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On the C^{∞}-Goursat problem for equations with constant coefficients

By

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§ 1. Introduction

The Goursat problem in the class of analytic functions has been treated by several authors, for instance Gårding [1], Hörmander [4], Miyake [5] and Hasegawa [2]. However, the same problem in the class of C^{∞}-functions has not yet been treated from the general point of view. In this paper, we consider general equations in the case when the initial hyperplane is simple characteristic. In [3], we treated the second order equations and obtained a necessary snd a sufficient condition for the $\mathcal{E}$-wellposedness of the Goursat problem.

Let

$$P(\partial_t, \partial_x, \partial_y)=P_m(\partial_t, \partial_x, \partial_y)+P_{m-1}(\partial_t, \partial_x, \partial_y)$$
$$+R_{m-2}(\partial_t, \partial_x, \partial_y)$$

be a partial differential operator of order m with constant coefficients, where P_m and P_{m-1} are homogeneous parts of order m and $m-1$ of P respectively. $\partial_t, \partial_x, \partial_y$ stand for $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ respectively. Mor precisely, we consider this operator in $(t, x, y)\in \boldsymbol{R}^1_+\times \boldsymbol{R}^1\times \boldsymbol{R}^l$.

We impose on the principal part the following assumption;

(A. 1) $t=0$ is simple characteristic for P. Namely, the coefficient of ∂_t^m vanishes, and moreover if we denote the terms containing ∂_t^{m-1} of P_m by $(a_0\partial_x+\sum_{j=1}^{l} a_j\partial_{y_j})\partial_t^{m-1}$, then $a_0\xi+\sum a_j\eta_j\neq 0$.

Our problem is the following: Let the equation be

$$P(\partial_t, \partial_x, \partial_y)u(t, x, y)=f(t, y, x)\in \mathcal{E}_{t, x, y} \quad (t\geqq 0) \tag{1.1}$$

and the data, say Goursat data, be

$$\begin{cases} \partial_t^i u(0, x, y)=u_i(x, y)\in \mathcal{E}_{x, y}, & 0\leqq i\leqq m-2 \\ u(t, 0, y)=\varphi(t, y)\in \mathcal{E}_{t, y}, & (t\geqq 0) \end{cases} \tag{1.2}$$

We say that the Goursat problem is $\mathcal{E}$-wellposed if for any data $\{u_i\}_{0\le i\le m-2}$, φ and f, there exists a unique solution $u(t,x,y)\in\mathcal{E}_{t,x,y}$, for $(t,x,y)\in\{(t,x,y)\mid t\geqq 0, x\in\boldsymbol{R}^1, y\in\boldsymbol{R}^l\}$. Of course we should impose among $\{u_i\}$ and φ the following compatibility condition;

$$\text{(C)} \qquad \partial_t^i\varphi(0,y)=u_i(0,y), \quad 0\leqq i\leqq m-2.$$

Let us remark the following fact: If the Goursat problem is $\mathcal{E}$-wellposed, then by Banach's closed graph theorem, the linear mapping

$$(u_0, u_1\cdots, u_{m-2}, \varphi, f)\longmapsto u$$

is continuous from $\prod^{m-2}\mathcal{E}_{x,y}\times\mathcal{E}_{t,y}\times\mathcal{E}_{t,x,y}$ into $\mathcal{E}_{t,x,y}$.

Hereafter we assume, besides (A. 1),

(A. 2) $a_0\neq 0$,

(A. 3) $P_m(\tau,\xi,\eta)$ and $P_{m-1}(\tau,\xi,\eta)$ are real polynomials.

Although we assumed the coefficients are constants, the result obtained below could be extended to operators with variable coefficients. Our main aim is to elucidate the fundamental charactor of the C^∞-Goursat problem.

§ 2. Statement of the results

Let

$$P_m(\tau,\xi,\eta)=b_1(\xi,\eta)\tau^{m-1}+b_2(\xi,\eta)\tau^{m-2}+\cdots+b_m(\xi,\eta).$$

Let $\tau=\tau_i(\xi,\eta)$ $(1\leqq i\leqq m-1)$ be the roots of $P_m(\tau,\xi,\eta)=0$, when $b_1(\xi,\eta)\neq 0$ (ξ,η real). We have the following analogue to the hyperbolic equation.

Theorem 1. *In order that the Goursat problem is $\mathcal{E}$-wellposed, it is necessary that all the roots $\tau_i(\xi,\eta)$ are real for all ξ,η real.*

Moreover we have the following result.

Theorem 2. *In order that the Goursat problem is $\mathcal{E}$-wellposed, it is necessary that the principal symbol $P_m(\tau,\xi,\eta)$ is divisible by $b_1(\xi,\eta)$. Namely*

$$P_m(\tau,\xi,\eta)=b_1(\xi,\eta)\mathring{Q}_{m-1}(\tau,\xi,\eta),$$

where $\mathring{Q}_{m-1}$ is a polynomial of homogeneous degree $m-1$.

Concerning the homogeneous part P_{m-1} of P, when we impose the following assumption;

(A. 4) $\mathring{Q}_{m-1}(\partial_t,\partial_x,\partial_y)$ is strictly hyperbolic in the t-direction. Namely the roots $\tau=\tau_i(\xi,\eta)$ of $\mathring{Q}_{m-1}(\tau,\xi,\eta)=0$ are all real and distinct,

we have the following fact, which could be compared with the Levi condition.

Theorem 3. *Under the assumption (A. 4) on P_m, if the Goursat problem is $\mathcal{E}$-wellposed, then P_{m-1} has the following form;*

$$P_{m-1}(\tau, \xi, \eta)=c\mathring{Q}_{m-1}(\tau, \xi, \eta)+b_1(\xi, \eta)Q_{m-2}(\tau, \xi, \eta) \tag{2.1}$$

where c is constant, $b_1(\xi, \eta)$ is that appeared in Th. 2 and Q_{m-2} is a polynomial of homogeneous degree $m-2$.

Remark. If we don't assume (A. 4), namely we don't assume that the roots $\tau_i(\xi, \eta)$ *are distinct,* then the situation would be fairly complicated.

Now we can say the converse of the above theorems.

Theorem 4. *Under the assumption (A. 4) and (2. 1) of theorem 3, the Goursat problem is $\mathcal{E}$-wellposed.*

Before proving these theorems, we make the following reduction of the equation. In view of the assumption (A. 1) and (A. 2), we make the change of independent variables,

$$x'=\frac{1}{a_0}x\,,\quad y'_j=y_j-\frac{a_j}{a_0}x\quad (1\leqq j\leqq l)$$

Them, $\partial_x=\frac{1}{a_0}\partial_{x'}-\frac{1}{a_0}\sum a_j\partial_{y'_j}$, $\partial_{y_j}=\partial_{y'_j}$. Thus the operator $a_0\partial_x+\sum a_j\partial_{y_j}$ is transformed to $\partial_{x'}$. Next, let the coefficient of ∂_t^{m-1} be c, we put

$$u=e^{-cx'}\tilde{u}$$

then the coefficient of $\partial_t^{m-1}\tilde{u}$ disappears. Let us remark that, for this change of independent varisbles, the hyperplanes $t=0$ and $x=0$ where the Goursat data are given are transformed to $t'=0$ and $x'=0$ respectively. So, denoting t', x', y' and $\tilde{u}$ anew by t, x, y and u, the equation (1. 1) with $f=0$ takes the form

$$\partial_t^{m-1}\partial_x u=\sum_{\substack{i+j+|\alpha|\leq m\\ i\leq m-2}} a_{ij\alpha}\partial_t^i\partial_x^j\partial_y^\alpha u\,. \tag{2.2}$$

§ 3. Proof of Theorem 1

We follow the argument of Mizohata in [6] which was used to treat the Cauchy problem. However, we should remark that the direct use of Fourier transformation does not work.

For simplicity, we change the notations: we write $x_1, \cdots, x_l, x_{l+1}$ instead of $x, y_1, \cdots, y_l$. So the operator P is denoted by

$$P(\partial_t, \partial_x)=P_m(\partial_t, \partial_x)+R_{m-1}(\partial_t, \partial_x)\,.$$

we are going to prove the theorem by contradiction. More precisely, we assume the Goursat problem is $\mathcal{E}$-wellposed and that there exists a point ξ^0, whose first component $\xi_1^0\neq 0$, such that

$$P_m(\tau, \xi_0)=0$$

has a root $\tau=\tau_1(\xi^0)$ satisfying

$$\operatorname{Im}\tau_1(\xi^0)\neq 0\,.$$

We can assume, if neceaasry, by replacing ξ^0 by $-\xi^0$,

$$\operatorname{Im}\tau_1(\xi^0)<0 \quad (|\xi^0|=1,\ \xi_1^0\neq 0) \tag{3.1}$$

Localization and inequalities

Take a function $\beta(x)\in C^\infty$ with compact support. We assume $0\leqq\beta(x)\leqq 1$ and $\beta(x)\equiv 1$ in a neighborhood of the origin. Apply $\beta(x)$ to $P[u]=0$. Then

$$P[\beta u]=[P,\ \beta]u\,. \tag{3.2}$$

Next, let $\alpha(\xi)$ be C^∞ function with support contained in a small neighborhood of ξ^0. We assume $0\leqq\alpha(\xi)\leqq 1$, and $\alpha(\xi)\equiv 1$ in a neighborhood of ξ^0. Although we make precise the size of the supp $[\alpha]$ later, we assume from the beginning that on the support of α, the ξ_1 coordinates never vanishes.

Let $\alpha_n(\xi)=\alpha\left(\frac{\xi}{n}\right)$ and define pseudo-diff. operator $\alpha_n(D)$ by

$$\alpha_n(D)u(x)=\mathcal{F}^{-1}[\alpha_n(\xi)\hat{u}(\xi)]\,, \qquad \text{for} \quad u\in\mathcal{S}'$$

Apply $\alpha_n(D)$ to (3.2), then

$$P[\alpha_n\beta u]=\alpha_n[P,\ \beta]u\,. \tag{3.3}$$

Since $P=P_m+R_{m-1}$,

$$P_m[\alpha_n\beta u]=\alpha_n[P,\ \beta]u-R_{m-1}[\alpha_n\beta u]\,. \tag{3.4}$$

By assumption, P_m has the form

$$P_m(\partial_t,\ \partial_x)=\partial_{x_1}\partial_t^{m-1}+\sum_{j=2}^{m}q_j(\partial_x)\partial_t^{m-j}$$

where $q_j(\xi)$ is a polynomial of homogeneous degree j. Since on the support of $\alpha_n(\xi)$, the symbol $i\xi_1$ of ∂_{x_1} does not vanish, we can apply, in the dual space, $(i\xi_1)^{-1}$ to $\alpha_n(\xi)v(\xi)$. So we define $(i\xi_1)^{-1}(D)$ by

$$(i\xi_1)^{-1}(D)(\alpha_n v)=\mathcal{F}^{-1}[(i\xi_1)^{-1}\alpha_n(\xi)v(\xi)]\,.$$

Of course, we have

$$(i\xi_1)^{-1}\partial_{x_1}(\alpha_n v)=\partial_{x_1}(i\xi_1)^{-1}(\alpha_n v)=\alpha_n v\,.$$

Namely $(i\xi_1)^{-1}(D)$ is the inverse of ∂_{x_1}. Now, we apply $(i\xi_1)^{-1}(D)$ to (3.4). Then

$$\left[\partial_t^{m-1}+\sum_{j=2}^{m}(i\xi_1)^{-1}(D)q_j(\partial_x)\partial_t^{m-j}\right](\alpha_n\beta u) \tag{3.5}$$
$$=(i\xi_1)^{-1}(D)\{\alpha_n[P,\ \beta]u-R_{m-1}[\alpha_n\beta u]\}\,.$$

Now the coefficients $(i\xi_1)^{-1}(D)q_j(\partial_x)$ is a pseudo-differential operator in x of degree $j-1$. This implies in particular that $t=0$ is no longer characteristic to the operator of the left-hand side.

