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Automorphic forms and algebraic extensions of number fields, II

By

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Introduction

This is a continuation of the previous papers [7], [8] with the same title. Let F be a totally real algebraic number field which is a prime cyclic extension of the rational number field $\boldsymbol{Q}$ and satisfies the conditions $1\sim3$ in § 1, $\mathfrak{o}$ its maximal order and σ a generator of the Galois group $\mathrm{Gal}(F/\boldsymbol{Q})$. In [7], [8], we defined a subspace $\mathcal{S}_\kappa(SL_2(\mathfrak{o})^+)$ of the space $S_\kappa(GL_2(\mathfrak{o})^+)$ of Hilbert cusp forms of weight κ with respect to $GL_2(\mathfrak{o})^+$ by means of an action T_σ of σ and Hecke operators on $S_\kappa(GL_2(\mathfrak{o})^+)$, and gave the traces of Hecke operators on this subspace by using a twisted trace formula on $S_\kappa(GL_2(\mathfrak{o})^+)$. Moreover we showed the identity between the twisted trace formula and the ordinary trace formula on spaces of cusp forms of one variable, and using this identity we proved a generalization of Doi-Naganuma's result [1], [6] on lifting of cusp forms. In this paper, we shall generalize the above result to the case of congruence subgroups $\Gamma_0(\mathfrak{n})$ with some integral ideal $\mathfrak{n}$ of F. For an integral ideal with ${}^\sigma\mathfrak{n}=\mathfrak{n}$, we can define a subspace $\mathcal{S}_\kappa(\Gamma_0(\mathfrak{n}))$ of $S_\kappa(\Gamma_0(\mathfrak{n}))$ in the similar way, and can calculate the traces of Hecke operators on this subspace by using a twisted trace formula (Theorem 4.2). As in the above case, we can show the identity between the twisted trace formula and the ordinary trace formula for Hecke operators for spaces of cusp forms of one variable. By virtue of this identity, we can generalize the above result on lifting of cusp forms in the case of congruence subgroups.

Our result has been generalized in adelic and representation-theoretic setting by Shintani [12] and Langlands [4]. But we think it is not meaningless to give an explicit result in the classical case.

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Notation

The symbols $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ denotes respectively the ring of rational integers, the rational number field, the real number field and the complex number field.

The symbol $\mathfrak{H}$ denotes the complex upper half plane $\mathfrak{H}=\{z\in \boldsymbol{C}|\mathrm{Im}(z)>0\}$. For an associative ring S with an identity element, we denote by $S^{\times}$ the group of all invertible elements in S, and by $M_r(S)$ the ring of r by r matrices with coefficients in S, and put $GL_r(S)=M_r(S)^{\times}$. For subsets S_{ij} $(1\leqq i,\ j\leqq r)$ of S, (S_{ij}) $(\subset M_r(S))$ denotes the set $\{s=(s_{ij})\in M_r(S)\,|\,s_{ij}\in S_{ij}\}$.

§ 1. Preliminaries

In this section, we shall recall some definitions and results in [7], [8] and consider a generalization of them. Let F be a totally real algebraic number field which is a cyclic extension of $\boldsymbol{Q}$ with a prime degree l, and $\mathfrak{o}$ its maximal order. We assume the following conditions on F as in [7], [8]:

1) The class number of F is one.
2) $\mathfrak{o}$ has a unit of any signature distribution.
3) The extension $F/\boldsymbol{Q}$ is tamely ramified.

Let q denote the conductor of the extension $F/\boldsymbol{Q}$, then q is a prime with $q\equiv 1$ mod. l. We choose a generator σ of the Galois group $\mathrm{Gal}(F/\boldsymbol{Q})$ and fix it. We also fix an embedding of F into $\boldsymbol{R}$ and consider F a subfield of $\boldsymbol{R}$. Then all the distinct embeddings of F into $\boldsymbol{R}$ are given by σ^i, $0\leqq i\leqq l-1$. Using these embeddings, we consider $GL_2(F)$ a subgroup of the l-fold product of $GL_2(\boldsymbol{R})$ by

$$g \longrightarrow (g^{(1)}, g^{(2)}, \cdots, g^{(l)}),$$

where $x^{(i)}=x^{\sigma^{i-1}}$ for $x\in F$ and $g^{(i)}=\begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix}$ for $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in GL_2(F)$. Let $GL_2(\mathfrak{o})^+$ (resp. $GL_2(F)^+$) be the subgroup of $GL_2(\mathfrak{o})$ (resp. $GL_2(F)$) consisting of all elements with totally positive determinants. For an integral ideal $\mathfrak{n}$ of F, we denote by $\mathfrak{R}_F(\mathfrak{n})$ the $\mathfrak{o}$-order $\begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{n} & \mathfrak{o} \end{pmatrix}$ of $M_2(F)$ and by $\Gamma_0(\mathfrak{n})$ the subgroup

$$\Gamma_0(\mathfrak{n})=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in GL_2(\mathfrak{o})^+ \,|\, c\equiv 0 \bmod. \mathfrak{n}\right\}$$

of $GL_2(\mathfrak{o})^+$. Since $GL_2(\boldsymbol{R})^+=\{g\in GL_2(\boldsymbol{R})|\det g>0\}$ acts on $\mathfrak{H}$ by $gz=\dfrac{az+b}{cz+d}$ for $g\in GL_2(\boldsymbol{R})^+$ and $z\in\mathfrak{H}$, $GL_2(\mathfrak{o})^+$ and its subgroup $\Gamma_0(\mathfrak{n})$ act on the l-fold product $\mathfrak{H}^l$ of $\mathfrak{H}$ through the above embedding of $GL_2(F)$ into $GL_2(\boldsymbol{R})^l$.

For an even positive integer κ, we denote by $S_\kappa(\Gamma_0(\mathfrak{n}))$ the space of all Hilbert cusp forms of weight κ with respect to $\Gamma_0(\mathfrak{n})$, that is, the space of all holomorphic functions $f(z)$ on $\mathfrak{H}^l$ satisfying

i) $f(\gamma z)=\prod_{i=1}^{l}(c^{(i)}z_i+d^{(i)})^{\kappa}f(z)$ for all $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(\mathfrak{n})$ and $z=(z_2, z_2, \cdots, z_l)\in\mathfrak{H}^l$

ii) $f(z)$ vanishes at each cusp of $\Gamma_0(\mathfrak{n})$.

Let us consider the action of Hecke ring on this space. For a place v of F, we denote by F_v the completion of F at v. We shall use $\mathfrak{p}$ to denote finite places. Let $\mathfrak{o}_\mathfrak{p}$ be the ring of all $\mathfrak{p}$-adic integers in $F_\mathfrak{p}$ and let F_A and $F_A^\times$ be the adele ring and the idele group of F respectively. For a finite place $\mathfrak{p}$ which does not divide $\mathfrak{n}$, we denote by $\mathfrak{U}_\mathfrak{p}$ the subgroup $GL_3(\mathfrak{o}_\mathfrak{p})$ of $GL_2(F_\mathfrak{p})$, and for a prime ideal $\mathfrak{p}$ which divides $\mathfrak{n}$, we denote by $\mathfrak{U}_\mathfrak{p}$ the group of all invertible elements of the ring $\mathfrak{R}_F(\mathfrak{n})\otimes_\mathfrak{o}\mathfrak{o}_\mathfrak{p}$. We denote the subgroup $\prod_\mathfrak{p}\mathfrak{U}_\mathfrak{p}\times\prod_v GL_2(F_v)$ by $\mathfrak{U}_F$, where $\mathfrak{p}$ runs through all finite places and v runs through all infinite places. We denote by Δ_F the subgoup of $GL_2(F_A)$ consisting of all elements satisfying the condition that $\alpha_\mathfrak{p}\in\mathfrak{U}_\mathfrak{p}$ for all $\mathfrak{p}$ dividing $\mathfrak{n}$, where $\alpha_\mathfrak{p}$ denotes the $\mathfrak{p}$-component of α. Since $\mathfrak{U}_F$ and $\alpha\mathfrak{U}_F\alpha^{-1}$ are commensurable with each other for $\alpha\in\Delta_F$, we can define the Hecke ring $R(\mathfrak{U}_F, \Delta_F)$ with respect to $\mathfrak{U}_F$ and Δ_F as in Shimura [10]. Namely, $R(\mathfrak{U}_F, \Delta_F)$ is a free $\boldsymbol{Z}$-module generated by all double cosets $\mathfrak{U}_F\alpha\mathfrak{U}_F$ $(\alpha\in\Delta_F)$ with a structure of ring as well. We can make $R(\mathfrak{U}_F, \Delta_F)$ act on $S_\kappa(\Gamma_0(\mathfrak{n}))$ in the following way. For a double coset $\mathfrak{U}_F\alpha\mathfrak{U}_F$ with $\alpha\in\Delta_F$, let $\mathfrak{U}_F\alpha\mathfrak{U}_F\cap GL_2(F)^+=\bigcup_\nu\Gamma_0(\mathfrak{n})\alpha_\nu$ be a disjoint union. We define the action $\mathfrak{T}$ of $R(\mathfrak{U}_F, \Delta_F)$ by

$$\mathfrak{T}(\mathfrak{U}_F\alpha\mathfrak{U}_F)f=N(\det\alpha)^{\kappa/2-1}\sum_\nu f|[\alpha_\nu]_\kappa \qquad \text{for} \quad f\in S_\kappa(\Gamma_0(\mathfrak{n}))\,.$$

