtue of (1). Hence, we get $(D\cdot C)>0$. q.e.d.

§2. A characterization of $\mathbf{P}^n$

In this section, we will give a characterization for a non-singular algebraic variety X to be isomorphic to a projective space by using global vector fields on X.

Theorem 8. *Let X be an n-dimensional, non-singular projective algebraic variety defined over an algebraically closed field k of characteristic 0. If there is a non-zero global vector field on X vanishing on an ample irreducible*) effective divisor D in X, i.e., $H^0(X, T_X\otimes\mathcal{O}_X(-D))\neq 0$, then X is isomorphic to $\mathbf{P}^n$ and D is a hyperplane in $\mathbf{P}^n$.*

Proof. Let $G=\operatorname{Aut}(X)^0$ and let $G'=\{g\in G|$ every point of D is fixed by $g\}$. Then, G' is a linear algebraic subgroup of G and the tangent space of G' at the unit element $=H^0(X, T_X\otimes\mathcal{O}_X(-D))$. Therefore, we consider the following two cases, (I) and (II).

(I) G_m acts non-trivially on X and $D\subset X^{G_m}$: Since $\operatorname{Pic}(G_m)=0$, there is a G_m-linearization on $\mathcal{O}_X(D)$ and we fix this linearization on $\mathcal{O}_X(D)$. Let $R=\bigoplus_{\nu\geq 0}H^0(X, \mathcal{O}_X(\nu D))$, $R_\nu=H^0(X, \mathcal{O}_X(\nu D))$. Then R is a finitely generated graded ring over k and

*) The irreducibility can be omitted. We have only to assume that $H^0(D, \mathcal{O}_D)\cong k$.

each R_ν, the homogeneous part of degree ν in $R(\nu\in Z,\ \nu\geq 0)$, is a rational G_m-module. Now let $\{F_0, F_1,\dots, F_r\}$ be a minimal set of G_m-semi-invariant homogeneous generators of R over k, $F_0(\in R_1)$ being the element corresponding to D. For a semi-invariant element $F(\neq 0)$, we denote the weight of F by $\chi(F)$, i.e., $\tau(t)F=t^{\chi(F)}F$ $(t\in G_m)$. We prove that $R=k[F_0,\dots, F_r]$ is a polynomial ring over k, $r=n$, $\deg F_i=1$ $(0\leq i\leq n)$ and $\chi(F_1)/\deg F_1=\cdots=\chi(F_n)/\deg F_n$, $\chi(F_0)/\deg F_0\neq\chi(F_1)/\deg F_1$.

Lemma 9. $\chi(F_1)/\deg F_1=\cdots=\chi(F_r)/\deg F_r$, $\chi(F_0)/\deg F_0\neq\chi(F_1)/\deg F_1$.

Proof. $\{\bar F_1,\dots, \bar F_r\}$ is a minimal set of semi-invariant homogeneous generators of the quotient ring $R/(F_0)$ where $\bar F_i$ is the image of F_i in $R/(F_0)$ $(1\leq i\leq r)$. Let (Y, L) be a polarized algebraic scheme over k with a G_m-action such that $H^0(Y, \mathcal{O}_Y)=k$ and L has a G_m-linearization. Then the action of G_m on Y is trivial if and only if there are characters χ_ν of G_m such that the action of G_m on $H^0(Y, L^{\otimes\nu})$ is a multiplication by χ_ν for every $H^0(Y, L^{\otimes\nu})\neq 0$ $(\nu\in Z)$ such that all the χ_ν/ν are equal to each other. Therefore, $\chi(F_1)/\deg F_1=\cdots=\chi(F_r)/\deg F_r$ because the action of G_m on $D\cong\mathrm{Proj}(R/(F_0))$ is trivial and $\chi(F_0)/\deg(F_0)\neq\chi(F_1)/\deg F_1$ because the action of G_m on X is non-trivial. q.e.d.

Hence, F_0 is transcendental over $\{F_1,\dots, F_r\}$.