Let us remark that the characteristic equation of the left-hand side of (3.5) is

$$\tau^{m-1}+\sum_{j=2}^{m}(i\xi_1)^{-1}q_j(i\xi)\tau^{m-j}=(i\xi_1)^{-1}P_m(\tau, i\xi)=0\,.$$

Let $\tau_1^0, \tau_2^0, \cdots, \tau_{m-1}^0$ be the roots of this equation when $\xi=\xi^0$. By assumption there exists positive numbers δ and ε such that

$$\text{(3.6)}\qquad \begin{aligned}&\operatorname{Re}\tau_i-\varepsilon\geqq 3\delta\,, &&\text{for}\quad 1\leqq i\leqq N_1\\ &\operatorname{Re}\hat{\tau}_j-\varepsilon\leqq -3\delta\,, &&\text{for}\quad N_1+1\leqq j\leqq m-1\,.\end{aligned}$$

Now, we consider the right-hand side of (3.3). Let it be

$$\text{(3.7)}\qquad \Big[\partial_t^{m-1}+\sum_{j=2}^{m}(i\xi_1)^{-1}(D)q_j\partial_t^{m-j}+(i\xi_1)^{-1}(D)R_{m-1}(\partial_t, \partial_x)\Big](\alpha_n\beta u)$$
$$=(i\xi_1)^{-1}(D)\alpha_n[P, \beta]u\,.$$

The right-hand side is

$$(i\xi_1)^{-1}(D)\sum_{j=1}^{l+1}\frac{\partial P}{\partial\xi_j}(\partial_t, \partial_x)(\alpha_n\partial_{x_j}\beta\cdot u)$$
$$-(i\xi_1)^{-1}(D)\sum_{|\nu|\geqq 2}\frac{(-1)^{|\nu|}}{\nu\,!}P^{(\nu)}(\partial_t, \partial_x)(\alpha_n\partial_x^\nu\beta\cdot u)\,.$$

Now, in view of the fact that the term containing ∂_t^{m-1} as a factor is $\partial_t^{m-1}\partial_{x_1}$, this can be written as

$$(i\xi_1)^{-1}(D)\partial_t^{m-1}(\alpha_n\partial_{x_1}\beta\cdot u)+\sum_{|\nu|\geqq 1}{}' C_\nu(\partial_t, D)(\alpha_n\partial_x^\nu\beta\cdot u)\,,$$

where C_ν is of order $m-1-|\nu|$, differential in t and pseudo-differential in x, and the term contrining ∂_t^{m-1} does not appear. For simplicity, we denote the left-hand side of (3.7) by

$$[\partial_t^{m-1}+a(\partial_t, D)](\alpha_n\beta u)\,,$$

where $a(\partial_t, D)$ is of order $m-1$ and has the same property as C_ν. So the relation (3.7) becomes

$$\text{(3.8)}\qquad [\partial_t^{m-1}+a(\partial_t, D)](\alpha_n\beta u)$$
$$=(i\xi_1)^{-1}\partial_t^{m-1}(\alpha_n\partial_{x_1}\beta\cdot u)+\sum_{|\nu|\geqq 1}{}' C_\nu(\partial_t, D)(\alpha_n\partial_x^\nu\beta\cdot u)\,.$$

This can be written, denoting $a(\partial_t, D)$ by $-C_0(\partial_t, D)$, as

$$\text{(3.9)}\qquad \partial_t^{m-1}(\alpha_n\beta u)=(i\xi_1)^{-1}\partial_t^{m-1}(\alpha_n\partial_{x_1}\beta\cdot u)+\sum_{|\nu|\geqq 0}{}' C_\nu(\partial_t, D)(\alpha_n\partial_x^\nu\beta\cdot u)$$

where C_ν is of order $m-1-|\nu|$. If we use this relation for $\partial_t^{m-1}(\alpha_n\partial_{x_1}\beta\cdot u)$, (3.9) can be written again

$$\partial_t^{m-1}(\alpha_n\beta u)=(i\xi_1)^{-2}\partial_t^{m-1}(\alpha_n\partial_{x_1}^2\beta\cdot u)$$
$$+(i\xi_1)^{-1}\sum_{|\nu|\geqq 0}C_\nu(\partial_t, D)(\alpha_n\partial_{x_1}\partial_x^\nu\beta\cdot u)+\sum_{|\nu|\geqq 0}{}' C_\nu(\partial_t, D)(\alpha_n\partial_x^\nu\beta\cdot u)\,.$$

If we repeat this process, we have

$$\partial_t^{m-1}(\alpha_n\beta u)=(i\xi_1)^{-k}\partial_t^{m-1}(\alpha_n\partial_{x_1}^k\beta\cdot u)$$
$$+\sum_{s=0}^{k}(i\xi_1)^{-s}\sum_{|\nu|\geq 0}C_\nu(\partial_t, D)(\alpha_n\partial_{x_1}^s\partial_x^\nu\beta\cdot u),$$

where k is an arbitrary positive integer. We can interpret this formula as follows: for an arbitrary positive integer k, $\partial_t^{m-1}(\alpha_n\beta u)$ can be expressed as

$$\partial_t^{m-1}(\alpha_n\beta u)=(i\xi_1)^{-k}\partial_t^{m-1}(\alpha_n\partial_{x_1}^k\beta\cdot u) + \sum_{|\nu|\geq 0}d_{\nu,k}(\partial_t, D)(\alpha_n\partial_x^\nu\beta\cdot u) \tag{3.10}$$

where $d_{\nu,k}$ is differential in t, pseudo-differential in x of order $m-1-|\nu|$, and is of the form

$$\sum_{j=0}^{m-2}b_j(D)\partial_t^j$$

where order $b_j(D)\leqq m-1-|\nu|-j$.

Now, we consider the equivalent system to (3.5). Let

$$[\partial_t^{m-1}+\sum(i\xi_1)^{-1}(D)q_j(\partial_x)\partial_t^{m-j}](\alpha_n\beta u)=f,$$

where

$$f=(i\xi_1)^{-1}(D)\{\alpha_n[P, \beta]u-R_{m-1}(\alpha_n\beta u)\}.$$

Let

$$U={}^t((\Lambda+1)^{m-2}(\beta u), (\Lambda+1)^{m-3}\partial_t(\beta u), \cdots, \partial_t^{m-2}(\beta u))\equiv E(\Lambda, \partial_t)(\beta u),$$

then (3.5) becomes

$$\partial_t(\alpha_n U)=H\Lambda\alpha_n U+B\alpha_n U+F \tag{3.11}$$

where $F={}^t(0,\cdots,0,f)$, B is a bounded operator in L^2, and

$$H(\xi)=\begin{pmatrix} 0 & 1 & & & \mathbf{0} \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & \ddots & 1 \\ h_{m-1} & \cdots & & h_2 & h_1 \end{pmatrix} \quad h^j(\xi)=|\xi|^{-j}q_{j+1}(i\xi)/(i\xi_1),$$

$H(\xi)$ is homogeneous degree 0 in ξ.

By the definition of H, we have

$$\det(\tau I-H(\xi))=(i\xi_1)^{-1}P_m(\tau, i\xi) \qquad \text{for} \quad \xi_1\neq 0.$$

In the same way as [6] p. 117, we can find a non-singular matrix N_0 such that

$$N_0H(\xi^0)N_0^{-1}=\begin{pmatrix} \mathring{\tau} & & \mathbf{0} \\ & \mathring{\tau} & \\ a_{ij}^0 & & \ddots \ \mathring{\tau}_{m-1} \end{pmatrix}=\mathscr{D}_0$$

where $|a_{ij}^0|<\frac{1}{4}\delta/m$, δ being defined in (3.6). Then (3.11) become

$$\partial_t\alpha_n(N_0U)=[\mathcal{D}_0+N_0\{H(\xi)-H(\xi^0)\}N_0^{-1}]\Lambda\alpha_n(N_0U) \tag{3.12}$$
$$+N_0BN_0^{-1}(\alpha_nN_0U)+N_0F\,.$$

Next, if we restrict the neighborhood V_{ξ^0} of ξ^0 small, for all $\xi\in V_{\xi^0}$, by denoting

$$N_0\{H(\xi)-H(\xi^0)\}N_0^{-1}=\mathcal{D}_\varepsilon(\xi)=(a_{ij}^{(\varepsilon)}(\xi))_{1\le i,j\le m-1}\,,$$

we have

$$|a_{ij}^{(\varepsilon)}(\xi)|<\frac{\delta}{4m}\quad(1\leqq i,\ j\leqq m-1)\,.$$

We take $\alpha(\xi)$ such that $\operatorname{supp}[\alpha]\subset V_{\xi^0}$. Put,

$$\exp(-\varepsilon\Lambda t)N_0\alpha_nU=V^{(n)}={}^t(v_1^{(n)},\cdots,v_{m-1}^{(n)})\,,$$

then (3.12) become

$$\partial_tv^{(n)}=(\mathcal{D}_0+\mathcal{D}_\varepsilon-\varepsilon)\Lambda v^{(n)}+N_0BN_0^{-1}v^{(n)}+\exp(-\varepsilon\Lambda t)N_0F\,.$$

And define

$$S(t)=\sum_{i=1}^{N_1}\|v_i^{(n)}\|^2-\sum_{j=N_1+1}^{m-1}\|v_j^{(n)}\|^2\,,$$

where $\|\cdot\|$ stands for the L^2-norm in the x-space. We omit the suffix (n), then

$$S'(t)=\sum_i\operatorname{Re}\left(\frac{dv_i}{dt},v_i\right)-\sum_j\operatorname{Re}\left(\frac{dv_j}{dt},v_j\right).$$

The calculation gives $\left(\text{assuming }\frac{2}{3}<\text{distance }(0,\operatorname{supp}\alpha(\xi))<\frac{3}{4}\right)$

$$S'(t)\geqq\delta n\|v^{(n)}\|^2-C\|v^{(n)}\|\cdot\|\tilde{F}\|\geqq\delta'n\|v^{(n)}\|^2-\frac{C'}{n}\|\tilde{F}\|^2 \tag{3.13}$$

where $\delta'(<\delta)$ and C' are positove constants independent of n and $\tilde{F}=\exp(-\varepsilon\Lambda t)\times N_0F$.

In view of the form $\tilde{F}$, we have

$$S'(t)\geqq\delta'n\|v^{(n)}\|^2-\frac{C}{n}\|\exp(-\varepsilon\Lambda t)f\|^2\,,$$

where C is a positive constant.