Here $N(\det\alpha)$ is the norm of the ideal of F determined by $\det\alpha\in F_A^\times$, and for $\alpha=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in GL_2(F)^+$, $f\in S_\kappa(\Gamma_0(\mathfrak{n}))$ we put

$$f|[\alpha]_\kappa=f(\alpha z)\prod_{i=1}^{l}(c^{(i)}z_i+d^{(i)})^{-\kappa}(\det\alpha^{(i)})^{\kappa/2}\,.$$

Then this action $\mathfrak{T}$ gives a representation of $R(\mathfrak{U}_F, \Delta_F)$ in the space $S_\kappa(\Gamma_0(\mathfrak{n}))$, and it is known that there exists a basis consisting of common eigen-functions for all $\mathfrak{T}(e)$, $e\in R(\mathfrak{U}_F, \Delta_F)$. Now let us define an action T_σ of σ on $\mathfrak{H}^l$ by permutation of variables, namely,

$$T_\sigma(z_1, z_2, \cdots, z_l)=(z_2, \cdots, z_l, z_1)\,.$$

If $\mathfrak{n}$ satisfies the condition ${}^\sigma\mathfrak{n}=\mathfrak{n}$, we can define an action of σ on $S_\kappa(\Gamma_0(\mathfrak{n}))$ by

$$(T_\sigma f)(z)=f(T_\sigma z)\,.$$

Here we used the same letter T_σ to denote the actions of σ on $\mathfrak{H}^l$ and on $S_\kappa(\Gamma_0(\mathfrak{n}))$. Let $S_\kappa^0(\Gamma_0(\mathfrak{n}))$ be the space of the new forms in $S_\kappa(\Gamma_0(\mathfrak{n}))$ (see T. Miyake [5]), then $S_\kappa^0(\Gamma_0(\mathfrak{n}))$ is stable under T_σ and we define $\mathcal{S}_\kappa^0(\Gamma_0(\mathfrak{n}))$ by

$$\mathcal{S}_\kappa^0(\Gamma_0(\mathfrak{n}))=\{f\in S_\kappa^0(\Gamma_0(\mathfrak{n}))|\mathfrak{T}(e)T_\sigma f=T_\sigma\mathfrak{T}(e)f \qquad \text{for all} \quad e\in R(U_F, \Delta_F)\}\,.$$

Then $S_\kappa^0(\Gamma_0(\mathfrak{n}))$ is stable under the action of $R(U_F, \Delta_F)$ and the definition of $S_\kappa^0(\Gamma_0(\mathfrak{n}))$ is independent of the choice of σ. Let $\mathfrak{q}$ be a prime ideal of F such that $\mathfrak{q}^l=(q)$, and δ be a totally positive element of $\mathfrak{o}$ such that $(\delta)=\mathfrak{q}$, and put $\mathfrak{n}=(N\delta^\nu)$ with $N\in\boldsymbol{Z}$, $(N, q)=1$, and a non-negative integer ν. For $M|N$, λ, $0\leqq\lambda\leqq\nu$ and $f\in S_\kappa(\Gamma_0((M\delta^\lambda)))$, $f(d\delta^\mu z)$ is contained in $S_\kappa(\Gamma_0(\mathfrak{n}))$ for $d|N/M$, $0\leqq\mu\leqq\nu-\lambda$, where $d\delta^\mu z=((d\delta^\mu)z_1, {}^\sigma(d\delta^\mu)z_2, \cdots, {}^{\sigma^{l-1}}(d\delta^\mu)z_l)$. We define a subspace $\mathcal{S}_\kappa(\Gamma_0(\mathfrak{n}))$ of $S_\kappa(\Gamma_0(\mathfrak{n}))$ by

$$\mathcal{S}_\kappa(\Gamma_0(\mathfrak{n}))=\sum_{\substack{d\mid N/M\\ 0\leq\mu\leq\nu-\lambda}}\mathcal{S}_\kappa^0(\Gamma_0((M\delta^\lambda)))^{d\delta\Gamma'}.$$

If we denote the action of $R(\mathfrak{U}_F, \Delta_F)$ on $\mathcal{S}_\kappa(\Gamma_0(\mathfrak{n}))$ by $\mathfrak{T}_{\mathcal{S}}$, then by virtue of T. Miyake's result ([5]), we can prove the following in the similar way as Proposition 1.3 of [8].

Proposition 1.1. *The notation being as above, then we have*

$$\operatorname{tr}\mathfrak{T}_{\mathcal{S}}(e)=\operatorname{tr}T_\sigma\mathfrak{T}(e)=\operatorname{tr}\mathfrak{T}(e)T_\sigma$$

for all $e\in R(\mathfrak{U}_F, \Delta_F)$.

§2. Selberg's trace formula

In this section, we shall reduce the calculation of $\operatorname{tr}\mathfrak{T}_{\mathcal{S}}(e)$ to the determination of some twisted conjugacy classes by means of Selberg's trace formula. For $z, z'\in\mathfrak{H}^l$, put

$$k(z, z')=\prod_{i=1}^{l}\left(\frac{z_i-\bar{z}'_i}{2\sqrt{-1}}\right)^{-\kappa},$$

and for $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in GL_2(F)^+$ and $z\in\mathfrak{H}^l$, put

$$j(\gamma, z)=\prod_{i=1}^{l}(c^{(i)}z_i+d^{(i)})(\det\gamma^{(i)})^{-1/2}.$$

For $\Gamma_0(\mathfrak{n})$ we put

$$K(z, z')=\left(\frac{\kappa-1}{4\pi}\right)^l\sum_{\gamma\in\Gamma_0(\mathfrak{n})\ \mathrm{mod.}\ Z(\Gamma_0(\mathfrak{n}))}k(\gamma z, z')j(\gamma, z)^{-\kappa},$$

where $Z(\Gamma_0(\mathfrak{n}))$ denotes the centre of $\Gamma_0(\mathfrak{n})$. If $\kappa\geq 4$, this gives the kernel function of the space $S_\kappa(\Gamma_0(\mathfrak{n}))$ ([2], [9]), and by Proposition 1.1, for a double coset $\mathfrak{U}_F\alpha\mathfrak{U}_F$ in $R(\mathfrak{U}_F, \Delta_F)$, we obtain

$$\operatorname{tr}\mathfrak{T}_{\mathcal{S}}(U_F\alpha U_F)=\left(\frac{\kappa-1}{4\pi}\right)^l\int_{\mathcal{F}}\sum_{\gamma\in\mathfrak{U}_F\alpha\mathfrak{U}_F\cap GL_2(F)^+\ \mathrm{mod.}\ Z(\Gamma_0(\mathfrak{n}))}k(\gamma T_\sigma z, z)j(\gamma, T_\sigma z)^{-\kappa}\prod_{i=1}^{l}y_i^\kappa dz,$$

where $\mathcal{F}$ denotes a fundamental doman of $\mathfrak{H}^l$ with respect to $\Gamma_0(\mathfrak{n})$ and dz denotes the invariant measure $\prod_{i=1}^{l}y_i^{-2}dx_idy_i$ with $z_i=x_i+\sqrt{-1}y_i$. In [8], we treated only the case where $\mathfrak{n}=\mathfrak{o}$ hence $\Gamma_0(\mathfrak{n})=GL_2(\mathfrak{o})^+$ has a unique cusp, but we can proceed in the similar way as in [8]. Before giving the result, we must recall some definitions and notations. In general, let G be a group on which a cyclic group generated by σ, $\sigma^l=1$, acts, and H be its subgroup such as ${}^\sigma H=H$. Then we define σ-twisted conjugacy $\underset{H}{\approx}$ in G with respect to H by

$$g\underset{H}{\approx}g'\Leftrightarrow g=h^{-1}g'^{\sigma}h\qquad\text{for}\quad h\in H.$$

We define a "norm" of an element $g\in G$ by

$$Ng=g^\sigma g^{\sigma^2}g\cdots{}^{\sigma^{l-1}}g. \tag{1}$$

If we denote by $\underset{H}{\sim}$ the usual conjugacy with respect to H, then we see easily that $Ng \underset{H}{\approx} Ng'$ if $g \underset{H}{\approx} g'$. Hence this norm map N defines a well defined map from twisted conjugacy classes to usual conjugacy classes. Here we take $GL_2(F)^+$ and $\Gamma_0(\mathfrak{n})$ as G and H respectively. We call $g \in GL_2(F)^+$ is of type v, e, p, if as an element of $GL_2(\boldsymbol{R})^+$, Ng is a scalar, an elliptic element, a parabolic element. If $g \in GL_2(F)^+$ fixes two cusps of $\Gamma_0(\mathfrak{n})$ and Ng is a hyperbolic element, we call g is of type h. If g is of type h, the characteristic polynomial of Ng has distinct two roots in $\boldsymbol{Q}$. For $g \in GL_2(F)$, put

(2) $$Z_\sigma(g) = \{x \in M_2(F) \mid g^\sigma x = xg\} .$$

then $Z_\sigma(g)$ is a $\boldsymbol{Q}$-algebra and we put

$$\Gamma(g) = Z_\sigma(g) \cap \Gamma_0(\mathfrak{n}) .$$

We denote by C_v, C_e, C_h, C_p a complete system of representatives of elements of type v, e, h, p in $\mathfrak{U}_F \alpha \mathfrak{U}_F \cap GL_2(F)^+$ with respect to $\underset{\Gamma_0(\mathfrak{n})}{\approx}$ respectively.