Lemma 10. *$r=n$ and $\{F_0, F_1,\dots, F_r\}$ is algebraically independent over k.*

Proof. Since F_0 is transcendental over $\{F_1,\dots, F_r\}$, $V_+(F_1,\dots, F_r)=\{P\}$ where P is a closed point in $X\cong\mathrm{Proj}\,R$ and $P\in D_+(F_0)=\mathrm{Spec}\,k[F'_1,\dots, F'_r]$ $(F'_i=F_i/F_0^{\deg F_i},\ 1\leq i\leq r)$. Now let $k[F'_1,\dots, F'_r]=k[Y_1,\dots, Y_r]/I$, where $\{Y_1,\dots, Y_r\}$ is algebraically independent over k and let M be the maximal ideal in $k[Y_1,\dots, Y_r]$ generated by $(Y_1,\dots, Y_r)$. Then the regular local ring $(\mathcal{O}_{X,P}, m_P)$ is equal to $(k[Y]_M/Ik[Y]_M, Mk[Y]_M/Ik[Y]_M)$. We claim that $r=n$ and $I=0$. Indeed if $I\not\subset M^2$, then we may assume that there is a non-trivial, G_m-semi-invariant relation such that

$$F'_1+\sum_{i\geq 2}a_iF'_i+f(F'_1,\dots, F'_r)=0\qquad (\deg f(Y_1,\dots, Y_r)\geq 2).$$

By virtue of Lemma 9, we can easily prove that $f(Y_1,\dots, Y_r)$ does not contain the monomial which is divisible by Y_1 and the relation

$$F_1+\sum_{i\geq 2}a_iF_i+f(F_2,\dots, F_r)=0$$

holds. This contradicts to the fact that $\{F_0, F_1,\dots, F_r\}$ is a minimal set of semi-invariant homogeneous generators of R. Hence $I\subset M^2$. Then $\dim_k(m_P/m_P^2)=\dim_k(Mk[Y]_M/M^2k[Y]_M)=n$ implies that $r=n$ and $I=0$. Therefore, $\{F_0,\dots, F_r\}$ is algebraically independent over k. q.e.d.

Lemma 11. $\deg F_i=1$ $(0\leq i\leq n)$.

Proof. For every $i(1\leq i\leq r)$, put D_i to be the divisor defined by F_i in X. Then D_i is linearly equivalent to $\deg F_iD$. Let C_i be the curve defined by $(F_1,\dots, \widehat{F_i},\dots, F_n)$ $(1\leq i\leq n)$. Then $(D_i\cdot C_i)=1$ and $\deg F_i=1$. q.e.d.

By the above results, we have completed the proof of the assertion in case (I). We consider the other case.

(II) G_a acts non-trivially on X and $D \subset X^{G_a}$: Since $\mathrm{Pic}(G_a)=0$, there is a G_a-linearization on $\mathcal{O}_X(D)$ and we fix this linearization on $\mathcal{O}_X(D)$ and as in the case (I), we consider the finitely generated graded ring $R=\bigoplus_{\nu\geq 0} H^0(X, \mathcal{O}_X(\nu D))$, $R_\nu = H^0(X, \mathcal{O}_X(\nu D))$. Let $\{F_0, F_1, \dots, F_r\}$ be a minimal set of homogeneous generators of R, $F_0(\in R_1)$ being the element corresponding to D. Since G_a acts on $D \cong \mathrm{Proj}(R/(F_0))$ trivially, the action of G_a on the quotient ring $R/(F_0)$ is trivial, i.e., $\tau(t)F - F \in (F_0)$ $(t \in G_a)$ for every homogeneous element F in R. Now we define $\Delta(F) = (\tau(1)F - F)/F_0(\in R)$ and $\chi(F) = \deg_t[\tau(t)F]$ for every homogeneous element F. Since F_0 is G_a-invariant, $\tau(t)(\Delta(F)) = (\tau(t+1)F - \tau(t)F))/F_0$ and $\chi(\Delta(F)) = \chi(F) - 1$ if $\chi(F) \neq 0$. By the induction on the degree of F, we see that $\deg F \geq \chi(F)$ in general.

Lemma 12. $\max\{\chi(F_i)/\deg F_i\} = 1$.