Proof of Theorem 1

We assume (2.2) to be $\mathcal{E}$-wellposed. At first we define a series of solutions $u_n(t,x)$ of the Goursat problem. Namely we define their Goursat data. Let $\hat{\phi}(\xi)$ be a function whose support is located in a small neighborhood of ξ^0. On the support of $\hat{\phi}(\xi)$, $\alpha(\xi)=1$. And assume $\int|\hat{\phi}(\xi)|^2d\xi=1$. We define

$$\hat{\phi}_{n+1}(\xi)=\tilde{\hat{\phi}}(\xi-n\xi^0)\,. \tag{3.14}$$

Namely

$$\phi_{n+1}(x)=\exp(in\xi^0 x)\phi(x). \tag{3.15}$$

We define $u_n(t, x)$ by the solution of $Pu_n=0$ which satisfy the following Goursat data.

$$\begin{cases} N_0{}^t((\Lambda+1)^{m-2}u_n(0, x), (\Lambda+1)^{m-3}\partial_t u_n(0, x), \cdots, \partial_t^{m-2}u_n(0, x)) \\ \qquad ={}^t(\phi_n(x), 0, \cdots, 0) \\ u_n(t, 0, x')=\varphi_n(t, x')\equiv \sum_{i=0}^{m-2}\partial_t^i u_n(0, 0, x')t^i/i! \end{cases} \tag{3.16}$$

Obviously Goursat data (3.16) satisfy compatibility condition (C). (3.16) can be written as follows.

$$\begin{cases} \partial_t^i u_n(0, x)=\mathcal{F}^{-1}(C_i\phi_n(\xi)/(|\xi|+1)^{m-2-i}) \quad 0\leqq i\leqq m-2, \\ u_n(t, 0, x')=\varphi_n(t, x'). \end{cases} \tag{3.11}$$

In the same way as [6] p. 119, we have

$$\|\alpha_n N_0 E(\Lambda, \partial_t)\beta u(0, x)\|=c+o\left(\frac{1}{n}\right), \tag{3.18}$$

c is a positive constant.

Now, we put $u=u_n(t, x)$ in (3.5). By hypothesis of $\mathcal{E}$-whllposedness, there exist a positive integer h and a neighborhood (in x-space) Ω of $x=0$ and small T' such that

$$\max_{x\in\Omega}|\partial_t^i u_n(x, t)|\leqq O(n^h) \qquad \text{for} \quad 0\leqq t\leqq T', \quad 0\leqq i\leqq m-1. \tag{3.19}$$

By taking the support of $\beta(x)$ small, we can assume that the support of $\beta(x)$ is contained in Ω, therefore we have

$$\|\beta(x)\partial_t^i u_n(t, x)\|\leqq O(n^h), \qquad \text{for} \quad 0\leqq t\leqq T', \quad 0\leqq i\leqq m-1. \tag{3.20}$$

Now, let us consider the right-hand side of (3.5). The term $(i\xi_1)^{-1}(D)\alpha_n\partial_{x_1}\beta\partial_t^{m-1}u$ is one which does not appear in [6] when considering the Cauchy problem. Here we use (3.10) for $\partial_t^{m-1}(\alpha_n\partial_{x_1}\beta u_n)$. In view of (3.20), if we choose $k=h$ then

$$\|(i\xi_1)^{-k}\partial_t^{m-1}(\alpha_n\partial_{x_1}^k\beta\cdot u_n)\|\leqq C.$$

After using the relation (3.10), the right-hand side of (3.5) has only the terms which appear in [6] and a term whose L^2-norm is bounded with respect to n.

Hereafter we can consider in the nearly same way as [6] p. 121~124. Of course on the way the terms $\partial_t^{m-1}(\alpha_n\partial_{x_1}^{\nu}\beta\cdot u_n)$ appear. We treat these terms in the above way.

After all we have

$$S_n(t)\geqq C\cdot\exp\left(\frac{\delta'}{2}nt\right)-O\left(\frac{1}{n}\right). \tag{3.21}$$

For the definition of $S_n(t)$, see [6] p. 124. On the other hand, by hypothesis of $\mathcal{E}$-wellposedness, $S_n(t)$ must has polynomial order with respect to n. Thus the proof of theorem 1 is complete.

§ 4. The proof of Theorem 2

Consider the equation (2.2). At first we prove Theorem when $y\in R^1$. In this case (2.2) can be written as follows.

$$\partial_t^{m-1}\partial_x u=\sum_{\substack{i+j+k\leq m\\ i\leq m-2}} a_{ijk}\partial_t^i\partial_x^j\partial_y^k u\,. \tag{4.1}$$

Then Theorem 2 claims that; if $\mathcal{E}$-wellposed then all terms in the principal part of (4.1) have the factor ∂_x, more precisely $a_{m-i,0,i}=0$ $2\leqq i\leqq m$.

Now, we give a rough sketch of the proof of Theorem 2. In first step we prove $a_{m-2,0,2}=0$. More precisely, assuming $a_{m-2,0,2}\neq 0$ we construct a sequence of solutions of (4.1) which shows the continuity from Goursat data to solutions does not hold. This shows, in view of Banach's closed graph theorem, the Goursat problem is not well-posed if $a_{m-2,0,2}\neq 0$. In second step, we show further $a_{m-2,0,i}=0$ $(3\leqq i\leqq m)$ by using the conclusion of Theorem 1.

First step

We assum $a_{m-2,0,2}\neq 0$. (4.1), we pick up all terms containing ∂_t^{m-2} as factor, then

$$\begin{aligned}\partial_t^{m-1}\partial_x u=(&a_{m-2,2,0}\partial_x^2+a_{m-2,1,1}\partial_x\partial_y+a_{m-2,0,2}\partial_y^2+a_{m-2,1,0}\partial_x\\ &+a_{m-2,0,1}\partial_y+a_{m-2,0,0})\partial_t^{m-2}+\sum_{i\leq m-3} a_{ijk}\partial_t^i\partial_x^j\partial_y^k u\,.\end{aligned} \tag{4.2}$$

Putting $u=\exp(i\eta y)v(t,x)$, $v(t,x)$ must satisfy the following equation;

$$\begin{aligned}\partial_t^{m-1}\partial_x v=&[a_{m-2,0,2}(i\eta)^2+a_{m-2,0,1}(i\eta)+a_{m-2,0,0}]\partial_t^{m-2}v+(a_{m-2,2,0}\partial_x^2\\ &+a_{m-2,1,1}(i\eta)\partial_x+a_{m-2,1,0}\partial_x)\partial_t^{m-2}v+\sum_{i\leq m-3} a_{ijk}(i\eta)^k\partial_t^i\partial_x^j v\,.\end{aligned} \tag{4.3}$$

If necessary, changing x by $-x$, we can assume $a_{m-2,0,2}<0$. Now the first term of the right-hand side has the form:

$$(a\eta^2+ib\eta+c)\partial_t^{m-2}v\,,\qquad \text{where}\quad a>0\,,$$

we make the positive parameter η tend to $+\infty$. Then denoting

$$\zeta=\sqrt{a\eta^2+ib\eta+c}\,,$$

with $\operatorname{Re}\zeta>0$, we have

$$\zeta=\sqrt{a}\,\eta+i\frac{b}{2\sqrt{a}}+O\Big(\frac{1}{\eta}\Big)\qquad\text{when}\quad \eta\longrightarrow+\infty\,.$$

This implies in particular (let us remark b is also real),

$$|\zeta|=\zeta+O\Big(\frac{1}{\eta}\Big)=\zeta+O\Big(\frac{1}{|\zeta|}\Big)$$

Let us consider the solution v of (4.3) which satisfies the following conditions;

(4. 4)
$$\begin{cases} \partial_t^i v(0, x)=0 & 0\leqq i\leqq m-3 \\ \partial_t^{m-2} v(0, x)=1 & \\ v(t, 0)=t^{m-2}/(m-2)\,! & \end{cases}$$

In view of the first condition, we put $v(t, x)$

(4. 5)
$$v(t, x)=\sum_{p\geqq 0, q\geqq 0} v_{pq}\frac{t^{m-2+p}}{(m-2+p)\,!}\,\frac{x^q}{q\,!}.$$

The second condition implies

$$v_{00}=1.\quad v_{0,q}=0,\qquad \text{for all}\quad q\geqq 1.$$

The third condition implies

$$v_{00}=1,\quad v_{p,0}=0,\qquad \text{for all}\quad p\geqq 1.$$

Comparing the coefficient of $t^r x^s$,

(4. 6)
$$v_{r+1, s+1}=\zeta^2 v_{r, s}+(a_{m-2, 2, 0}v_{r, s+2}+a_{m-2, 1, 1}(i\eta)v_{r, s+1}+a_{m-2, 1, 0}v_{r, s+1}) + \sum_{i\leqq m-3} a_{ijk}(i\eta)^k v_{r+i-(m-2), s+j}\,.$$

To make clear the recurence relation, we consider the terms corresponding to $i=m-3$ in the summation.

$$\sum_{j, k} a_{m-3, j, k}(i\eta)^k v_{r-1, s+j}$$

Since $j+k\leqq 3$, $j\leqq 3$. So

$$\sum_{j=0}^{3}(\sum_{k\leqq 3-j} a_{m-3, j, k}(i\eta)^k)v_{r-1, s+j}\,.$$

So, if we denote $a_{m-3, j}(\eta)=\sum_{k\leqq 3-j} a_{m-3, j, k}(i\eta)^k$, we can rewrite

$$\sum_{j=0}^{3} a_{m-3, j}(\eta)v_{r-1, s+j}\,,$$

where we have

$$|a_{m-3, j}(\eta)|\leqq A'\eta^{3-j}\,.$$

So (4. 6) can be written in the form,

(4. 7)
$$v_{r+1, s+1}=\zeta^2 v_{r, s}+a_1(\eta)v_{r, s+1}+a_0 v_{r, s+2}+\sum_{j=0}^{3} a_{m-3, j}(\eta)v_{r-1, r+j} + \sum_{j=0}^{4} a_{m-4, j}(\eta)v_{r-2, s+j}+\cdots+\sum_{j=0}^{i} a_{m-i, j}(\eta)v_{r-i+2, r+j}+\cdots$$

where $a_1(\eta)=a_{m-2, 1, 1}(i\eta)+a_{m-2, 1, 0}$, $a_0=a_{m=2, 2, 0}$ and $a_{m-i, j}(\eta)$ is a polynomial in η of order$\leqq i-j$. This shows that $v_{p, q}$ is determined uniquely. At first, we see easily that

(4. 8)
$$\begin{cases} v_{pp}=\zeta^{2p} & \text{for}\quad p\geqq 0 \\ v_{pq}=0 & \text{for}\quad q>p\,. \end{cases}$$

We are going to estimate v_{pq} $(q<p)$, where η is large. For this purpose let us remark that there exists a positive constant A such that for η large

$$|a_1(\eta)|\leqq A|\zeta|,\quad |a_0|\leqq A,$$
$$|a_{m-i,j}(\eta)|\leqq A|\zeta|^{i-j}.\quad \text{(Remark that } \eta<\text{const.}|\zeta|).$$

In view of this, we consider the following associate majorant:

$$v'_{r+1,s+1}=|\zeta|^2v'_{r,s}+A|\zeta|v'_{r,s+1}+Av'_{r,s+2}+\sum_{j=0}^{3}A|\zeta|^{3-j}v'_{r-1,s+j}$$
$$+\sum_{j=0}^{4}A|\zeta|^{4-j}v'_{r-2,s+j}+\cdots+\sum_{j=0}^{i}A|\zeta|^{i-j}v'_{r-i+2,s+j}+\cdots \tag{4.9}$$

Then we have

Lemma 4.1. *It holds*

$$|v'_{rs}|\leqq\frac{r!}{s!}|\zeta|^{2s}(C|\zeta|)^{r-s}\qquad\text{for}\quad s\leqq r.$$

where C is an appropriate positive constant (independent of η).