Theorem 2. 1. *The notation being as above, assume $\kappa \geqq 4$. Then we have*

$$\operatorname{tr} \mathfrak{T}_S(\mathfrak{U}_F \alpha \mathfrak{U}_F) = \frac{1}{2|H^1(\mathrm{Gal}(F/\boldsymbol{Q}), \mathfrak{o}^\times)|}(t_v + t_e + t_h + t_p) ,$$

where

$$t_v = \frac{\kappa - 1}{4\pi} \sum_{g \in C_v} v(\Gamma(g) \backslash \mathfrak{H})(\det Ng)^{\kappa/2-1}$$

$$t_e = -\frac{1}{2} \sum_{g \in C_e} \frac{1}{[\Gamma(g) : \{\pm 1\}]} \frac{\zeta(Ng)^{\kappa-1} - \eta(Ng)^{\kappa-1}}{\zeta(Ng) - \eta(Ng)}$$

$$t_h = -\sum_{g \in C_h} \frac{(\mathrm{Min.}(|\zeta(Ng)|, |\eta(Ng)|))^{\kappa-1}}{|\zeta(Ng) - \eta(Ng)|}$$

$$t_p = \lim_{s \to 0} \sum_{g \in C_p} \frac{\lambda_1(g)^s (\lambda_1(g)\lambda_2(g))^s \cdots (\lambda_1(g) \cdots \lambda_{l-1}(g))^s}{2\pi}$$

$$\times \left(\frac{-\sqrt{-1}\, |m(g)|}{A(g)} \right)^{1+ls} (\det Ng)^{\kappa/2-1} .$$

Here $v(\Gamma(g) \backslash \mathfrak{H})$ denotes the volume of a fundamental domain of $\Gamma(g)$ in $\mathfrak{H}$ with respect to the invariant measure $dz = y^{-2} dx dy$, $z = x + \sqrt{-1} y$, and $\zeta(Ng)$, $\eta(Ng)$ denote the distinct two roots of the characteristic polynomial of Ng. For g of type p, choose $h \in GL_2(F)^+$ which transforms the cusp of $\Gamma_0(\mathfrak{n})$ fixed by Ng to the infinite point $(\sqrt{-1}\infty, \sqrt{-1}\infty \cdots, \sqrt{-1}\infty)$. Then $hg^\sigma h^{-1} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a, b, d \in F$, $h\Gamma(g)h^{-1} = \left\{ \pm \begin{pmatrix} 1 & n\, m(g) \\ 0 & 1 \end{pmatrix} \middle| n \in \boldsymbol{Z} \right\}$ with $m(g) \in F$, and put

$$\lambda_i(g) = {}^{\sigma^{i-1}}(a/d) \qquad \mu_i(g) = {}^{\sigma^{i-1}}(b/d)$$
$$A(g) = \lambda_1(g) + \lambda_1(g)\mu_2(g) + \cdots + \lambda_1(g) \cdots \lambda_{l-1}(g)\mu_l(g) .$$

Since we can prove this in the same way as Theorem 1′ of [8], we omit the details. On elements of type p, we note the following. If Ng fixes a cusp x, then ${}^{\sigma}Ng$ fixes the cusp ${}^{\sigma}x$ and we have $g^{\sigma}x=x$ since $g^{\sigma}(Ng)g^{-1}=Ng$. Hence $hg^{\sigma}h^{-1}$ fixes the infinite point and is of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$.

§ 3. Twisted conjugacy classes

In this section, we shall determine some twisted conjugacy classes. Let k be a local field of characteristic 0, and $\boldsymbol{r}$ be its maximal order. In this section, we denote by F the l-fold product of k, or a cyclic extension of k of degree l, and by $\boldsymbol{o}$ the l-fold product of $\boldsymbol{r}$ when $F\cong k\oplus k\oplus\cdots\oplus k$ (l-times), and the maximal order of F otherwise. In the case where the extension F/k is ramified, we assume it is tamely ramified. We choose a generator σ of $\mathrm{Gal}(F/k)$ if F is a field and fix it. If $F\cong k\oplus\cdots\oplus k$, let σ, $\sigma^{l}=1$, act on F by permutation, namly,

$$ {}^{\sigma}(x_1, x_2, \cdots, x_l)=(x_2, \cdots, x_l, x_1)\,. $$

We denote by p a prime element of k and by π a prime element of F when F is a field. If F is the unramified extension of k we take $\pi=p$. For a non-negative integer ν, we denote by $\mathfrak{R}_F(\nu)$ (resp. $\mathfrak{R}_k(\nu)$) an order of $M_2(F)$ (resp. $M_2(k)$) given by

$$ \mathfrak{R}_F(\nu)=\begin{cases} \begin{pmatrix} \boldsymbol{o} & \boldsymbol{o} \\ p^{\nu}\boldsymbol{o} & \boldsymbol{o} \end{pmatrix} & \text{if } F\cong k\oplus\cdots\oplus k \\ \begin{pmatrix} \boldsymbol{o} & \boldsymbol{o} \\ \pi^{\nu}\boldsymbol{o} & \boldsymbol{o} \end{pmatrix} & \text{otherwise} \end{cases} $$

$$ \left(\text{resp. } \mathfrak{R}_k(\nu)=\begin{pmatrix} \boldsymbol{r} & \boldsymbol{r} \\ p^{\nu}\boldsymbol{r} & \boldsymbol{r} \end{pmatrix}\right). $$

Since ${}^{\sigma}\mathfrak{R}_F(\nu)=\mathfrak{R}_F(\nu)$, we can consider twisted conjugacy by taking $G=M_2(F)^{\times}$, $H=\mathfrak{R}_F(\nu)^{\times}$, and for $g\in M_2(F)^{\times}$ we can define Ng and $Z_{\sigma}(g)$ by (1) and (2). We note the characteristic polynomial of Ng is contained in $k[X]$, since $g^{\sigma}(Ng)g^{-1}=Ng$.

Let $f(X)=X^2-sX+n$ be a quadratic polynomial in $\boldsymbol{r}[X]$. In the following, we assume n is contained in $N_{F/k}(\boldsymbol{o}^{\times})$, where $N_{F/k}(x)=x^{\sigma}x\cdots{}^{\sigma^{l-1}}x$ for $x\in F$. For $g\in M_2(F)$, let $f_g(X)$ denote the characteristic polynomial of g. If Ng is not a scalar and $f_{Ng}(X)=f(X)$, then we have a canonical isomorphism $\tilde{\phi}_g$ from $k[X]/(f(X))$ to $Z_{\sigma}(g)$ such that $\tilde{\phi}_g(\tilde{X})=Ng$. Here $\tilde{X}$ is the class of X in $k[X]/(f(X))$. For an $\boldsymbol{r}$-order Λ of $k[X]/(f(X))$ containing $\tilde{X}$, we define a number $c_{\sigma}(f, \Lambda, \mathfrak{R}_F(\nu))$ as the cardinality of the following twisted conjugacy classes

$$ \{\bar{g}\in\mathfrak{R}_F(\nu)^{\times}\,|\,f_{N\bar{g}}=f,\ \tilde{\phi}_{\bar{g}}(\Lambda)=Z_{\sigma}(\bar{g})\cap\mathfrak{R}_F(\nu)\}\,/\underset{\mathfrak{R}_F(\nu)^{\times}}{\approx}\,. $$

By means of the usual conjugacy, we define the number $c(f, \Lambda, R_k(\nu))$ as the cardinality of

$$ \{g\in\mathfrak{R}_k(\nu)^{\times}\,|\,f_g=f,\ \phi_g(\Lambda)=k[g]\cap\mathfrak{R}_k(\nu)\}\,/\underset{\mathfrak{R}_k(\nu)^{\times}}{\sim}\,. $$

Here $k[g]$ is the k-algebra generated by g and ψ_g is the canonical isomorphism from $k[X]/(f(X))$ to $k[g]$ given by $\psi_g(\tilde{X})=g$. Then this number $c(f, \Lambda, R_F(\nu))$ has been well studied by Hijikata ([3], especially Theorem 2.3) and on $c_\sigma(f, \Lambda, R_k(\nu))$ we can prove the following.

Proposition 3.1. *The notation and the assumption being as above, suppose* $l|\nu$ *and* $\nu\neq 0$ *if* F *is a ramified extension of* k. *Then we have*

i) *If* F *is the* l*-fold product of* k, *or the unramified extension of* k *of degree* l,

$$c_\sigma(f, \Lambda, \mathfrak{R}_F(\nu))=c(f, \Lambda, \mathfrak{R}_k(\nu)).$$

ii) *If* F *is a tamely ramified cyclic extension of* k *of degree* l,

$$c_\sigma(f, \Lambda, \mathfrak{R}_F(\nu))=\sum_{i=1}^{l}\frac{\chi_i(\alpha)+\chi_i(\beta)}{2}\{c(f, \Lambda, \mathfrak{R}_k(\nu/l)) \\ +(l-1)c(f, \Lambda, \mathfrak{R}_k(\nu/l+1))\},$$

where χ_i, $2\leqq i\leqq l-1$, *are all characters of* $(\boldsymbol{r}/p\boldsymbol{r})^\times$ *of order* l *and* χ_1 *is the identity character, and* α, β *are the roots of* $f(X)\equiv 0 \bmod. p$ *in* $\boldsymbol{r}$. *If there does not exist any roots, we put* $(\chi_i(\alpha)+\chi_i(\beta))/2=0$.