Proof. Assuming that $\max\{\chi(F_i)/\deg F_i\} = b/a \leq 1$, $(a, b)=1$, $a>1$, we will get a contradiction. For a general point P in X, $\tau(\infty)(P) \in V_+(F_i \mid \chi(F_i)/\deg F_i < b/a)$. Since a and b are coprime, $\deg F_i$ is divisible by a if $\chi(F_i)/\deg F_i = b/a$. Hence $\tau(\infty)(P) \in V_+(R_{Na+1})$ for every $N>0$. Since D is ample, $\sqrt{R_N R}$ is an irrelevant prime for every sufficiently large integer N and this is a contradiction. q.e.d.

Operating Δ if necessary, we may assume that there is an element $F(\neq 0)$ with $\deg F = 1$ and $\chi(F) = 1$. Hence, after changing generators appropriately, we may furthermore assume that $\tau(t)F_1 = F_1 + tF_0$ $(t \in G_a)$ and $\deg F_1 = 1$.

Lemma 13. For every homogeneous element F, there exists a unique set of G_a-invariant homogeneous elements $\{G_0, G_1, \dots, G_m\}$ such that

$$F = \sum_{\nu=0}^{m} \frac{G_\nu}{\nu!} F_1\{F_1 - F_0\} \cdots \{F_1 - (\nu-1)F_0\} \qquad (m = \chi(F)).$$

Proof. We prove the assertion by the induction on $\chi(F)$. If $\chi(F)=0$, i.e., F is G_a-invariant, it is obvious. Applying the induction hypothesis on $\Delta(F)$, we have a unique set $\{G_1, \dots, G_m\}$ (G_i: G_a-invariant and homogeneous) such that $\Delta(F) = \sum_{\nu=0}^{m-1} \frac{G_{\nu+1}}{\nu!} F_1 \cdots \{F_1 - (\nu-1)F_0\} = \Delta\left[\sum_{\nu=1}^{m} \frac{G_\nu}{\nu!} F_1 \cdots \{F_1 - (\nu-1)F_0\}\right]$. Hence $G_0 = F - \sum_{\nu=1}^{m} \frac{G_\nu}{\nu!} F_1 \cdots \{F_1 - (\nu-1)F_0\}$. q.e.d.

For every homogeneous element F, we denote by $\alpha(F)$ the element G_0 given in Lemma 13. Then Lemma 13 implies that $F - \alpha(F) \in (R_+^2)_\nu + k[F_0, F_1]_\nu$ $(\nu = \deg F)$, where $R_+ =$ the ideal $(F_0, F_1, \dots, F_r)$ in R. Applying the above relation, we can take a good minimal set of homogeneous generators of R.

Lemma 14. *There is a minimal set of homogeneous generators $\{F_0, F_1, F_2, \dots, F_r\}$ of R such that $\{F_0, F_2, \dots, F_r\}$ are G_a-invariant and $\tau(t)F_1 = F_1 + tF_0 (t \in G_a)$.*

Proof. Let $\tilde{F}_0 = F_0$, $\tilde{F}_1 = F_1$ and $\tilde{F}_i = \alpha(F_i)$, $\deg F_i \leq \deg F_{i+1}$ $(i \geq 2)$. Then F_i

$\in k\tilde{F}_i+(R_+^2)_\nu+k[F_0, F_1]_\nu$ $(\nu=\deg F_i)$. Therefore, we can prove that $k[\tilde{F}_0, \tilde{F}_1,\dots, \tilde{F}_i]=k[F_0, F_1,\dots, F_i]$ for every i $(0\le i\le r)$ by the induction on i and $\{\tilde{F}_0, \tilde{F}_1,\dots, \tilde{F}_r\}$ is the desired minimal set of homogeneous generators of R. q.e.d.

By virtue of Lemma 14, F_1 is transcendental over $\{F_0, F_2,\dots, F_r\}$ and $V_+(F_0, F_2,\dots, F_r)=\{P\}$ (P is a closed point in X). By the same argument used in the case (I), we can prove that $r=n$, $\{F_0,\dots, F_r\}$ is algebraically independent over k and $\deg F_i=1$ $(0\le i\le n)$. Therefore, X is isomorphic to $\mathbf{P}^n$ and D corresponds to a hyperplane in $\mathbf{P}^n$. q.e.d.