Proof. At first by (4.8) the lemma is true for $s=r$. So we shall prove this for $s<r$. We prove this by induction on r. Suppose the estimate is true for v'_{is} $i\leqq r$. Then

$$|v'_{r+1,s+1}|\leqq|\zeta|^2\frac{r!}{s!}|\zeta|^{2s}(C|\zeta|)^{r-s}+A|\zeta|\frac{r!}{(s+1)!}|\zeta|^{2s+2}(C|\zeta|)^{r-s-1}$$
$$+A\frac{r!}{(s+2)!}|\zeta|^{2s+4}(C|\zeta|)^{r-s-2}$$
$$+A\sum_{j=0}^{3}|\zeta|^{3-j}\frac{(r-1)!}{(s+j)!}|\zeta|^{2s+2j}(C|\zeta|)^{r-s-j-1}$$
$$+A\sum_{j=0}^{4}|\zeta|^{4-j}\frac{(r-2)!}{(s+j)!}|\zeta|^{2s+2j}(C|\zeta|)^{r-s-j-2}+\cdots$$
$$+A\sum_{j=0}^{i}|\zeta|^{i-j}\frac{(r-i+2)!}{(s+j)!}|\zeta|^{2s+2j}(C|\zeta|)^{r-s-j-(i-2)}+\cdots$$
$$=\frac{(r+1)!}{(s+1)!}|\zeta|^{2s+2}(C|\zeta|)^{r-s}\Big[\frac{s+1}{r+1}+\frac{1}{r+1}\frac{A}{C}+\frac{1}{(r+1)(s+2)}\frac{A}{C^2}$$
$$+A\sum_{j=0}^{3}\frac{1}{(r+1)r}\frac{(s+1)!}{(s+j)!}C^{-j-1}+A\sum_{j=0}^{4}\frac{(r-2)!}{(r+1)!}\frac{(s+1)!}{(s+j)!}C^{-j-2}+\cdots$$
$$+A\sum_{j=0}^{i}\frac{(r-i+2)!}{(r+1)!}\frac{(s+1)!}{(s+j)!}C^{-j-i+2}+\cdots\Big].$$

The quantity between [ ] is majorized by

$$\frac{r}{r+1}+\frac{1}{r+1}\frac{A}{C}+\frac{1}{r+1}\frac{A}{C^2}+A\sum_{j=0}^{3}\frac{1}{r+1}C^{-j-1}+A\sum_{j=0}^{4}\frac{1}{r+1}C^{-j-2} \tag{4.10}$$

$$+\cdots+A\sum_{j=0}^{i}\frac{1}{r+1}C^{-j-i+2}+\cdots$$

Now we choose $C(>1)$ in such a way that

$$\frac{A}{C}\sum_{j=0}^{\infty}C^{-j}<\frac{1}{m}.$$

Namely, $C>mA+1$.

Then, in view of $\frac{r}{r+1}=1-\frac{1}{r+1}$, the above quantity is less than 1. This completes the proof of Lemma.

Now we consider

$$\partial_t^{m-2}v=\sum_{p,q}\frac{t^p}{p!}\frac{x^q}{q!}v_{pq}.$$

For $t, x\geqq 0$, we have

$$|\partial_t^{m-2}v(t, x;\eta)|\geqq\Big|\sum_{p\geq 0}v_{pp}\frac{t^p x^p}{(p!)^2}\Big|-\Big|\sum_{q<p}v_{pq}\frac{t^p x^q}{p!q!}\Big|$$

$$\geqq\Big|\sum_{p\geq 0}\frac{\zeta^{2p}}{(p!)^2}t^p x^p\Big|-\sum_{q<p}v'_{pq}\frac{t^p x^q}{p!q!}\quad(\text{Since } |v_{pq}|<v'_{pq}\ \text{ for } q<p)$$

$$=\Big|\sum_{p\geq 0}\frac{\zeta^{2p}}{(p!)^2}t^p x^p\Big|-\sum_{q<p}t^p\frac{x^q}{(q!)^2}|\zeta|^{2q}(C|\zeta|)^{p-q}.$$

This second term is

$$\sum_{q\geq 0}\frac{|\zeta|^{2q}}{(q!)^2}x^q\sum_{p;\,p>q}t^p(C|\zeta|)^{p-q}=\sum_{q\geq 0}\frac{|\zeta|^{2q}}{(q!)^2}x^q t^q\sum_{j\geq 1}(C|\zeta|)^j t^j.$$

In conclusion

(4.11)
$$|\partial_t^{m-2}v(t, x;\eta)|\geqq\sum_{q\geq 0}\frac{|\zeta|^{2q}}{(q!)^2}x^q t^q\Big(1-\sum_{j\geq 1}(C|\zeta|)^j t^j\Big)$$
$$-\Big(\sum_{q\geq 0}\frac{|\zeta|^{2q}}{(q!)^2}x^q t^q-\sum_{q\geq 0}\frac{\zeta^{2q}}{(q!)^2}x^q t^q\Big).$$

Now let us recall the definition of the Bessel function of imaginary argument

$$I_0(z)=\sum_{j\geq 0}\frac{1}{(j!)^2}\Big(\frac{z}{2}\Big)^{2j}$$

and its asymptotic formula for $z\to\infty$ in the sector $-\frac{\pi}{2}+\delta<\arg z<\frac{\pi}{2}-\delta$

$$I_0(z)\sim\frac{e^z}{\sqrt{2\pi z}}\Big(1+\frac{1}{8z}+O\Big(\frac{1}{z^2}\Big)\Big).$$

Now, the right-hand side of (4.11) is expressed as

(4.12)
$$I_0(2|\zeta|\sqrt{xt})\Big(1-\sum_{j\geq 1}(C|\zeta|)^j t^j\Big)-\{I_0(2|\zeta|\sqrt{xt})-I_0(2\zeta\sqrt{xt})\}$$

We take $t=t_\zeta=\{C|\zeta|\}^{-1}\times\frac{1}{3}$. Then $1-\sum_{j\geq 1}(C|\zeta|)^j t^j=\frac{1}{2}$. We fix $x_0>0$. Then, since $2|\zeta|\sqrt{x_0}\sqrt{t_\zeta}=\delta\sqrt{|\zeta|}$ (δ: positive constant) thus the first term increases

$$\sim\frac{1}{\sqrt{2\pi\delta\sqrt{|\zeta|}}}\exp(\delta|\zeta|^{1/2})\{1+O(|\zeta|^{-1/2})\}.$$

Next,

$$\zeta\sqrt{x_0 t_\zeta}=(|\zeta|+O(1/|\zeta|))\sqrt{x_0 t_\zeta}=\frac{\delta}{2}|\zeta|^{1/2}+O(1/|\zeta|^{3/2})$$

Thus

$$\exp(2\zeta\sqrt{x_0 t_\zeta})=\exp(\delta|\zeta|^{1/2})\exp(O(|\zeta|^{-3/2}))$$
$$=\exp(\delta|\zeta|^{1/2})\{1+O(|\zeta|^{-3/2})\}$$

This shows

$$I_0(2|\zeta|\sqrt{x_0 t_\zeta})-I_0(2\zeta\sqrt{x_0 t_\zeta})=\exp(\delta|\zeta|^{1/2})\{O(|\zeta|^{-3/2})\}$$

Thus we have

$$|\partial_t^{m-1}v(t_\zeta, x_0\,;\eta)|\geqq\exp(\delta'|\zeta|^{1/2}),\quad(0<\delta'<\delta) \tag{4.13}$$

for η therefore $|\zeta|$ large.

Now, recall $u=\exp(i\eta y)v(t, x)$. By (4.4) we have the following;

$$\begin{cases}\partial_t^i u(0, x, y)=0 & 0\leqq i\leqq m-3\\ \partial_t^{m-2}u(0, x, y)=e^{i\eta y}\\ u(t, 0, y)=e^{i\eta y}t^{m-2}/(m-2)!\end{cases} \tag{4.14}$$

By the assumption of $\mathcal{E}$-wellposedness the grouth order of $\partial_t^{m-1}u(t, x, y)$ is at most polynomial of η therefore $|\zeta|$. On the otherhand, by (4.13), $\partial_t^{m-1}v(t_\zeta, x_0. y)$ has exponential order of $|\zeta|$. These can not be compatible.

Second step

Denote

$$P_m(\tau, \xi, \eta)=\xi\mathring{Q}_{m-1}(\tau, \xi, \eta)+\mathring{P}_m(\tau, \eta)$$

Owing to the first step, the degree of $\mathring{P}_m(\tau, \eta)$ with respect to τ is at most $m-3$. We shall prove "if $\mathring{P}_m(\tau, \eta)\not\equiv 0$, then the characteristic polynomial $P_m(\tau, \xi, \eta)=0$ has a non-real root".

Considering the homogeneity of $P_m(\tau, \xi, \eta)$, we put

$$\tau/\eta=\tau',\qquad \xi/\eta=\lambda. \tag{4.15}$$

Then

$$P_m(\tau, \xi, \eta)=\eta^m\{\lambda\mathring{Q}_{m-1}(\tau', \lambda, 1)+\mathring{P}_m(\tau', 1)\}=0. \tag{4.16}$$

Suppose

$$\mathring{P}_m(\tau', 1)=a_{m-j}\tau'^{m-j}+\cdots+a_0,\quad a_{m-j}\neq 0,\quad j\geqq 3. \tag{4.17}$$

If necessary, changing x by $-x$, we can assume $a_{m-j}>0$. Again putting

$$\tau'^{-1}=\zeta \tag{4.18}$$

and denoting

$$\mathring{Q}_{m-1}(\tau', \lambda, 1)=\tau'^{m-1}+C_1(\lambda)\tau'^{m-2}+\cdots+C_{m-1}(\lambda),$$

(4.16) can be written as follows.

$$\begin{aligned}\lambda(1+C_1(\lambda)\zeta+C_2(\lambda)\zeta^2+\cdots+C_{m-1}(\lambda)\zeta^{m-1})\\ +\zeta^{j-1}(a_{m-j}+a_{m-j-1}\zeta+\cdots+a_0\zeta^{m-j})=0.\end{aligned} \tag{4.19}$$

Then

$$\zeta^{j-1}=-\frac{1+C_1(\lambda)\zeta+\cdots+C_{m-1}(\lambda)\zeta^{m-1}}{a_{m-j}+a_{m-j-1}\zeta+\cdots+a_0\zeta^{m-j}}\lambda \tag{4.20}$$

This show that (4.19) has the roots $\zeta(\lambda)$ such that

$$\zeta(\lambda)\sim\sqrt[j-1]{-\lambda/a_{m-j}} \quad \text{when} \quad \lambda\to 0. \tag{4.21}$$

If $j-1\geqq 2$ since a_{m-j} is real and positive one of branches satisfies

$$|\operatorname{Im}\zeta(\lambda)|\geqq\delta\sqrt[j-1]{|\lambda|}. \tag{4.22}$$

Thus we complete the second step.

Finally we consider $y\in R^l$. Recall (2.2)

$$\partial_t^{m-1}\partial_x u=\sum_{\substack{i+j+|\alpha|\leq m\\ i\leq m-2}} a_{ij\alpha}\partial_t^i\partial_x^j\partial_y^\alpha u. \tag{2.2}$$

Theorem 2 means that $\mathcal{E}$-wellposed then

$$\sum_{|\alpha|=i}|a_{m-i,0,\alpha}|=0, \qquad 2\leqq i\leqq m.$$

If $\sum_{|\alpha|=i}|a_{m-i,0,\alpha}|\neq 0$, by suitable change of independent variables, we can consider the coefficient of $\partial_t^{m-i}\partial_{y_1}^i\neq 0$. In first step we consider such $u=u(t, x, y_1)$ which is independent of $\{y_i; 2\leqq i\leqq l\}$. So we can make the same process as $l=1$. In second step, we consider such $\eta=(\eta_1, 0\cdots 0)$. So we can make the same process as $l=1$. Thus we complete the proof of Theorem 2.