Proof. First we treat the case i).

Lemma 3.2. *The notation being as above, let* F *be* $k\oplus\cdots\oplus k$ *(*l*-times) or the unramified extension of* k *of degree* l. *Let* g *be an element of* $\mathfrak{R}_F(\nu)$ *with* $f_g=f$ *and* $k[g]\cong k[x]/(f(x))$. *If there exists* $\bar{g}\in\mathfrak{R}_F(\nu)^\times$ *such that* $N\bar{g}=g$ *and* $\tilde{\psi}_{\bar{g}}(\Lambda)=Z_\sigma(\bar{g})\cap\mathfrak{R}_F(\nu)$, *then* g *is conjugate to an element* g' *of* $\mathfrak{R}_k(\nu)^\times$ *by* $\mathfrak{R}_F(\nu)^\times$.

Proof. For $F=k\oplus\cdots\oplus k$, this assertion is obvious, because for $g=(g_1, \cdots, g_l)$, $g_i\in\mathfrak{R}_k(\nu)$, $Ng=(g_1g_2\cdots g_l, g_2g_3\cdots g_lg_1, \cdots, g_lg_1\cdots, g_{l-1})$ and $g=hg'h^{-1}$ with $g'=g_1g_2\cdots g_l\in\mathfrak{R}_k(\nu)$ and $h=(1, g_1, g_1g_2, \cdots, g_1g_2\cdots g_{l-1})\in\mathfrak{R}_F(\nu)^\times$. Now assume F is the unramified extension, then for an $\boldsymbol{o}$-order $\bar{\Lambda}$ of $F[X]/(f(X))$ we have by Lemma 3.12 in [8],

$$\bar{\Lambda}\cap k[X]/(f(X))=\Lambda \Leftrightarrow \bar{\Lambda}=\boldsymbol{o}[\Lambda]$$

and if $[\Lambda:\boldsymbol{r}[\tilde{X}]]^\rho=[\boldsymbol{r}:p\boldsymbol{r}]^\rho$ with a non-negative integer ρ, then $[\bar{\Lambda}:\boldsymbol{o}[\tilde{X}]]=[\boldsymbol{o}:\pi\boldsymbol{o}]^\rho$. Put

$$\Omega_F(\nu, \bar{\Lambda})=\{\xi\in\boldsymbol{o}\,|\,f(\xi)\equiv 0 \bmod. \pi^{\nu+2\rho}\} \tag{3}$$

$$\Omega'_F(\nu, \bar{\Lambda})=\begin{cases}\{\eta\in\boldsymbol{o}\,|\,f(\eta)\equiv 0 \bmod. \pi^{\nu+2\rho+1}\} \\ \qquad\qquad \text{if } s^2-4n\equiv 0 \bmod. \pi^{2\rho+1} \text{ and } \nu>0 \\ \phi \qquad\quad \text{if } s^2-4n\not\equiv 0 \bmod. \pi^{2\rho+1} \text{ or } \nu=0.\end{cases} \tag{4}$$

For $\xi\in\Omega_F(\nu, \bar{\Lambda})$ and $\eta\in\Omega'_F(\nu, \bar{\Lambda})$, put

$$\varphi_\xi(\tilde{X})=\begin{pmatrix}\xi & \pi^\rho \\ \pi^{-\rho}f(\xi) & s-\xi\end{pmatrix} \tag{5}$$

$$\varphi'_\eta(\tilde{X})=\begin{pmatrix}s-\eta & \pi^{-\rho}f(\eta) \\ \pi^\rho & \eta\end{pmatrix}. \tag{6}$$

Then by Theorem 2.3 of [3], $\varphi_{\xi}(\tilde{X})$ and $\varphi'_{\eta}(\tilde{X})$ give a complete system of representatives of the classes of $g\in\mathfrak{R}_F(\nu)$ satisfying $\psi_g: F[X]/(f(X))\cong F[g]$ and $F[g]\cap\mathfrak{R}_F(\nu)=\psi_g(\bar{\Lambda})$ with respect to $\mathfrak{R}_F(\nu)^{\times}$-conjugacy. Hence we may assume $g=\varphi_{\xi}(\tilde{X})$ or $\varphi'_{\eta}(\tilde{X})$ for some $\xi\in\Omega_F(\nu,\bar{\Lambda})$ or $\eta\in\Omega'_F(\nu,\bar{\Lambda})$ by noting $h^{-1}gh=N(h^{-1}\bar{g}^{\sigma}h)$ for $h\in R_F(\nu)^{\times}$. Since $\bar{g}^{\sigma}(N\bar{g})\bar{g}^{-1}=N\bar{g}$, we have $\varphi_{\xi}(\tilde{X}) \underset{\mathfrak{R}_F(\nu)^{\times}}{\sim} \varphi_{\sigma\xi}(\tilde{X})$ and $\varphi'_{\eta}(\tilde{X}) \underset{\mathfrak{R}_F(\nu)^{\times}}{\sim} \varphi'_{\sigma\eta}(X)$. By Lemma 2.5 of [3], we have $\xi\equiv{}^{\sigma}\xi$ mod. $\pi^{\nu+\rho}$ and $\eta\equiv{}^{\sigma}\eta$ mod. $\pi^{\nu+\rho}$. Hence there exist ξ_0 and η_0 in r which satisfy $\xi\equiv\xi_0$ mod. $\pi^{\nu+\rho}$ and $\eta\equiv\eta_0$ mod. $\pi^{\nu+\rho}$ respectively. Since $\varphi_{\xi}(\tilde{X}) \underset{\mathfrak{R}_F(\nu)^{\times}}{\sim} \varphi_{\xi_0}(\tilde{X})$, $\varphi'_{\eta}(\tilde{X}) \underset{\mathfrak{R}_F(\nu)^{\times}}{\sim} \varphi'_{\eta_0}(\tilde{X})$ and ${}^{\sigma}\varphi_{\xi_0}(\tilde{X})=\varphi_{\xi_0}(\tilde{X})$, ${}^{\sigma}\varphi'_{\eta_0}(\tilde{X})=\varphi'_{\eta_0}(\tilde{X})$, we proved our lemma.

Now we return to the proof of Prop. 3.1. By means of the norm map N, we have a well-defined map

$$N:\ \{\bar{g}\in\mathfrak{R}_F(\nu)^{\times}\,|\,f_{N\bar{g}}=f,\ Z_{\sigma}(\bar{g})\cap\mathfrak{R}_F(\nu)=\tilde{\psi}_{\bar{g}}(\Lambda)\}/\underset{\mathfrak{R}_F(\nu)^{\times}}{\approx}$$
$$\longrightarrow \{g\in\mathfrak{R}_F(\nu)^{\times}\,|\,f_g=f,\ k[g]\cap\mathfrak{R}_F(\nu)=\psi_g(\Lambda)\}/\underset{\mathfrak{R}_F(\nu)^{\times}}{\sim}.$$

By the above lemma, the classes on the right hand side which are contained in the image of the norm map have representatives in $\mathfrak{R}_k(\nu)^{\times}$. For $g_1, g_2\in\mathfrak{R}_k(\nu)^{\times}$ with $k[g_i]\cong k[X]/(f(X))$, assume $h^{-1}g_1h=g_2$ with $h\in\mathfrak{R}_F(\nu)^{\times}$. Then we have ${}^{\sigma}hh^{-1}g_1h^{\sigma}h^{-1}=g_1$, Hence ${}^{\sigma}hh^{-1}\in F[g_1]\cap\mathfrak{R}_F(\nu)^{\times}$. We note for $x\in F[g_1]$,

$$Nx=N_{F/k}x$$

where $N_{F/k}x=\prod_{i=0}^{l-1}({}^{\sigma^{i-1}}a+{}^{\sigma^{i-1}}bg_1)$ for $x=a+bg_1$ with $a, b\in F$. Since $N_{F/k}(h^{\sigma}h^{-1})=N(h^{\sigma}h^{-1})=1$, and $H^1(\langle\sigma\rangle, (F[g_1]\cap\mathfrak{R}_F(\nu))^{\times})=1$ by Lemma 3.14 of [8], there exists $u\in(F[g_1]\cap R_F(\nu))^{\times}$ such that $h^{\sigma}h^{-1}=u^{\sigma}u^{-1}$. Therefore $u^{-1}h\in\mathfrak{R}_k(\nu)^{\times}$ and $(uh)^{-1}g_1(uh)=g_2$, that is, $g_1 \underset{\mathfrak{R}_k(\nu)^{\times}}{\sim} g_2$. From this we see N induces the following map

$$N:\ \{\bar{g}\in\mathfrak{R}_F(\nu)^{\times}\,|\,f_{N\bar{g}}=f,\ Z_{\sigma}(\bar{g})\cap R_F(\nu)=\tilde{\psi}_{\bar{g}}(\Lambda)\}/\underset{\mathfrak{R}_F(\nu)^{\times}}{\approx}$$
$$\longrightarrow \{g\in\mathfrak{R}_k(\nu)^{\times}\,|\,f_g=f,\ k[g]\cap\mathfrak{R}_k(\nu)=\psi_g(\Lambda)\}/\underset{\mathfrak{R}_k(\nu)^{\times}}{\sim}.$$