§3. Application

T. Mabuchi has succeeded in proving that the conjecture $(H\text{–}3)$ holds true under the assumption that the second Betti number $=1$ [9]. Our Theorem 4 implies that the second Betti number $=1$, if the tangent vector bundle T_X of X is ample. Combining his result with ours, we can now prove that the conjecture $(H\text{–}3)$ is true. In this section, applying our previous results, we shall give another proof which is simpler than Mabuchi's [9]. It seems that ours might work in higher dimensional cases. The keys to our proof are the results of Bialynicki-Birula's on G_m-actions [2] and the arguments of Mabuchi's.

Theorem 15. *If X is a 3-dimensional, non-singular projective algebraic variety with ample tangent vector bundle T_X defined over an algebraically closed field of characteristic 0, then X is isomorphic to $\mathbf{P}^3$.*

Proof. Let $P=P(T_X)$ be the projective fiber bundle of T_X over X and let L be the tautological line bundle of T_X. Then L is ample because T_X is ample and the canonical line bundle of P is isomorphic to $L^{\otimes -3}$. $H^i(X, T_X)=H^i(P, L)=H^{5-i}(P, L^{\otimes -4})=0$ for i $(1\le i\le 3)$ by Serre's duality and Kodaira's vanishing theorem. Hence $\dim H^0(X, T_X)=\chi(X, T_X)=\frac{1}{2}(c_1^3-2c_1c_2+c_3)+5$, c_i $(1\le i\le 3)$ being the i-th Chern class of T_X, by the Riemann-Roch theorem. $c_1^3-2c_1c_2+c_3$ is a positive integer ([3]). Hence, $\dim H^0(X, T_X)\ge 6$. Now let $G=\mathrm{Aut}^0(X)$. Since the irregularity of $X(=h^1(X, \mathcal{O}_X))$ is 0, G is a linear algebraic group and $\dim G=\dim H^0(X, T_X)\ge 6$. We consider the following two cases.

(I) $G\supset G_m$

We use the useful results of Bialynicki-Birula's on G_m-actions [2]. As for the definitions of $(+)$-decomposition (resp. $(-)$-decomposition) of X and G_m-fibrations $\gamma_i^+: X_i^+\to X_i^{G_m}$ (resp. $\gamma_i^-: X_i^-\to X_i^{G_m}$), we refer to his paper. Let X^{G_m} be the fixed point scheme of X and let $X^{G_m}=\bigcup_{i=1}^r X_i^{G_m}$ be the decomposition of connected components. Then every component $X_i^{G_m}$ is smooth [6]. Following the Bialynicki-Birula's results ([2], Theorem 4.3 and Corollary 1), let $X=\bigcup_{i=1}^r X_i^+$ (resp. $X=\bigcup_{i=1}^r X_i^-$), $(X_i^+)^{G_m}=X_i^{G_m}$ (resp. $(X_i^-)^{G_m}=X_i^{G_m}$) be the unique $(+)$-decomposition of X (resp. $(-)$-decomposition of X), $\gamma_i^+: X_i^+\to X_i^{G_m}$ (resp. $\gamma_i^-: X_i^-\to X_i^{G_m}$) a G_m-fibration and let $U = X_1^+$ be the dense G_m-invariant locally closed subscheme of X. For simplicity, we put $Y = X_1^{G_m} = U^{G_m}$ and denote the G_m-fibration by γ: $U \to Y$. Since γ is a smooth morphism, we have a surjective homomorphism: $T_U = T_X|U \to \gamma^*(T_Y)$ (U being an open subcheme of X). Restricting these vector bundles to Y, we see that there is a surjective homomorphism: $T_X|Y \to T_Y$ and hence T_Y is ample. Since the action of G_m is non-trivial, $\dim Y = 0$, 1 or 2.