§5. Proof of Theorem 3

We prove Th. 3 by the same principle as §3. However, in this case the reasoning becomes fairly delicate. We regard P_m+P_{m-1} as the principal part of P. Consider the equation (2.2), then

$$\begin{aligned}P(\tau, \xi, \eta)&=\tau^{m-1}\xi-\sum_{i\leq m-2}a_{ij\alpha}\tau^i\xi^j\eta^\alpha\\ &\equiv P_m(\tau, \xi, \eta)+P_{m-1}(\tau, \xi, \eta)+R_{m-2}(\tau, \xi, \eta).\end{aligned} \tag{5.1}$$

where P_m, P_{m-1} are the polynomials of homogeneous degree m and $m-1$ respectively. R_{m-2} is a polynomial of degree $m-2$. Let us remark that the coefficient of τ^{m-1} in P_{m-1} vanishes and by Theorem 2 we can assume $P_m=\xi\mathring{Q}_{m-1}$. Let

$$P_{m-1}=\xi Q_{m-2}(\tau,\ \xi,\ \eta)+\mathring{P}_{m-1}(\tau,\ \eta) \tag{5.2}$$

where Q_{m-2} and $\mathring{P}_{m-1}$ are the polynomials of homogeneous degree $m-2$ and $m-1$ respectively, then

$$\begin{aligned} P&=\xi(\mathring{Q}_{m-1}+Q_{m-2})+\mathring{P}_{m-1}(\tau,\ \eta)+R_{m-2} \\ &\equiv\xi Q_{m-1}+\mathring{P}_{m-1}(\tau,\ \eta)+R_{m-2}\,. \end{aligned} \tag{5.3}$$

where the coefficient τ^{m-1} in P_{m-1} vanishes and by the assumption (A. 4), the roots $\tau=\tau_i(\xi,\ \eta)$ $(1\leqq i\leqq m-1)$ of $\mathring{Q}_{m-1}(\tau,\ \xi,\ \eta)=0$ are all real and distinct.

We shall prove that if $\mathring{P}_{m-1}(\tau,\ \eta)\not\equiv 0$ then the Goursat problem is not $\mathcal{E}$-well-posed. Suppose $\mathring{P}_{m-1}(\tau,\ \eta)\not\equiv 0$, then there exists η^0 such that $\mathring{P}_{m-1}(\tau\ '\eta^0)\not\equiv 0$. Let us consider the characteristic roots of $P_m(\tau,\ i\xi,\ i\eta)+P_{m-1}(\tau,\ i\xi,\ i\eta)=0$ in a neighborhood of $\xi=\sqrt{\lambda},\ \eta=\lambda\eta^0$ $(\lambda>0$, large). We have the following lemma.

Lemma 5. 1. *If $\mathring{Q}_{m-1}(\tau\,;\xi,\ \eta)=0$ has real distinct roots $\lambda_i(\xi,\ \eta)$ $(1\leqq i\leqq m-1)$ for all real $(\xi,\ \eta)\neq(0,\ 0)$, and if $\mathring{P}_{m-1}(\tau,\ \eta)\not\equiv 0$, then there exists an $\eta^0(|\eta^0|=1)$ such that $\mathring{P}_{m-1}(\tau,\ \eta^0)\not\equiv 0$ and the roots $\{\tau_i\}_{1\leq i\leq m-1}$ of*

$$i\xi Q_{m-1}(\tau,\ i\xi,\ i\eta)+\mathring{P}_{m-1}(\tau,\ i\eta)=0\,. \tag{5.4}$$

have the following estimate for $\xi,\ \eta\in V_\lambda$.

$$\begin{cases} \operatorname{Re}\tau_i>\delta\sqrt{\lambda}, & 1\leqq i\leqq N_1\quad N_1\geqq 1\,,\quad \delta>0 \\ |\operatorname{Re}\tau_i|\leqq const. & N_1+1\leqq i\leqq N_2\,, \\ \operatorname{Re}\tau_i<-\delta\sqrt{\lambda}, & N_2+1\leqq i\leqq m-1 \qquad for\ \ \lambda(>0)\ large, \end{cases} \tag{5.5}$$

where $V_\lambda=\{(\xi,\ \eta)\,;|\xi-\varepsilon_0\sqrt{\lambda}|<\varepsilon\sqrt{\lambda},|\eta-\lambda\eta^0|<\varepsilon\lambda\}$, $\varepsilon_0=+1$ or -1, the choice being defined later.

Proof of Lemma

Putting $\tau=i\tau'$ then considering the homogeneity of $\mathring{Q}_{m-1}$, Q_{m-2} and $\mathring{P}_{m-1}$, (5. 4) becomes

$$\xi\{\mathring{Q}_{m-1}(\tau',\ \xi,\ \eta)-iQ_{m-2}(\tau',\ \xi,\ \eta)\}-i\mathring{P}_{m-1}(\tau',\ \eta)=0\,. \tag{5.6}$$

Now, let

$$\mathring{Q}_{m-1}(\tau',\ \xi,\ \eta)=\prod_{i=1}^{m-1}(\tau'-\lambda_i(\xi,\ \eta))\,. \tag{5.7}$$

$\{\lambda_i(\xi,\ \eta)\}_{1\leq i\leq m-1}$ are all real and distinct for all $(\xi,\ \eta)\neq(0,\ 0)$. Put $\lambda_i(0,\ \eta)=\lambda_i(\eta)$. By the assumption there exists $\eta^0(|\eta^0|=1)$ such that $\mathring{P}_{m-1}(\tau',\ \eta^0)\not\equiv 0$. Next, since the degree in τ' of $\mathring{P}_{m-1}(\tau,\ \eta^0)$ is at most $m-2$, there exists at least one i $(1\leqq i\leqq m-1)$ such that $\mathring{P}_{m-1}(\lambda_i(\eta^0),\ \eta^0)\neq 0$. Moreover we can assume, if necessary by changing slightly η^0 we have

(5.8) $\mathring{P}_{m-1}(\lambda_k(\eta), \eta) \neq 0 \quad \text{or} \quad \mathring{P}_{m-1}(\lambda_k(\eta), \eta) \equiv 0$

for $\eta \in V_0 = \{\eta ; |\eta - \eta^0| < \varepsilon, |\eta| = 1\}$ where ε is small.

Let us write

$$\eta = |\eta|\eta', \quad |\eta'| = 1.$$

Therefore

$$(5.9) \quad \begin{cases} \lambda_i(\xi, \eta) = \lambda_i(\xi/|\eta|, \eta')|\eta| \\ Q_{m-2}(\tau', \xi, \eta) = Q_{m-2}(\tau'/|\eta|, \xi/|\eta|, \eta')|\eta|^{m-2} \\ \mathring{P}_{m-1}(\tau', \eta) = \mathring{P}_{m-1}(\tau'/|\eta|, \eta')|\eta|^{m-1}. \end{cases}$$

Put

$$(5.10) \quad \tau' = \lambda_i(\xi, \eta) + \zeta(\xi, \eta)|\eta|.$$

By (5.9) and (5.10), (5.6) becomes

$$(5.11) \quad \zeta \prod_{j \neq i} (\lambda_i(\xi/|\eta|, \eta') - \lambda_j(\xi/|\eta|, \eta') + \zeta) \\ - i|\eta|^{-1} Q_{m-2}(\lambda_i(\xi/|\eta|, \eta') + \zeta, \xi/|\eta|, \eta') \\ = i\mathring{P}_{m-1}(\lambda_i(\xi/|\eta|, \eta') + \zeta, \eta')/\xi.$$

Put $\xi/|\eta| = s$, where we consider $s \to 0$ when $|\eta| \to \infty$. Then (5.11) becomes

$$\zeta \prod_{j \neq i} (\lambda_i(s, \eta') - \lambda_j(s, \eta') + \zeta) - i\frac{s}{\xi} Q_{m-2}(\lambda_i(s, \eta') + \zeta, s, \eta') \\ = i\mathring{P}_{m-1}(\lambda_i(s, \eta') + \zeta, \eta')/\xi.$$

Namely

$$\zeta = \frac{i\mathring{P}_{m-1}(\lambda_i(s, \eta') + \zeta, \eta') + isQ_{m-2}(\lambda_i(s, \eta') + \zeta, s, \eta')}{\prod_{j \neq i} (\lambda_i(s, \eta') - \lambda_j(s, \eta') + \zeta)} \cdot \frac{1}{\xi}$$

Rewrite this by

$$\zeta = f(\zeta ; s, \eta')/\xi.$$

Moreover putting

$$1/\xi = u.$$

Then

$$(5.12) \quad \zeta = f(\zeta ; s, \eta')u,$$

where $f(\zeta ; s, \eta')$ is a holomorophic function in a neighborhood of $(0 ; 0, \eta^0)$ with respect to ζ, s and η. By means of Lagrange's Theorem, ζ is a holomorphic function of u (considering s, η' are holomorphic parameter) and

$$(5.13) \quad \zeta = \sum_{j=1}^{\infty} c_j(s, \eta')u^j,$$

$$c_j(s, \eta') = \frac{1}{j} \cdot \frac{1}{2\pi i} \int_{z=\delta_0} \frac{f(z ; s, \eta')^j}{z^j} dz \quad (j \geqq 1).$$

where δ_0 is positive and suitable small. By calculation we have

$$c_1(0, \eta')=i\mathring{P}_{m-1}(\lambda_i(\eta'), \eta')/\prod_{j\neq i}(\lambda_i(\eta')-\lambda_j(\eta')).$$

Because of (5.8), we have

$$|c_1(0, \eta')|\geqq\delta'(>0) \quad \text{or} \quad c_1(0, \eta')\equiv 0, \qquad \eta'\in V_0.$$

Moreover considering the Taylor expansion of ζ with respect to s, we have

$$\zeta=\sum_{ij}c_{ij}(\eta')s^i u^j, \qquad \text{where} \quad c_{01}(\eta')=c_1(0, \eta').$$

Namely

$$\zeta\{i\mathring{P}_{m-1}(\lambda_i(\eta'), \eta')/\prod_{j\neq i}(\lambda_i(\eta')-\lambda_j(\eta'))\}u+\sum_{i+j\geq 2}c_{ij}(\eta')s^i u^j. \tag{5.14}$$

Particularly we put $s=\rho u$, therefore $\xi/|\eta|=\rho/\xi$ so $\xi^2=\rho|\eta|$. Then (5.14) becomes the following:

$$\zeta=\{i\mathring{P}_{m-1}(\lambda_i(\eta'), \eta')/\prod_{j\neq i}(\lambda_i(\eta')-\lambda_j(\eta'))\}u+\sum_{i+j\geq 2}c_{ij}(\eta')\rho^i u^{i+j}. \tag{5.15}$$

Finally we have

$$\begin{aligned}\tau_i(\xi, \eta)&=i\tau'_i(\xi, \eta)\\ &=i\lambda_i(\xi, \eta)-(\mathring{P}_{m-1}(\lambda_i(\eta'), \eta')/\prod_{j\neq i}(\lambda_i(\eta')-\lambda_j(\eta'))\frac{\xi}{\rho}\\ &\quad+i\sum_{i+j\geq 2}c_{ij}(\eta')\rho^{i-1}\left(\frac{1}{\xi}\right)^{i+j-2}.\end{aligned} \tag{5.16}$$

Moreover considering that $c_{ij}(\eta')$ is of homogeneous degree 1 we put $c_{ij}(\eta')|\eta| =c_{ij}(\eta)$ so we have

$$\begin{aligned}\tau_i(\xi, \eta)&=i\lambda_i(\xi, \eta)\\ &\quad-\{\mathring{P}_{m-1}(\lambda_i(\eta), \eta)/\prod_{j\neq i}(\lambda_i(\eta)-\lambda_j(\eta))\}\frac{1}{\xi}\\ &\quad+i\sum_{i+j\geq 2}c_{ij}(\eta)(\xi/|\eta|)^i(1/\xi)^j.\end{aligned} \tag{5.16'}$$

We can assume $\mathring{P}_{m-1}(\lambda_1(\eta'), \eta')\neq 0$, $\eta'\in V_0$. Now, let us define the sign of ε_0 by the following;

$$-(\mathring{P}_{m-1}(\lambda_1(\eta'), \eta')/\prod_{j\neq 1}(\lambda_1(\eta')-\lambda_j(\eta'))\times\varepsilon_0>0.$$

Then by (5.16) we obtain (5.5). The proof of the lemma thus complete.