We show this map is bijective. The surjectivity of this map easily follows from Lemma 3.14 of [8]. For $g\in\{g\in\mathfrak{R}_k(\nu)\,|\,f_g=f,\ k[g]\cap\mathfrak{R}_k(\nu)=\psi_g(\Lambda)\}$ and $\bar{g}_1, \bar{g}_2\in\mathfrak{R}_F(\nu)^{\times}$, suppose $N\bar{g}_1=N\bar{g}_2=g$. Then by the relation $\bar{g}_i{}^{\sigma}(N\bar{g}_i)\bar{g}_i^{-1}=N\bar{g}_i$, $i=1,2$, we see $\bar{g}_1, \bar{g}_2\in(F[g]\cap\mathfrak{R}_F(\nu))^{\times}$ and $N_{F/k}(\bar{g}_1\bar{g}_2^{-1})=1$. By Lemma 3.14 of [8], there exists $h\in(F[g]\cap\mathfrak{R}_F(\nu))^{\times}$ such that $\bar{g}_1\bar{g}_2^{-1}=h^{-1\sigma}h$, hence $\bar{g}_1=h^{-1}\bar{g}_2{}^{\sigma}h$, and we proved the injectivity of the above map.

Now we proceed to the case ii). First we note N induces the following map

$$(7)\qquad N:\ \{\bar{g}\in\mathfrak{R}_F(\nu)^{\times}\,|\,f_{N\bar{g}}=f,\ Z_{\sigma}(\bar{g})\cap\mathfrak{R}_F(\nu)=\tilde{\psi}_{\bar{g}}(\Lambda)\}/\underset{\mathfrak{R}_F(\nu)^{\times}}{\approx}$$
$$\longrightarrow \bigcup_{\bar{\Lambda}}\{g\in\mathfrak{R}_F(\nu)^{\times}\,|\,f_g=f,\ F[g]\cap\mathfrak{R}_F(\nu)=\psi_g(\bar{\Lambda})\}/\underset{\mathfrak{R}_F(\nu)^{\times}}{\sim}.$$

Here $\bar{\Lambda}$'s are all $\boldsymbol{o}$-orders of $F[X]/(f(X))$ which satisfy $\bar{\Lambda}\cap k[X]/(f(X))=\Lambda$. We note $k[X]/(f(X))$ is one of the followings; a) $k\oplus k$, b) the unramified quadratic extension of k, c) a ramified quadratic extension of k, d) $k+k\Delta$ with $\Delta^2=0$ and such $\boldsymbol{o}$-orders $\bar{\Lambda}$ are given by the following lemma.

Lemma 3. 3. *The notation being as above, assume F is a tamely ramified cyclic extension of k of degree l. For an $\mathfrak{r}$-order Λ of $k[X]/(f(X))$ and an $\mathfrak{o}$-order $\bar{\Lambda}$ of $F[X]/(f(X))$ containing $\tilde{X}$, let ρ and $\bar{\rho}$ be a non-negative integer such that $[\Lambda:\mathfrak{o}[\tilde{X}]]=[\mathfrak{r}:\mathfrak{p}\mathfrak{r}]^{\rho}$ and $[\bar{\Lambda}:\mathfrak{o}[\tilde{X}]]=[\mathfrak{o}:\pi\mathfrak{o}]^{\bar{\rho}}$ respectively. Then we have*

i) *Assume Λ is not the maximal order of $k[X]/(f(X))$ if $k[X]/(f(X))$ is of type a, b, or c, then*

$$\bar{\Lambda}\cap k[X]/(f(X))=\Lambda \Leftrightarrow l\rho\leqq\bar{\rho}\leqq l\rho+l-1\,.$$

ii) *If $k[X]/(f(X))$ is of type a, b, or c and Λ is the maximal order,*

$$\bar{\Lambda}\cap k[X]/(f(X))=\Lambda \Leftrightarrow \begin{cases} l\rho=\bar{\rho} & \text{if } k[X]/(f(X)) \text{ is of type a, or b} \\ l\rho\leqq\bar{\rho}\leqq l\rho+[l/2] & \text{if } k[X]/(f(X)) \text{ of type c,}\end{cases}$$

where $[l/2]$ denotes the largest integer which does not exceed $l/2$.

This is just the restatement of Lemma 3.21 of [8]. For $\bar{\Lambda}$, we define $\Omega_F(\nu, \bar{\Lambda})$ (resp. $\Omega'_F(\nu, \bar{\Lambda})$) by (3) (resp. (4)) taking $\bar{\rho}$ with $[\Lambda:\mathfrak{o}[\tilde{X}]]=[\mathfrak{o}:\pi\mathfrak{o}]^{\bar{\rho}}$ as ρ in (3) (resp. (4)), and define $\varphi_{\xi}(\tilde{X})$, $\xi\in\Omega_F(\nu, \bar{\Lambda})$ (resp. $\varphi'_{\eta}(\tilde{X})$, $\eta\in\Omega'_F(\nu, \bar{\Lambda})$) by (5) (resp. (6)). Then $\varphi_{\xi}(\tilde{X})$, $\xi\in\Omega_F(\nu, \bar{\Lambda}) \bmod. \pi^{\nu+\bar{\rho}}$, and $\varphi_{\eta}(\tilde{X})$, $\eta\in\Omega'_F(\nu, \bar{\Lambda}) \bmod. \pi^{\nu+\bar{\rho}}$ give a complete system of the representatives of $\{g\in\mathfrak{R}_F(\nu)^{\times}\,|\,f_g=f,\ F[g]\cap\mathfrak{R}_F(\nu) =\phi_g(\bar{\Lambda})\}/\widetilde{\mathfrak{R}_F(\nu)^{\times}}$. If it is contained in the image of the map (7), by the same argument as in the case i), we have ${}^{\sigma}\xi\equiv\xi \bmod. \pi^{\nu+\bar{\rho}}$, or ${}^{\sigma}\eta\equiv\eta \bmod. \pi^{\nu+\bar{\rho}}$. Since F/k is tamely ramified, there exist $\xi_0\in\mathfrak{r}$, $\eta_0\in\mathfrak{r}$ such as $\xi\equiv\xi_0 \bmod. \pi^{\nu+\bar{\rho}}$, $\eta\equiv\eta_0 \bmod. \pi^{\nu+\bar{\rho}}$. Let g be one of $\varphi_{\xi_0}(\tilde{X})$ and $\varphi'_{\eta_0}(\tilde{X})$, then there exists $h_0\in R_F(\nu)^{\times}$ such that ${}^{\sigma}g=h_0^{-1}gh_0$. We note that we can take as h_0 a diagonal matrix contained in $R_F(\nu)^{\times}$ with $Nh_0=1$. For $x\in M_2(F)$, put ${}^{\sigma_0}x=h_0{}^{\sigma}xh_0^{-1}$ and $N_0x=x^{\sigma_0}x\cdots{}^{\sigma_0 l-1}x$. Then we see $Nx=N_0(xh_0^{-1})$, and the restriction of N_0 to $F[g]$ coincides with the norm map $N_{F/k}$ from $F[g]$ to $k[g]$. If there exists $\bar{g}\in R_F(\nu)^{\times}$ such that $N\bar{g}=g$, then we see $\bar{g}h_0^{-1}\in F[g]$ and $N_0(gh_0^{-1})=g$. Assume Λ is not the maimal order, then by Proposition 3.25~3.27 in [8], it is easy to see that g is contained in the image of the map (7) if and only if $\sum_{i=1}\chi_i(\alpha)=\sum_{i=1}\chi_i(\beta)=l$ for the roots α, β of $f(X)\equiv 0 \bmod. \mathfrak{p}$. Suppose there exist $\bar{g}_1$, $\bar{g}_2\in\mathfrak{R}_F(\nu)^{\times}$ such that $N\bar{g}_1=N\bar{g}_2=g$, then we have $g_1h_0^{-1}$, $g_2h_0^{-1}\in(F[g]\cap\mathfrak{R}_F(\nu))^{\times}$ and $N_{F/k}(g_1h_0^{-1}(g_2h_0^{-1})^{-1})=1$. If $g_1=\gamma^{-1}g_2\gamma$ with $\gamma\in\mathfrak{R}_F(\nu)^{\times}$, then $\gamma\in(F[g]\cap\mathfrak{R}_F(\nu))^{\times}$ and $g_1h_0^{-1}=\gamma^{-1}g_2h_0^{-1\sigma_0}\gamma$. Hence we see the inverse image of g is in one to one correspondence with $H^1(\langle\sigma\rangle, (F[g]\cap\mathfrak{R}_F(\nu))^{\times})$, and by Lemma 3.22 of [8], this group is isomorphic to the cyclic group of order l. Now let us count the number of $\varphi_{\xi_0}(\tilde{X})$ and $\varphi'_{\eta_0}(\tilde{X})$. As we have seen, this is equal to