(i) $\dim Y = 2$. By virtue of our Theorem 8, $X \simeq \mathbf{P}^3$ and $Y \simeq$ a hyperplane in $\mathbf{P}^3$.

(ii) $\dim Y = 1$. Y is a non-singular curve with the ample tangent bundle. Hence, $Y \simeq \mathbf{P}^1$. Let H be the closure of $\gamma^{-1}(P)$ for a point P in Y. Then, the intersection number $(H \cdot Y) = 1$ and so H is the ample generator of $\mathrm{Pic}(X) = \mathbf{Z}$ (cf. Theorem 4). Put $c_1 = \alpha H$ (α being a positive integer). We see that $\alpha \geq 4$ by considering the exact sequence: $0 \to T_Y \to T_X|Y \to N_{Y/X} \to 0$ and the fact that $Y \simeq \mathbf{P}^1$. By virtue of Kobayashi-Ochiai's theorem ([8], Corollaries to Theorem 1.1 and Theorem 2.1), $X \simeq \mathbf{P}^3$.

(iii) $\dim Y = 0$. In this case, $U \simeq \mathbf{A}^3$ (3-dimensional affine space) and the action of G_m on U is positive definite [2], i.e., $\tau(t)X_1 = t^a X_1$, $\tau(t)X_2 = t^b X_2$, $\tau(t)X_3 = t^c X_3$ ($t \in G_m$; a, b, c being positive integers) for an affine coordinate system $\{X_1, X_2, X_3\}$ of $\mathbf{A}^3$. Let P_0 be the origin (0, 0, 0) of $\mathbf{A}^3 = U$ and let $H = X - U$. Since $\mathrm{Pic}(X) \simeq \mathbf{Z}$ and $U \simeq \mathbf{A}^3$, H is irreducible and is the ample generator of $\mathrm{Pic}(X)$. Now let us consider the G_m-invariant locally closed (+)-strata of X contained in H. Assume that X_2^+ is the (+)-stratum which is open in H. Let $Z = (X_2^+)^{G_m}$, W the (−)-stratum such that $W^{G_m} = Z$ and γ': $W \to Z$ the G_m-fibration. Let P be a point in Z and let C be the closure of $\gamma'^{-1}(P)$ in X. Then C is a rational curve such that $(C \cdot H) = 1$. For a closed subscheme V in X, let us denote by $T_Q(V)$ the tangent space of V at a non-singular point Q in V. Now let $T_P(X) = T_P(X)^0 \oplus T_P(X)^+ \oplus T_P(X)^-$ be the decomposition of $T_P(X)$ into the eigenspaces with respect to the action of G_m on $T_P(X)$ (See [2]). Then, $T_P(Z) = T_P(X)^0$, $T_P(H) = T_P(X)^0 \oplus T_P(X)^-$, $T_P(W) = T_P(X)^+ \oplus T_P(X)^0$ and $T_P(C) = T_P(X)^-$. Since $\dim_k T_P(H) = 2$, we see that $\dim_k T_P(C) = 1$ and C is a rational curve such that $(C \cdot H) = 1$ because C and H meet transversally at P. Let f: $\tilde{C} \to C$ be the desingularization of C and let $c_1 = \alpha H$ (α being a positive integer). Since $\tilde{C} \simeq \mathbf{P}^1$, $f^*(T_X)$ decomposes into three ample line bundles and so $\alpha \geq 3$. Thus, we see that $X \simeq \mathbf{P}^3$ by virtue of Kobayashi-Ochiai's theorem.

(II) $G =$ a unipotent algebraic group.

In this case, we prove that X is isomorphic to $\mathbf{P}^3$. This is a contradiction because $G = PGL(3)$. Therefore, case (II) does not occur. First, we state an easy lemma on unipotent algebraic groups.

Lemma 16. *Let G be a connected unipotent algebraic group defined over an algebraically closed field of characteristic* 0 *and let K be a connected closed subgroup of G. Then, we get the following*

(i) *If* $\mathrm{codim}_G K = 1$, *then K is normal in G and* $K \supset [G, G]$.