Localization in the (x, y) space

Let $\beta(x, y)$ be $C^{\infty}_{x,y}$ function with compact support. Apply β to $Pu=0$. Then $\beta Pu=0$. By the formula

$$\beta Pu=\sum_{|\nu|\geq 0}\frac{(-1)^{|\nu|}}{\nu!}P^{(\nu)}(\beta^{(\nu)}u)$$

we have

$$P[\beta u]=-\sum_{|\nu|\geq 1}\frac{(-1)^{|\nu|}}{\nu!}P^{(\nu)}[\beta^{(\nu)}u]. \tag{5.17}$$

Now we take a function $\alpha(\xi, \eta)$ with compact support. Apply this to (5.17), because of constant coefficients we have

$$P[\alpha\beta u]=-\sum_{|\nu|\geq 1}\frac{(-1)^{|\nu|}}{\nu!}P^{(\nu)}[\alpha\beta^{(\nu)}u]. \tag{5.18}$$

For convenience we say that the order of differential operator ∂_t, ∂_y is 1 and the order of ∂_x is 1/2. For example the order of $\partial_t^i\partial_x^j\partial_y^\alpha$ is $i+|\alpha|+j/2$. The order of left-hand side of (5.18) is $m-1+1/2$. The order of $\frac{\partial}{\partial\xi}P$ is $m-1$. And the another terms in the right-hand side of (5.18) is at most $m-1-1/2$. So the most delicate part in the right-hand side is $\frac{\partial}{\partial\xi}P[\alpha\beta_x u]$. The same reasoning used in § 3 can not be applied to estimate this term. This difficulty was overcome in the following way*). Recall (5.3),

$$P=\xi Q_{m-1}(\tau, \xi, \eta)+\mathring{P}_{m-1}(\tau, \eta)+R_{m-2}(\tau, \xi, \eta).$$

Then

$$\frac{\partial P}{\partial\xi}=Q_{m-1}+\xi\frac{\partial Q_{m-1}}{\partial\xi}+\frac{\partial R_{m-2}}{\partial\xi}. \tag{5.19}$$

The order of Q_{m-1} is $m-1$ and the order of $\xi\frac{\partial Q_{m-1}}{\partial\xi}+\frac{\partial R_{m-2}}{\partial\xi}$ is at most $m-1-1/2$. Replacing in (5.18) $\beta\to\beta_x$ we have

$$P[\alpha\beta_x u]=-\sum_{|\nu|\geq 1}(-1)^{|\nu|}P^{(\nu)}[\alpha(\beta_x)^{(\nu)}u]. \tag{5.20}$$

Let us assume that on the support of $\alpha(\xi, \eta)$ ξ does not vanish. Now we define the pseudo-differential operator ∂_x^{-1} by

$$\partial_x^{-1}f\equiv\mathcal{F}^{-1}\Big(\frac{1}{i\xi}\hat{f}(\xi, \eta)\Big), \tag{5.21}$$

where $f(x, y)$ is assumed that its Fouriertransform $\hat{f}(\xi, \eta)$ has its support away from the hyperplane $\xi=0$. The operator ∂_x^{-1} is the same as $(i\xi_1)^{-1}$ in § 3. We regard the order of ∂_x^{-1} as $-1/2$. Using the operator ∂_x^{-1}, (5.20) can be written as follows:

$$\begin{aligned}Q_{m-1}(\alpha\beta_x u)=&-\partial_x^{-1}(\mathring{P}_{m-1}+R_{m-2})[\alpha\beta_x u]\\&-\sum_{|\nu|\geq 1}\frac{(-1)^{|\nu|}}{\nu!}\partial_x^{-1}P^{(\nu)}[\alpha(\beta_x)^{(\nu)}u].\end{aligned} \tag{5.22}$$

By (5.18) and (5.19) we have

$$\begin{aligned}P[\alpha\beta u]=&\Big[Q_{m-1}+\partial_x\frac{\partial Q_{m-1}}{\partial\xi}+\frac{\partial R_{m-2}}{\partial\xi}\Big](\alpha\beta_x u)\\&-\sum_{|\nu|\geq 1}{}'\frac{(-1)^{|\nu|}}{\nu!}P^{(\nu)}(\alpha\beta^{(\nu)}u)\end{aligned}$$

*) This idea is due to Prof. Mizohata.

where $\sum'$ does not contain the term $\frac{\partial}{\partial\xi}P$, by (5.22)

$$P[\alpha\beta u]=-\sum_{|\nu|\geq 1}\frac{(-1)^{|\nu|}}{\nu!}\partial_x^{-1}P^{(\nu)}[\alpha(\beta_x)^{(\nu)}u]$$
$$-\partial_x^{-1}[\mathring{P}_{m-1}+R_{m-2}](\alpha\beta_x u)+\left[\partial_x\frac{\partial Q_{m-1}}{\partial\xi}+\frac{\partial R_{m-2}}{\partial\xi}\right](\alpha\beta_x u)$$
$$-\sum_{|\nu|\geq 1}'\frac{(-1)^{|\nu|}}{\nu!}P^{(\nu)}[\alpha\beta^{(\nu)}u].$$

Finally

$$P[\alpha\beta u]=\sum_{|\nu|\geq 1}C_\nu(\partial_t\partial_x\partial_y\partial_x^{-1})(\alpha\beta^{(\nu)}u), \tag{5.23}$$

where the order of the left-hand side is $m-1/2$ and the right-hand side is at most $m-1-1/2$. Let us notice that the right-hand side contains the term $\partial_x^{-1}\partial_t^{m-1}$.

Localization in the (ξ, η) space and reduction to the system

At first (if necessary replacing x by $-x$) we can assume ε_0 (in Lemma 5.1) $=+1$. Take a function $\alpha_0(\xi, \eta)\in C^{\infty}$ of small support which takes the value 1 in a neighborhood of $(\xi, \eta)=(0, 0)$. Define

$$\alpha_n(\xi, \eta)=\alpha_0\left(\frac{\xi-\sqrt{n}}{\sqrt{n}}, \frac{\eta-n\eta^0}{\sqrt{n}}\right). \tag{5.24}$$

Assuming the support of $\alpha_0(\xi, \eta)$ remains in $|\xi|, |\eta|<\varepsilon'$ ($\varepsilon'>0$, small), then the support of $\alpha_n(\xi, \eta)$ remain

$$\begin{cases} (1-\varepsilon')\sqrt{n}\leqq\xi\leqq(1+\varepsilon')\sqrt{n} \\ n-\varepsilon'\sqrt{n}\leqq|\eta|\leqq n+\varepsilon'\sqrt{n} \end{cases} \tag{5.25}$$

Recall (5.23) and replace α by α_n, then

$$P[\alpha_n\beta u]=\sum_{|\nu|\geq 1}C_\nu(\partial_t\partial_x\partial_y\partial_x^{-1})[\alpha_n\beta^{(\nu)}u]\equiv f_{(n)}. \tag{5.26}$$

By (5.3) we can rewrite (5.26) in the following form:

$$[Q_{m-1}+\partial_x^{-1}\mathring{P}_{m-1}][\alpha_n\beta u]=\partial_x^{-1}[f_{(n)}-R_{m-2}(\alpha_n\beta u)]. \tag{5.27}$$

Let

$$E(\Lambda)=\begin{pmatrix} (\Lambda+1)^{m-2} & & \\ & (\Lambda+1)^{m-3} & \\ & & \ddots \\ & & & 1 \end{pmatrix}.$$

Putting $\tilde{u}={}^t(u, \partial_t u, \cdots, \partial_t^{m-2}u)$ and $U=E(\Lambda)(\beta\tilde{u})$ we obtain an equivalent system to (5.27):

$$\partial_t(\alpha_n U)=H\alpha_n U+F, \tag{5.28}$$

where $F={}^t(0, \cdots, 0, \partial_x^{-1}[f_{(n)}-R_{m-2}(\alpha_n\beta u)])$. Denote

$\Omega=\{(\xi, \eta); \xi>R$ and $\eta'\in V_0$ where $\eta=\eta'|\eta|$, and $1-\varepsilon''<\xi^2/|\eta|<1+\varepsilon''\}$

where R is a large constant and ε'' is a small constant. Let us diagonalize H in Ω. Taking account of the fact that $Q_{m-1}(\tau, \xi, \eta)=0$ has real distinct roots (if we choose R sufficiently large) we can find $(m-1)\times(m-1)$ matrix $N(\xi, \eta)$ having the following properties.

i) $N(\xi, \eta)HN^{-1}(\xi, \eta)\begin{pmatrix} \tau_1(\xi, \eta) & & \mathbf{0} \\ & \tau_2(\xi, \eta) & \\ & \ddots & \\ \mathbf{0} & & \tau_{m-1}(\xi, \eta) \end{pmatrix} \equiv D$

ii) Each component of $N(\xi, \eta)$ and $N^{-1}(\xi, \eta)$ is C^∞fuction of order 0 with respect to (ξ, η), for $(\xi, \eta)\in\Omega$.

iii) $|\partial_\xi^{\nu_1}\partial_\eta^{\nu_2}n_{ij}(\xi, \eta)|<\text{const.}(|\xi|+|\eta|)^{-\frac{\nu_1+|\nu_2|}{2}}$ for $(\xi, \eta)\in\Omega$ where $n_{ij}(\xi, \eta)$ is the symbol of (ij) element of N.

For example we take

$$N=\begin{pmatrix} 1 & 1 & \cdots\cdots & 1 \\ \tilde\tau_1 & \tilde\tau_2 & \cdots\cdots & \tilde\tau_{m-1} \\ \tilde\tau_1^2 & \tilde\tau_2^2 & \cdots\cdots & \tilde\tau_{m-1}^2 \\ & \cdots\cdots\cdots\cdots & & \\ \tilde\tau_1^{m-2} & \tilde\tau_2^{m-2} & & \tilde\tau_{m-1}^{m-2} \end{pmatrix}$$

where $\tilde\tau_i=\tau_i/(1+\Lambda)$. This N satisfies the above properties i), ii) and iii).