$$\sum_{\bar{\Lambda}}|\Omega_F(\nu, \bar{\Lambda})^{\langle\sigma\rangle}/\pi^{\nu+\bar{\rho}}|+\sum_{\bar{\Lambda}}|\Omega'_F(\nu, \bar{\Lambda})^{\langle\sigma\rangle}/\pi^{\nu+\bar{\rho}}|,$$

where $\Omega_F(\nu, \bar{\Lambda})^{\langle\sigma\rangle}=\{\xi\in\mathfrak{r}\,|\,f(\xi)\equiv 0 \bmod. \pi^{\nu+2\bar{\rho}}\}$ (resp. $\Omega'_F(\nu, \Lambda)^{\langle\sigma\rangle}=\{\eta\in\mathfrak{r}\,|\,f(\eta)\equiv 0 \bmod. \pi^{\nu+2\rho+1}\}$) and $\Omega_F(\nu, \bar{\Lambda})^{\langle\sigma\rangle}/\pi^{\nu+\bar{\rho}}$ (resp. $\Omega'_F(\nu, \bar{\Lambda})^{\langle\sigma\rangle}/\pi^{\nu+\bar{\rho}}$) denotes a complete system of representatives of $\Omega_F(\nu, \bar{\Lambda})^{\langle\sigma\rangle} \bmod. \pi^{\nu+\bar{\rho}}$ (resp $\Omega'_F(\nu, \bar{\Lambda})^{\langle\sigma\rangle} \bmod. \pi^{\nu+\bar{\rho}}$). For a non-negative integer μ and an $\mathfrak{r}$-order Λ of $k[X]/(f(X))$ with $[\Lambda:\mathfrak{r}[\tilde{X}]]=[\mathfrak{r}:\mathfrak{p}\mathfrak{r}]^{\rho}$, we define $\Omega_k(\mu, \Lambda)$, $\Omega_k(\mu, \Lambda)$ in the same way as (3), (4), namely,

$$\Omega_k(\mu, \Lambda)=\{\xi\in \boldsymbol{r} \mid f(\xi)\equiv 0 \bmod. \mathfrak{p}^{\mu+2\rho}\}$$

$$\Omega'_k(\mu, \Lambda)=\begin{cases}\{\eta\in \boldsymbol{r}\mid f(\eta)\equiv 0 \bmod. \mathfrak{p}^{\mu+2\rho+1}\}, & \text{if } \nu>0 \\ & \text{and } s^2-4n\equiv 0 \bmod. \mathfrak{p}^{2\rho+1} \\ \phi & \text{if } \nu=0 \text{ or } s^2-4n\not\equiv 0 \bmod. \mathfrak{p}^{2\rho+1}.\end{cases}$$

Define ρ and $\bar{\rho}$ as in Lemma 3.3, then if $l\neq 2$, we obtain by direct calculation,

$$\Omega_F(\nu, \Lambda)^{<\sigma>}/\pi^{\nu+\bar{\rho}}=\begin{cases}\Omega_k(\nu/l, \Lambda)/\mathfrak{p}^{\nu/l+\rho}, & \bar{\rho}=l\rho \\ \Omega_k(\nu/l+1, \Lambda)/\mathfrak{p}^{\nu/l+1+\rho}, & l\rho+1\leqq\bar{\rho}\leqq l\rho+[l/2] \\ \Omega'_k(\nu/l+1, \Lambda)/\mathfrak{p}^{\nu/l+1+\rho}, & l\rho+[l/2]+1\leqq\bar{\rho}\leqq l\rho+l-1\end{cases}$$

$$\Omega'_F(\nu, \Lambda)^{<\sigma>}/\pi^{\nu+\bar{\rho}}=\begin{cases}\Omega'_k(\nu/l, \Lambda)/\mathfrak{p}^{\nu/l+\rho}, & \bar{\rho}=l\rho \\ \Omega_k(\nu/l+1, \Lambda)/\mathfrak{p}^{\nu/l+1+\rho}, & l\rho+1\leqq\bar{\rho}\leqq l\rho+[l/2] \\ \Omega'_k(\nu/l+1, \Lambda)/\mathfrak{p}^{\nu/l+1+\rho}, & l\rho+[l/2]+1\leqq\bar{\rho}\leqq l\rho+l-1\end{cases}$$

From this we have

$$\begin{aligned}\sum_{\bar{\Lambda}}|\Omega_F(\nu, \Lambda)^{<\sigma>}/\pi^{\nu+\bar{\rho}}|+\sum_{\bar{\Lambda}}|\Omega'_F(\nu, \Lambda)^{<\sigma>}/\pi^{\nu+\bar{\rho}}| \\ =|\Omega_k(\nu/l, \Lambda)/\mathfrak{p}^{\nu/l+\rho}|+|\Omega_k(\nu/l, \Lambda)/\mathfrak{p}^{\nu/l+\rho}| \\ +(l-1)(|\Omega_k(\nu/l+1, \Lambda)/\mathfrak{p}^{\nu/l+1+\rho}|+|\Omega_k(\nu/l+1, \Lambda)/\mathfrak{p}^{\nu/l+1+\rho}|).\end{aligned}$$

We can check this equality holds also for $l=2$. Since $c(f, \Lambda, \mathfrak{R}_k(\nu/l))=|\Omega_k(\nu/l, \Lambda)/\mathfrak{p}^{\nu/l+\rho}|+|\Omega_k(\nu/l, \Lambda)/\mathfrak{p}^{\nu/l+\rho}|$, and $c(f, \Lambda, \mathfrak{R}_k(\nu/l+1))=|\Omega_k(\nu/l+1, \Lambda)/\mathfrak{p}^{\nu/l+\rho}|+|\Omega_k(\nu/l+1, \Lambda)/\mathfrak{p}^{\nu/l+1+\rho}|$, we proved the assertion ii) in this case. When Λ is the maximal order, we must divide the cases according to the type of $k[X]/(f(X))$. If $k[X]/(f(X))\cong k\oplus k$, we obtain in the similar way as above

$$c_\sigma(f, \Lambda, \mathfrak{R}_F(\nu))=\sum_{i=1}^{l}\frac{\chi_i(\alpha)+\chi_i(\beta)}{2}lc(f, \Lambda, \mathfrak{R}_k(\nu/l)).$$

But since we have $c(f, \Lambda, \mathfrak{R}_k(\nu/l))=c(f, \Lambda, \mathfrak{R}_k(\nu/l+1))$ in this case, we obtain our result. We can treat the other cases in the similar way, and we omit the details. Thus our proposition has been proved.

We can prove the following proposition in the similar way as above.

Proposition 3.4. *The notation being as above, assume F is a tamely ramified cyclic extension of k of degree l and $\nu\equiv 1 \bmod. l$. Then we have*

$$c_\sigma(f, \Lambda, \mathfrak{R}_F(\nu))=l\sum_{i=1}^{l}\frac{\chi_i(\alpha)+\chi_i(\beta)}{2}c(f, \Lambda, \mathfrak{R}_k((\nu-1)/l+1)).$$

Now we treat the elements $\bar{g}\in R_F(\nu)^\times$ such that $N\bar{g}\in F^\times$. Let $\bar{g}$ be such an element and $f_{N\bar{g}}(X)=(X-\alpha)^2$ be the characteristic polynomial of $N\bar{g}$, then we see $\alpha\in N_{F/k}(\boldsymbol{o}^\times)$ since $\bar{g}\in\mathfrak{R}_F(\nu)$ with $\nu\geqq 1$. For $\alpha\in \boldsymbol{r}^\times$, it is easy to see that there exists $g\in\mathfrak{R}_F(\nu)^\times$ such that $N\bar{g}=\alpha$ if and only if $\alpha\in N_{F/k}(\boldsymbol{o}^\times)$. For $\bar{g}$ with $N\bar{g}=\alpha$, $Z_\sigma(g)$ is isomorphic to $M_2(k)$, and for an $\boldsymbol{r}$-order Λ of we define $c_\sigma(\alpha, \Lambda, \mathfrak{R}_F(\nu))$ in the following way,

$$c_\sigma(\alpha, \Lambda, \mathfrak{R}_F(\nu)) = |\{\bar g \in \mathfrak{R}_F(\nu)^\times \mid N\bar g \in F^\times,\ f_{N\bar g} = (X-\alpha)^2,\ \tilde\varphi_{\bar g}(\Lambda) \sim Z_\sigma(\bar g) \cap \mathfrak{R}_F(\nu)\} / \underset{\mathfrak{R}_F(\nu)}{\approx}|\,,$$

where $\tilde\varphi_{\bar g}$ is an isomorphism from $M_2(k)$ to $Z_\sigma(\bar g)$ and $\sim$ means $\tilde\varphi_{\bar g}(\Lambda) = \gamma^{-1}(Z_\sigma(\bar g) \cap \mathfrak{R}_F(\nu))\gamma$ for some $\gamma \in Z_\sigma(\bar g)^\times$. Then we can prove