(ii) *If* $\mathrm{codim}_G K = 2$, *then* $N(K)$ (= *the normalizer of K in G*) *is normal in G and* $K \supset [N(K), N(K)]$.

Proof. Since G is nilpotent as an abstract group, $N(K)\supsetneqq K$ for every subgroup K of G. Using this fact, one can prove the lemma easily.

Let H be an ample generator of $\mathrm{Pic}(X)=\mathbf{Z}$. We see $H^i(X, \mathcal{O}_X(H))=0$ for every $i(1\leq i\leq 3)$ by Serre duality and Kodaira vanishing theorem. Thus $h^0(X, \mathcal{O}_X(H))=\chi(X, \mathcal{O}_X(H))=1+\frac{1}{12}(c_1^2+c_2)H+\frac{1}{4}c_1H^2+\frac{1}{6}H^3$ by the Riemann-Roch theorem, and hence $h^0(X, \mathcal{O}_X(H))\geq 2$. Therefore, we may assume that H is effective, irreducible and G-invariant. For each point y in H, we denote by G_y the stabilizer group of y. First, we will get a contradiction assuming that G does not contain commutative 5-dimensional closed subgroup: Let $m=\max_{y\in H}\{\dim O(y)\}$, $O(y)$ being the G-orbit of y. Then, $m=0$, 1 or 2.

(i) $m=0$. Since every point in H is G-invariant, $X\simeq\mathbf{P}^3$ by virtue of Theorem 8.

(ii) $m=1$. Every $G_y\,(y\in H)$ is normal in G and G_y contains $[G, G]$ by virtue of Lemma 16. Since $[G, G]\neq e$, and $[G, G]$ fixes every point in H, $X\simeq\mathbf{P}^3$.

(iii) $m=2$. Let $y(\in H)$ be a point such that $\dim O(y)=2$. By virtue of Lemma 16, $N(G_y)$ is normal and $G_y\supseteq[N(G_y), N(G_y)]$. Since $[N(G_y), N(G_y)]\neq e(\dim N(G_y)\geq 5)$ and $[N(G_y), N(G_y)]$ fixes every point in H, $X\simeq\mathbf{P}^3$.

Thus, G contains a 5-dimensional commutative closed subgroup K. Now let $n=\max_{x\in X}\{\dim O(x)\}$, $O(x)$ being the K-orbit of x and K_x the stabilizer group of x. Then, $n=1$, 2 or 3.

(i) $n=3$. Let x be a point of X such that $\dim O(x)=3$. Then $K_x(\dim K_x\geq 2)$ acts on X trivially because K is commutative. This is a contradiction.

(ii) $n=2$. Let x be a point of X such that $\dim O(x)=2$. Then $K_x(\dim K_x\geq 3)$ acts on the closure $\overline{O(x)}$ of $O(x)$ in X trivially. Hence $X\simeq\mathbf{P}^3$ by virtue of Theorem 8.

(iii) $n=1$. Let $X_i\,(1\leq i\leq 5)$ be the linearly independent global vector fields on X corresponding to the subgroup K of G and let $Y=$zero locus of X_1. We claim that $\dim Y=2$. Put $U=X-Y$. By our assumption, $X_2=fX_1$ on U where f is a regular function on U. If $\dim Y\leq 1$, then f is a regular function on X and f is a non-zero constant. Since X_1 and X_2 are linearly independent, this is a contradiction. Therefore G_a acts on Y trivially and $X\simeq\mathbf{P}^3$. q.e.d.

Finally, we give a theorem which might work for Hartshorne conjecture in higher dimensional case. Indeed, we can generalize the proof of the case (I) in Theorem 15 by using the Bialynicki-Birula's result cited above, and we get the following:

Theorem 17. *Let X be an n-dimensional non-singular projective algebraic variety defined over an algebraically closed field of characteristic* 0. *Assume the conjecture $(H-m)\,(1\leq m\leq n-2)$ is true and that X has a non-trivial G_m-action. Then, X is isomorphic to $\mathbf{P}^n$.*

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

AND
HARVARD UNIVERSITY

DEPARTMENT OF APPLIED MATHEMATICS
KONAN UNIVERSITY

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