Now denote $\Omega_n=\operatorname{supp}\alpha_n$. By the Lemma 5.1 we can assume;

$$\begin{cases} \text{Real part } \tau_i(\xi, \eta)\geqq(\delta+\varepsilon)\sqrt{n} & 1\leqq i\leqq N_1 \quad (N_1\geqq 1) \\ \text{Real part } \tau_i(\xi, \eta)\leqq(-\delta+\varepsilon)\sqrt{n}, & N_1+1\geqq i\geqq m-1 \\ \delta>0,\ \varepsilon>0 \ \text{small}, & (\xi, \eta)\in\Omega_n. \end{cases} \tag{5.29}$$

Put

$$\exp(-\varepsilon\sqrt{n}t)N\alpha_n U=V_n. \tag{5.30}$$

V_n satisfies the following equation:

$$(\partial_t+\varepsilon\sqrt{n})V_n-DV_n=G_n, \tag{5.31}$$

where $G_n=\exp(-\varepsilon\sqrt{n}t)NF$. Consider

$$S(t)=\sum_{i=1}^{N_1}\|V_i^{(n)}\|^2-\sum_{j=N_1+1}^{m-1}\|V_j^{(n)}\|^2,$$

where $V_i^{(n)}$ is the i-th component of $V^{(n)}$ and $\|\cdot\|$ is L^2-norm in (x, y) space. Then we have

$$S'(t)\geqq\delta\sqrt{n}\|V_n\|^2-\frac{C}{\sqrt{n}}\|G_n\|^2. \tag{5.32}$$

Proof of Theorem

We prove this by contradiction. We assume the Goursat problem (5.33) to be $\mathcal{E}$-wellposed.

$$(5.33)\qquad \begin{cases} Pu=f \\ \partial_t^i u(0,x,y)=u_i(x,y), \qquad 0\leqq i\leqq m-2 \\ u(t,0,y)=\varphi(t,y) \end{cases}$$

where $P(\tau,\xi,\eta)=\xi Q_{m-1}(\tau,\xi,\eta)+\mathring{P}_{m-1}(\tau,\eta)+R_{m-2}(\tau,\xi,\eta)$. At first we define a series of solutions $u_n(t,x,y)$ of (5.33) with $f=0$. Namely we define their Goursat data. Let $\hat{\phi}(\xi,\eta)$ be a C^{∞}function whose support is in a small neighborhood of $(\xi,\eta=(0,0)$. On the support of $\hat{\phi}(\xi,\eta)$, $\alpha_0(\xi,\eta)=1$, and we assume $\iint|\hat{\phi}(\xi,\eta)|^2 d\xi d\eta=1$. We define

$$(5.34)\qquad \hat{\phi}_n(\xi,\eta)=\hat{\phi}(\xi-\sqrt{n},\ \eta-n\eta^0)\,. \quad \text{Namely}$$

$$(5.35)\qquad \phi_n(x,y)=\exp(i\sqrt{n}x+in\eta^0 y)\phi(x,y)\,.$$

Now we define $u_n(t,x,y)$ as the solution of $Pu=0$ which satisfy the following Goursat data.

$$(5.36)\qquad \begin{cases} {}^t(u(0,x,y),\ \partial_t u(0,x,y),\ \cdots,\ \partial_t^{m-2}u(0,x,y)) \\ \quad =E^{-1}(\Lambda)N^{-1}\,{}^t(\phi_n(x,y),\ 0,\ \cdots,\ 0) \\ u_n(t,0,y)=\varphi_n(t,y)\equiv \sum_{i=0}^{m-2}\partial_t^i u_n(0,0,y)t^i/i!\,. \end{cases}$$

We want to show that*)

$$(5.37)\qquad N\alpha_n E(\Lambda)\beta u_n(0,x,y)=c+o\Big(\frac{1}{\sqrt{n}}\Big),$$

where c is a positive constant independent of n.

$$(5.38)\qquad \begin{aligned} N\alpha_n E\beta\tilde{u}_n(0,x,y)&=\beta\alpha_n NE\tilde{u}_n(0,x,y) \\ &\quad +(\alpha_n NE\beta-\beta\alpha_n NE)\tilde{u}_n(0,x,y) \end{aligned}$$

The first term of the right-hand side is, by definition, ${}^t(\beta\alpha_n\phi_n,\ 0,\ \cdots,\ 0)$. Since $\alpha_n(\xi,\eta)=1$ on the support of $\hat{\phi}_n(\xi,\eta)$ then $\alpha_n\phi_n=\phi_n$, hence

$$\beta(x,y)\phi_n=\beta(x,y)\exp(i\sqrt{n}x+in\eta^0 y)\phi(x,y)\,.$$

Since $\phi(x,y)$ is analytic we have

$$(5.39)\qquad \|\beta(x,y)\alpha_n NE\tilde{u}_n(0,x,y)\|=\Big\{\iint|\beta(x,y)\phi(x,y)|^2 dxdy\Big\}^{\frac{1}{2}}=c>0\,.$$

Now look at the last term of the right-hand side of (5.38). For simplicity we change the notations: we write $x_1, x_2, \cdots, x_{l+1}$ instead of $x, y_1, \cdots\ y_l$. So we write x instead of (x,y) and write ξ instead of (ξ,η).

---

*) In §3, N is a constant matrix but in this section N is pseudo-diff. op. So the reasoning becomes fairly delicrte.

We know that (c f [6] p. 120)

$$[\alpha_n NE,\ \beta(x)]=\sum_{1\leq|\nu|\leq p}\frac{(-1)^{|\nu|}}{\nu\,!}\beta^{(\nu)}(x)(x^{\nu}\cdot\alpha_n NE)\cdot E^{-1}N^{-1}+R_{n,p}\,.$$

The symbol of $(x^{\nu}\cdot\alpha_n NE)\cdot E^{-1}N^{-1}$ is $\partial_{\xi}^{\nu}\alpha_n N(\xi)E(\xi))\times E^{-1}(\xi)N^{-1}(\xi)$. Denoting the (i, j) element of this matrix by q_{ij} and its symbol by $q_{ij}(\xi)$, we have

$$|q_{ij}(\xi)|<C(1+|\xi|)^{-|\nu|/2} \qquad \text{for} \quad \xi\in\Omega\,. \tag{5.40}$$

Using this we obtain (5.37) by nearly same way as [6] p. 120.

Now we put $u=u_n(t, x, y)$ in (5.32). By hypothesis of $\mathcal{E}$-wellposedness, there exists a positive integer h and a neighborhood (in (x, y) space) G of $(x, y)=(0, 0)$ and a small T' such that

$$\max_{(x,y)\in G}|\partial_t^i u_n(t, x, y)|\leqq O(n^h) \qquad \text{for} \quad 0\leqq t\leqq T',\quad 0\leqq i\leqq m-1\,. \tag{5.41}$$

By taking the support of $\beta(x, y)$ small, we can assume that the support of $\beta(x, y)$ is contained in G. Therefore we have

$$\|\beta(x, y)\partial_t^i u_n(t, x, y)\|\leqq O(n^h) \qquad \text{for} \quad 0\leqq t\leqq T',\quad 0\leqq i\leqq m-1\,. \tag{5.42}$$

Now we consider the right-hand side of (5.31), G_n. Recall

$$G_n=N\exp(-\varepsilon\sqrt{n}\,t)F$$

$$F={}^t(0, \cdots, 0,\ \partial_x^{-1}(f_{(n)}-R_{m-2}(\alpha_n\beta u_n)))$$

$$f_{(n)}=\sum_{m\geq|\nu|\geq 1}C_\nu(\partial_x\partial_y\partial_t\partial_x^{-1})(\alpha_n\beta^{(\nu)}u_n)\,.$$

The most delicate term in G_n is $\partial_x^{-2}\partial_t^{m-1}(\alpha_n\beta_x u_n)$. For this we use (3.10) in §3. Namely

$$\partial_x^{-2}\partial_t^{m-1}(\alpha_n\beta_x u_n)=\partial_x^{-2}\partial_x^{-2h}\partial_t^{m-1}(\alpha_n\partial_x^{2h}\beta_x\cdot u_n)$$
$$+\sum_{|\nu|\geq 0}{}' d_{\nu,2h}(\partial_t,\ D)\partial_x^{-2}(\alpha_n\partial_x^{\nu}\beta_x\cdot u_n)\,.$$

Considering the support of α_n we haqe

$$\|\partial_x^{-2}\partial_x^{-2h}\partial_t^{m-1}(\alpha_n\partial_x^{2h}\beta_x\cdot u_n)\|$$
$$\leqq C\|\xi^{-2h-2}\alpha_n(\xi,\ \eta)\mathcal{F}(\partial_x^{2h+1}\beta\partial_t^{m-1}u_n)\|$$
$$\leqq C(1/\sqrt{n})^{2h+2}\|\partial_x^{2h+1}\beta\cdot\partial_t^{m-1}u_n\|$$
$$\leqq \text{const.}\,(1/n)^{h+1}n^h\leqq C/n\,.$$

Considering the order of $C_\nu(\partial_x\partial_y\partial_t\partial_x^{-1})$ and $d_{\nu,2h}(\partial_t,\ D)$ we see that G_n is expressed as linear combination of

$$\exp(-\varepsilon\sqrt{n}\,t)N\alpha_n E(\Lambda)(\beta^{(\nu)}\cdot\tilde{u}_n)$$

whose coefficients being bounded operators in L^2 and a bounded function with respect to n. Hereafter we can consider in the same way as [6].

§ 6. The proof of Theorem 4

Because of (2.1) and (2.2) from the beginning we consider the following P.

$$P(\tau, \xi, \eta)=\xi\mathring{Q}_{m-1}(\tau, \xi, \eta)+\xi\mathring{Q}_{m-2}(\tau, \xi, \eta)+R_{m-2}(\tau, \xi, \eta)$$
$$\equiv\xi Q_{m-1}+R_{m-2}$$
$$\text{where} \quad Q_{m-1}=\mathring{Q}_{m-1}+\mathring{Q}_{m-2}.$$

Moreover we can assume the Goursat data is zero. In fact, consider the following v;

$$v(t, x, y)=\sum_{k=0}^{m-2} u_k(x, y)t^k/k!+\varphi(t, y)-\sum_{k=0}^{m-2} u_k(0, y)t^k/k!.$$

By the compatibility condition (C), the $v(t, x, y)$ satisfies the Goursat data (1.2), and of course $v\in C^\infty$. In the prob. (1.1)—(1.2), put $u=v+\tilde{u}$, then

$$\begin{cases} P\tilde{u}=-Pv+f \\ \partial_t^i\tilde{u}(0, x, y)=0 \qquad 0\leqq i\leqq m-2 \\ \tilde{u}(t, 0, y)=0 \end{cases}$$

For simplicity we change the notations: we write $x_1, x_2, \cdots x_{l+1}$ and $\xi_1, \xi_2, \cdots, \xi_{l+1}$ instead of $x, y_1, \cdots, y_l$ and $\xi, \eta_1, \cdots, \eta_l$ respectively. After all we consider the following Goursat problem.

$$Q_{m-1}\partial_{x_1}u=R_{m-2}u+f \qquad f\in C^\infty_{t,x} \tag{6.1}$$

$$\begin{cases} \partial_t^i u(0, x)=0 \qquad 0\leqq i\leqq m-2 \\ u|_{x_1=0}=0 \end{cases} \tag{6.2}$$

$$x=(x_1, x')=(x_1, x_2, \cdots, x_l) \qquad x\in R^{l+1}.$$

Where Q_{m-1} is a differential operator of order $m-1$, moreover strongly hyperbolic with respect to t. And R_{m-2} is a differential operator of order $m-2$.

We shall prove Th. 4 by iteration, namely let v_0 be the solution of

$$Q_{m-1}v_0=f, \quad \partial_t^i v_0(0, x)=0 \quad 0\leqq i\leqq m-2,$$

and u_0 be the solution of

$$\partial_{x_1}u_0=v_0, \quad u_0|_{x_1=0}=0.$$

In general, for $j\geqq 1$, v_j be the solution of

$$Q_{m-1}v_j=R_{m-2}u_{j-1}, \quad \partial_t^i v_j(0, x)=0, \quad 0\leqq i\leqq m-2, \tag{6.3}$$

and u_j be the solution of

$$\partial_{x_1}u_j=v_j, \quad u_j|_{x_1=0}=0. \tag{6.4}$$

We want to prove that the series $u_0+u_1+\cdots$ converge.