Proposition 3.5. *The notation being as above, assume $l|\nu$ and $\nu \geqq 1$ if F is a tamely ramified cyclic extension of k of degree l. Then for $\alpha \in N_{F/k}(\mathfrak{o}^\times)$ and an $\mathfrak{r}$-order Λ of $M_2(k)$, we have*

i) *If F is $k \oplus k \oplus \cdots \oplus k$ (l-times) or the unramified extension of k of degree l*

$$c_\sigma(\alpha, \Lambda, \mathfrak{R}_F(\nu)) = \begin{cases} 1 & \text{if } \Lambda \sim \mathfrak{R}_k(\nu) \\ 0 & \text{otherwise}. \end{cases}$$

ii) *If F is a tamely ramified cyclic extension of k of degree l*

$$c_\sigma(\alpha, \Lambda, \mathfrak{R}_F(\nu)) = \begin{cases} \sum_{i=1}^{l} \chi_i(\alpha) & \text{if } \Lambda \sim \mathfrak{R}_k(\nu/l) \\ (l-1)\sum_{i=1}^{l} \chi_i(\alpha) & \text{if } \Lambda \sim \mathfrak{R}_k(\nu/l+1) \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Assume $N\bar g \in F^\times$ and $f_{N\bar g} = f$. Let $\bar\alpha$ be an element of $\mathfrak{o}^\times$ such that $N_{F/k}\bar\alpha = \alpha$. Then we have $\{\bar g \in \mathfrak{R}_F(\nu) \mid N\bar g \in F_\times, f_{N\bar g} = f\} = \{\bar\alpha \bar g \mid \bar g \in \mathfrak{R}_F(\nu)^\times, N\bar g = 1\}$. Hence the equivalence classes $\{\bar g \in \mathfrak{R}_F(\nu)^\times \mid N\bar g \in F^\times, f_{N\bar g} = f\} / \underset{\mathfrak{R}_F(\nu)^\times}{\approx}$ is in one to one correspondence with $H^1(\langle\sigma\rangle, \mathfrak{R}_F(\nu)^\times)$. On $H^1(\langle\sigma\rangle, \mathfrak{R}_F(\nu)^\times)$, we have

Lemma 3.6. *The notation being as above, we have*

i) *If F is $k \oplus \cdots \oplus k$ or the unramified extension of k of degree l*

$$|H^1(\langle\sigma\rangle, \mathfrak{R}_F(\nu)^\times)| = 1.$$

ii) *If F is a tamely ramified cyclic extension of k of degree l*

$$|H^1(\langle\sigma\rangle, \mathfrak{R}_F(\nu)^\times)| = l^2,$$

and a complete system of representatives of $H^1(\langle\sigma\rangle, \mathfrak{R}_F(\nu)^\times)$ is given by the cocycles $\{a_\tau^{i,j}, \tau \in \langle\sigma\rangle\}$, $1 \leqq i, j \leqq l$, defined by $a_\sigma^{i,j} = \begin{pmatrix} \pi^{-i\sigma}\pi^i & 0 \\ 0 & \pi^{-j\sigma}\pi^j \end{pmatrix}$.

This is an easy consequence of the fact that $H^1(\langle\sigma\rangle, \mathfrak{o}) = 1$ and $H^1(\langle\sigma\rangle, \mathfrak{o}^\times) = 1$ if F is $k \oplus \cdots \oplus k$ or the unramified extension of k, and $H^1(\langle\sigma\rangle, \mathfrak{o}) = 1$ and $H^1(\langle\sigma\rangle, \mathfrak{o}^\times) =$ the cyclic group of order l if F is a tamely ramified extension of k. We omit the details here. From this lemma, the assertion i) follows easily. Put $g_{i,j} = \alpha\begin{pmatrix} \pi^{-i\sigma}\pi^i & 0 \\ 0 & \pi^{-j\sigma}\pi^j \end{pmatrix}$, and let us determine $Z_\sigma(g_{i,j}) \cap \mathfrak{R}_F(\nu)$. If $i=j$, then we see $Z_\sigma(g_{i,j}) \cap \mathfrak{R}_F(\nu) = \mathfrak{R}_k(\nu/l)$. Since we have

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Z_\sigma(g_{i,j}) \Leftrightarrow a, d \in k,\ {}^\sigma b \pi^{-i+j\sigma}\pi^{i-j} = b$$
$${}^\sigma c \pi^{-j+i\sigma}\pi^{j-i} = c\,,$$

we see $Z_\sigma(g_{i,j})=\left\{\begin{pmatrix} a & \pi^{j-i}c \\ \pi^{i-j}c & d \end{pmatrix}\middle| a, b, c, d\in k\right\}$ and $Z_\sigma(g_{i,j})\cap\mathfrak{R}_F(\nu)\sim\tilde{\phi}_{\tilde{g}}(\mathfrak{R}_k(\nu/l+1))$, and our assertion has been proved.

In the similar way, we can prove the following.

Proposition 3.7. *The notation being as above, assume F is a tamely ramified cyclic extension of k of degree l and $\nu\equiv 1 \bmod . l$. Then for $\alpha\in\mathbf{r}^\times$ and an $\mathbf{r}$-order Λ of $M_2(k)$, we have*

$$c_\sigma(\alpha, \Lambda, \mathfrak{R}_F(\nu))=\begin{cases} l\sum_{i=1}\chi_i(\alpha) & \text{if } \Lambda\sim\mathfrak{R}_k((\nu-1)/l+1) \\ 0 & \text{otherwise}. \end{cases}$$

§4. Twisted trace formula and main result

In this section we follow the notation of §1 and 2. In particular, F denotes a totally real algebraic number field which satisfies the conditions $1\sim3$ in §1. For a prime p and the infinite prime ∞ of $\boldsymbol{Q}$, put $F_p=F\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_p$, $\mathfrak{o}_p=\mathfrak{a}\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_p$ and $F_\infty=F\otimes_{\boldsymbol{Q}}\boldsymbol{R}$, then F_p and F_∞ are the rings which we treated in §3. For an integral ideal $\mathfrak{a}$ with $(\mathfrak{a}, \mathfrak{n})=1$, we denote by $T(\mathfrak{a})$ the sum of the double cosets $\mathfrak{U}_F\alpha\mathfrak{U}_F$ such that the right $\mathfrak{R}_F(\mathfrak{n})$-ideal $\bigcap_{\mathfrak{p}}\alpha_{\mathfrak{p}}\mathfrak{R}_F(\mathfrak{n})_{\mathfrak{p}}$ is an integral ideal with norm $\mathfrak{a}$, then $T(\mathfrak{a})$ is an element of $R_F(\mathfrak{U}_F, \Delta_F)$. We denote by $\Xi(\mathfrak{a})$ the union of the $\mathfrak{U}_F$-double cosets which appear in $T(\mathfrak{a})$, and we put $\mathfrak{U}_{F,p}=\prod_{\mathfrak{p}|p}\mathfrak{U}_{\mathfrak{p}}$. Then $\Xi(\mathfrak{a})=\prod_p\Xi(\mathfrak{a})_p\times GL_2(F_\infty)$ with some $\mathfrak{U}_{F,p}$-double cosets $\Xi(\mathfrak{a})_p$, and we put $\Xi(\mathfrak{a})_+=\Xi(\mathfrak{a})\cap GL_2(F)^+$.

For $g\in GL_2(F)$ and a $\boldsymbol{Z}$-order Λ of $Z_\sigma(g)$, put

$$C_\sigma(g, \Lambda)=\{x^{-1}g^\sigma x\,|\,x\in GL_2(F),\ Z_\sigma(g)\cap x\mathfrak{R}_F(\mathfrak{n})x^{-1}=h^{-1}\Lambda h \quad h\in Z_\sigma(g)^\times\}\ .$$

For each prime p, we denote $Z_\sigma(g)_p=Z_p(g)\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_p$, $\Lambda_p=\Lambda\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_p$, $\mathfrak{R}_F(\mathfrak{n})_p=\mathfrak{R}_F(\mathfrak{n})\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_p$, and for $g\in GL_2(F_p)$, put

$$C_{\sigma,p}(g, \Lambda_p)=\{x^{-1}g^\sigma x\,|\,x\in GL_2(F_p),\ Z_\sigma(g)_p\cap x\mathfrak{R}_F(\mathfrak{n})_p x^{-1}=h^{-1}\Lambda_p h,\quad h\in Z_\sigma(g)_p^\times\}\ .$$

Then we can reduce the twisted conjugacy with respect to $\Gamma_0(\mathfrak{n})$ to the local one, namely, we can prove.

Proposition 4.1. *The notation being as above, for $g\in GL_2(F)$ assume the type number of Λ is one if $Z_\sigma(g)$ is a quaternion algebra over $\boldsymbol{Q}$. Then it holds*

$$|C_\sigma(g, \Lambda)\cap\Xi(\mathfrak{a})_+/\Gamma_0(\mathfrak{n})|=2/[\Lambda^\times ; \Lambda^1]\prod|C_{\sigma,p}(g, \Lambda_p)\cap\Xi(\mathfrak{a})_p/\widetilde{\mathfrak{R}_F(\mathfrak{n})_p^\times}|\ ,$$

where Λ^1 denotes the subgroup of $\Lambda^\times$ consisting of all elements with norm 1 *and $h(Z_\sigma(g), \Lambda)$ denotes the class number of Λ.*

We can prove this in the same wasy as in §4 of [8], and we omit the details here. In the following, we assume $\mathfrak{a}$ is divided by at most one prime factor of p if p decomposes into l distinct prime ideals in F. Then for such an integral ideal $\mathfrak{a}$, the number $|C_{\sigma,p}(g, \Lambda_p)\cap\Xi(\mathfrak{a})_p/\underset{\mathfrak{R}_F(\mathfrak{n})_p^\times}{\approx}|$ has been determined in §3 of [8] for $\mathfrak{n}=(1)$ and in §3 of this paper for $\mathfrak{n}=(N)$ with $N\in\boldsymbol{Z}$ or $\mathfrak{n}=(N)\mathfrak{q}$.