Now we introduce a dependence domain of the Goursat problem. Let

$$\lambda_{\max}=\max_{\substack{1\leq i\leq m-1\\ |\xi|=1}}\lambda_i(\xi)$$

where $\lambda_i(\xi)$ is a root of $\mathring{Q}_{m-1}(\tau,\xi)=0$. Let $\tilde{t}>0$ and let $\mathscr{D}(\tilde{t},\tilde{x})$ be the backward cone; $\{(t,x);|x-\tilde{x}|\leqq\lambda_{\max}(\tilde{t}-t)\}$ having $(\tilde{t},\tilde{x})$ as its apex satisfying $t\geqq0$. Denote by $D(\mathring{t},r)$ the set of all points formed by $\mathscr{D}(\mathring{t},x)$ when x runs through $|x|\leqq r$.

Hereafter take a $D(\mathring{t},r)$ and fix it. And denote $D(s)$ the intersection $D(\mathring{t},r)$ and the hyperplane $t=s$. Now, we have the following two lemmas;

Lemma 6.1.

$$\text{(6.5)}\qquad \begin{cases}\partial_{x_1}u=v(t,x) & v(t,x)\in H^0_{\text{loc}}\\ u|_{x_1=0}=0\end{cases}$$

The solution of the problem (6.5) has the following estimate;

$$\text{(6.6)}\qquad \|u(t)\|_{D(t)}\leqq C_1\|v(t)\|_{D(t)}$$

where $\|u(t)\|^2_{D(t)}=\int_{D(t)}|u(t,x)|^2dx$ *and* C_1 *is a constant depending on* $D(t)$ *but independent of* v.

Lemma 6.2.

$$\text{(6.7)}\qquad \begin{cases}Q_{m-1}v=g(t,x) & g(t,x)\in H^k_{\text{loc}}\\ \partial_t^i v(0,x)=0\,, & 0\leqq i\leqq m-2\,.\end{cases}$$

The solution of the proqlem (6.7) satisfies the following inequality;

$$\text{(6.8)}\qquad |||v(t)|||_{k,D(t)}\leqq C_2\int_0^t\|g(s)\|_{k,D(s)}ds\,,\qquad k=0,1,2,\cdots,$$

where $\|g(s)\|_{k,D(s)}=\sum_{|\alpha|\leq k}\|\partial_x^\alpha g(s)\|_{D(s)}$

and $|||v(t)|||_{k,D(t)}=\sum_{i=0}^{m-2}\|\partial_t^i v(t,x)\|_{m-2-i+k,D(t)}\,.$

Remark 6.1
In Lemma 6.1 we can replace $\|\cdot\|_{D(t)}$ by $|||\cdot|||_{k,D(t)}$. Namely

Lemma 6.1′
The solution of the problem (6.5) *has the following estimate: Assuming* $v(t,x)\in H^k_{\text{loc}}$ *we have*

$$\text{(6.6′)}\qquad |||u(t)|||_{k,D(t)}\leqq C_1'|||v(t)|||_{k,D(t)}\,.$$

The idea of using $D(t)$ is due to Prof. Mizohata, cf. [8].

The proof of Lemma 6.1
This lemma is a particular case of Poincare's inequality. Let

$$D_+(t)=D(t)\cap\{(t,\ x,\ y)\,;\ x_1\geqq 0\}$$

$$D_-(t)=D(t)\cap\{(t,\ x,\ y)\,;\ x_1\leqq 0\}\ .$$

By (6.5)

$$u(t,\ x)=\int_0^{x_1}v(t,\ s,\ x')ds\ .$$

Suppose $x_1>0$, by Schwarz's inequality

$$|u(t,\ x)|^2\leqq\left(\int_0^{x_1}|1\cdot v(t,\ s,\ x')|ds\right)^2\leqq\int_0^{x_1}1^2ds\int_0^{x_1}|v(t,\ s,\ x')|^2ds$$

When $(t,\ x)\in D_+(t)$, we have

$$|u(t,\ x)|^2\leqq x_1\int_0^{L(t,\,x')}|v(t,\ s,\ x')|^2ds$$

where $L(t,\ x')=\sup_{(t,\,x)\in D_+(t)}x_1$. Let $L=\sup_{(t,\,x)\in D_+(t)}L(t,\ x')$, then

$$\int_{D_+(t)}|u(t,\ x)|^2dx\leqq\int_0^L x_1dx_1\int_{D_+(t)}|v(t,\ x)|^2dx\ .$$

Therefore

$$\|u(t,\ x)\|^2_{D_+(t)}\leqq(1/2)L^2\|v(t,\ x)\|^2_{D_+(t)}\ .$$

In the same way we have

$$\|u(t,\ x)\|^2_{D_-(t)}\leqq(1/2)L^2\|v(t,\ x)^2_{D_-(t)}\ .$$

After all we have

$$\|u(t,\ x)\|_{D(t)}\leqq\sqrt{1/2}L\|v(t,\ x)\|_{D(t)}\ .\qquad \text{q.e.d.}$$

Lemma 6.2 is essentially same as theorem 6.12 in [7] p. 367. Considering the dependence domain of hyperbolic Q_{m-1}, we can obtain (6.8).

The proof of Th. 4

Consider $0\leqq t\leqq T$, denoting $M=\sup_{0\leqq s\leqq T}\|f(s)\|_{k,D(s)}$, by the lemma 6.2 we have

$$|||v_0|||_{k,D(t)}\leqq C_2\int_0^t\|f(s)\|_{k,D(s)}ds\leqq C_2Mt\ .$$

And by the lemma 6.1′ we obtain

$$|||u_0(t)|||_{k,D(t)}\leqq C_1|||v_0(t)|||_{k,D(t)}\ .$$

Next, let us estimate the solution v_1 of

$$Q_{m-1}v_1=R_{m-2}u_0\ ,\quad \partial_t^iv_1(0,\ x)=0\quad 0\leqq i\leqq m-2\ .$$

R_{m-2} is a diff. op. of order $m-2$, then

$$\|R_{m-2}u_0\|_{k,D(s)}\leqq C_3|||u_0(s)|||_{k,D(s)}\ .$$

By the lemma 6.2 and the above estimates we have

$$|||v_1|||_{k,D(t)} \leqq C_2 \int_0^t \|R_{m-2} u_0\|_{k,D(s)} ds$$

$$\leqq C_2 C_3 \int_0^t |||u_0(s)|||_{k,D(s)} ds \leqq C_1 C_2 C_3 \int_0^t |||v_0(s)|||_{k,D(s)} ds$$

$$\leqq C_1 C_2^2 C_3 M \int_0^t s\, ds = C_1 C_2^2 C_3 M t^2/2!\,.$$

And the solution u_1 of $\partial_{x_1} u_1 = v_1$, $u_1|_{x_1=0}=0$ has the following estimate;

$$|||u_1|||_{k,D(t)} \leqq (C_1 C_2)^2 C_3 M t^2/2!\,.$$

For general j, we have the following:

Lemma 6. 3.

$$|||u_j|||_{k,D(t)} \leqq (C_1 C_2)^{j+1} C_3^j M t^{j+1}/(j+1)! \qquad for \quad 0 \leqq t \leqq T\,. \tag{6.9}$$

When we replace $u_j \to \overset{\circ}{\partial}{}_t^i u_j$, we have an inequality nearly same as (6.9). Taking account of that $D(t, r)$ is arbitrary, by the Sobolev's lemma, the series $\sum_{k=0}^{\infty} u_k$ converge uniformly in arbitrary compact set in $R^{l+1} \times [0, T]$. In the same way $\sum_{k=0}^{\infty} \partial_t^i \partial_x^\alpha u_k(\forall i, \forall \alpha)$ converge uniformly in arbitrary compact set in $R^{l+1} \times [0, T]$. Let $u = \sum_{k=0}^{\infty} u_k$, u is a solution of the problem (6.1)—(6.2) and $u \in C_{t,x}^{\infty}$.

At the end, we shall prove that the solution of the problem (6.1)—(6.2) is unique. Let the prob. (6.10)

$$\begin{cases} Q_{m-1} \partial_{x_1} u = R_{m-2} u \\ \partial_t^i u(0, x) = 0 \qquad 0 \leqq i \leqq m-2 \\ u|_{x_1=0} = 0 \end{cases} \tag{6.10}$$

has a non trivial solution u. There exists $D(T, r)$ such that

$$\sup_{0 \leqq t \leqq T} |||u|||_{0,D(t)} = M > 0$$

By the lemma 6.1′ we have

$$|||u|||_{0,D(t)} \leqq C |||\partial_{x_1} u|||_{0,D(t)}\,.$$

By the lemma 6.2 we have

$$|||\partial_{x_1} u|||_{0,D(t)} \leqq C \int_0^t \|R_{m-2} u\|_{0,D(s)} ds \leqq C' \int_0^t |||u|||_{0,D(s)} ds\,.$$

So

$$|||u|||_{0,D(t)} \leqq C' \int_0^t |||u|||_{0,D(s)} ds \leqq C' M t\,. \tag{6.11}$$

Finally we have

$$|||u|||_{0,D(t)} \leqq M (C' t)^k / k! \qquad \text{for} \quad \forall k, \quad 0 \leqq t \leqq T\,. \tag{6.12}$$

Then $|||u|||_{0,D(t)} = 0$. Thus we complete the proof of Theorem 4.

Remark 6.2. For the variable coefficients the above reasoning is valid too.

Remark 6.3. The lemma 6.3 shows that the series $\sum_{j=0}^{\infty} u_j$ conuerge in $|||\cdot|||_{k,D(t)}$ sense. Then we have the following;

Proposition. *Consider the Goursat problem* (6.1)—(6.2). *For the data*

$$f \in E_t^0(H_{\text{loc}}^k) \qquad k=0,\ 1,\ 2,\ \cdots$$

the problem (6.1)—(6.2) *has a unique solution u such that*

$$u(t) \in H_{\text{loc}}^{m-2+k},\ \partial_t u(t) \in H_{\text{loc}}^{m-3+k},\ \cdots,\ \partial_t^{m-2} u(t) \in H_{\text{loc}}^k.$$

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Bibliography

[1] Gårding, L.: Une variante de la methode de majoration de Cauchy, acta Math. **114** (1965) 143-158.
[2] Hasegawa, Y.: On the initial-value problems with data on a characteristic hypersurface, J. Math. Kyoto Univ. vol **13**, No 3 (1973), 579-593.
[3] ———: On the C^{∞}-Goursat problem for 2nd order equations with real constant coefficients, Proc. Japan .cad., **51** (1975) 516-519.
[4] Hörmander, L.,: Linear partial differential operators, Springer-Verlag (1964).
[5] Miyake, M.: On the initial value problems with data on a characteristic surface for linear systems of first order equations, Publ. RIMS, Kyoto Univ., vol. **8**, No 2, (1972) 231-264.
[6] Mizohata, S.: Some remarks on the Cauchy problem, J. Math. Kyoto Univ., **1** (1961~1962) 109-127.
[7] ———: The theory of partial partial differential equations, Cambridge (1973).
[8] ———: Analyticity of solutions of hyperbolic systesm with analytic coefficients, comm. pure Appl. Math., Vol XIV, No 3 (1961) 547-559.