Before giving a formula for $\operatorname{tr}\mathfrak{T}_S(T(\mathfrak{a}))=\operatorname{tr}\mathfrak{T}(T(\mathfrak{a}))T_\sigma$, we introduce some notation. For $a\in\boldsymbol{Q}^\times$, assume there exists $g\in GL_2(F)$ such that $Ng=a$. Then $Z_\sigma(g)$ is a quaternion algebra over $\boldsymbol{Q}$ and the isomorphism class of $Z_\sigma(g)$ does not depend on the choice of g, and we denote it by $D(a)$. For each prime p and a $\boldsymbol{Z}_p$-order Λ_p of $D(a)_p=D(a)\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_p$, put

$$c_{\sigma,p}(a, \Lambda_p, \Xi(\mathfrak{a})_p)=|C_{\sigma,p}(g, \psi(\Lambda_p))\cap\Xi(\mathfrak{a})_p/\underset{\mathfrak{U}_{F,p}}{\approx}|,$$

where ψ is an isomorphism from $D(a)_p$ to $Z_\sigma(g)\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_p$. If there does not exist such g, we put $c_{\sigma,p}(a, \Lambda_p, \Xi(\mathfrak{a})_p)=0$. Then this number $c_{\sigma,p}(a, \Lambda_p, \Xi(\mathfrak{a})_p)$ does not depend on the choice of g. For $f(X)=X^2-sX+n\in\boldsymbol{Z}[X]$, let $K(f)$ denote the $\boldsymbol{Q}$-algebra $\boldsymbol{Q}[X]/(f(X))$, and for a prime p assume there exists $g\in GL_2(F_p)$ such that $f_{Ng}(X)=f(X)$ and $Ng\notin F^\times$, where f_{Ng} denote the characteristic polynomial of Ng. For a $\boldsymbol{Z}_p$-order of $K(f)_p=K(f)\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_p$, we put

$$c_{\sigma,p}(f, \Lambda_p, \Xi(\mathfrak{a})_p)=|C_{\sigma,p}(g, \psi(\Lambda_p))\cap\Xi(\mathfrak{a})_p/\underset{\mathfrak{U}_{F,p}}{\approx}|,$$

where ψ is the isomorphism from $K(f)_p$ to $Z_\sigma(g)_p$ defined by $\psi(\tilde{X})=Ng$. Then this number does not depened on the choice of g. If there does not exists $g\in GL_2(F_p)$ with $f_{Ng}=f$ and $Ng\notin F^\times$, we put $c_{\sigma,p}(f, \Lambda_p, \Xi(\mathfrak{a})_p)=0$ Now we can give a formula for $\operatorname{tr}\mathfrak{T}_S(T(\mathfrak{a}))$.

Theorem 4.2. *Let $\mathfrak{a}$ be an integral ideal of F with $(\mathfrak{a}, \mathfrak{n})=1$, and assume $\mathfrak{a}$ is divided by at most one prime factor of (p) if p decomposes into l distinct prime ideals in F. If $\kappa\geqq 4$, we have*

$$\operatorname{tr}\mathfrak{T}_S(T(\mathfrak{a}))=t_v+t_e+t_h+t_p\ ,$$

where t_v, t_e, t_h, and t_p are given as follows.

i) $$t_v=\delta(N\mathfrak{a})\frac{\kappa-1}{4\pi l}\sum_\Lambda\prod_p c_{\sigma,p}(\sqrt{N\mathfrak{a}}, \Lambda_p, \Xi(a)_p)[\Lambda_0^\times:\Lambda_p^\times]v(D(\sqrt{N\mathfrak{a}}))(N\mathfrak{a})^{\kappa/2-1}$$

where $N\mathfrak{a}=|\mathfrak{o}/\mathfrak{a}|$ and $\delta(N\mathfrak{a})=1$ or 0 according as $N\mathfrak{a}$ is a square or not. Λ runs through all classes of $\boldsymbol{Z}$-orders of $D(\sqrt{N\mathfrak{a}})$ with respect to the equivalence relation $\Lambda\sim\Lambda'\Leftrightarrow\Lambda=h^{-1}\Lambda'h$ $h\in D(\sqrt{N\mathfrak{a}})$ and Λ_0 denotes a maximal order of $D(\sqrt{N\mathfrak{a}})_p$ which contains Λ_p, and $v(D(\sqrt{N\mathfrak{a}}))$ is the volume of a fundamental domain in $\mathfrak{H}$ with respect to the group of all units of a maximal order of $D(\sqrt{N\mathfrak{a}})$ with norm 1.

ii) $$t_e=-\frac{1}{2l}\sum_f w_e(f)\sum_\Lambda\frac{h(K(f), \Lambda)}{[\Lambda:\{\pm 1\}]}\prod_p c_{\sigma,p}(f, \Lambda_p, \Xi(\mathfrak{a})_p)\ ,$$

where f runs through all polynomials $f(X)=X^2-sX+n$ in $\boldsymbol{Z}[X]$ such that $n=N\mathfrak{a}$ and $s^2-4n<0$. Let ξ, η be the two roots of $f=0$, then $w_e(f)=\dfrac{\xi^{\kappa-1}-\eta^{\kappa-1}}{\xi-\eta}$. Λ runs through all $\boldsymbol{Z}$-order of $K(f)$ which contains $\tilde{X}$.

iii) $t_h=-\frac{1}{l}\sum_f w_h(f)\sum_A \frac{h(K(f),\ A)}{[A:\{\pm 1\}]}\prod_p c_{\sigma,p}(f,\ A_p,\ \Xi(\mathfrak{a})_p)$,

where $f(X)=X^2-sX+n\in \boldsymbol{Z}[X]$, $n=N\mathfrak{a}$, *and* f *runs through all polynomials which have distinct two roots in* $\boldsymbol{Q}$. *Let* ζ, η *be the two roots of* $f=0$, *then* $w_h(f)=\frac{\mathrm{Min.}(|\zeta|^{\kappa-1}, |\eta|^{\kappa-1})}{|\zeta-\eta|}$ *and* A *runs through all* $\boldsymbol{Z}$*-orders of* $K(f)$ *which contain* $\tilde{X}$.

iv) $t_p=-\delta(\mathfrak{a})\frac{1}{2}a'\prod c_{\sigma,p}(f,\ A(\sqrt{N\mathfrak{a}}/a')_p,\ \Xi(\mathfrak{a})_p)(N\mathfrak{a})^{\kappa/2-1}$,

where $\delta(\mathfrak{a})$ *is* 1 *or* 0 *according as* $\mathfrak{a}$ *is a square or not, and* a' *is a positive integer determined by* $\mathfrak{a}\cap\boldsymbol{Z}=(a'^2)$. $f=(X-\sqrt{N\mathfrak{a}})^2=(X-a)^2$ *and* $A(a/a')$ *denotes a* $\boldsymbol{Z}$*-order of* $K(f)$ *such that* $[A(a/a'):\boldsymbol{Z}[\tilde{X}]]=a/a'$.

Proof. The calculation proceeds in the same way as in the proof of Theorem 2 in [8], and we omit the details. On t_p, we note the following. Let $g_i=\begin{pmatrix} a_i & b_i \\ c_i & d_i\end{pmatrix}$, $1\leqq i\leqq 2\prod c_{\sigma,p}(f,\ A((a/a'),\ \Xi(\mathfrak{a})_p)$, be a complete system of representatives of $C_\sigma(g,\ A(a/a'))\cap\Xi(\mathfrak{a})^+/\underset{\Gamma_0(\mathfrak{n})}{\approx}$ for $g\in GL_2(F)$ with $K(f)\cong\boldsymbol{Q}[g]$. For each g_i, choose $x_i\in GL_2(F)^+$ such that $g_i=x_i^{-1}\begin{pmatrix}\bar{a} & \bar{b}\\ 0 & \bar{a}\end{pmatrix}^\sigma x_i$ with $\bar{a}$, $\bar{b}\in F$. Then it is easy to see $g_i'=x_i^{-1}\begin{pmatrix}\bar{a} & -\bar{b}\\ 0 & \bar{a}\end{pmatrix}^\sigma x_i$ also gives a complete system of representatives of $C_\sigma(g,\ A(a/a'))\cap\Xi(\mathfrak{a})^+/\underset{\Gamma_0(\mathfrak{n})}{\approx}$, and we may assume $\lambda(g_j)=\lambda(g_i')$, $|m(g_i)|=|m(g_i')|$, and $A(g_i)=-A(g_i')$. For a positive integer t, let, $A((a/a')t)$ be a $\boldsymbol{Z}$-order of $K(f)$ such that $[A((a/a')t):\boldsymbol{Z}[\tilde{X}]]=(a/a')t$, then we have

$$|C_\sigma(g,\ A(a/a'))\cap\Xi(\mathfrak{a})_+/\underset{\Gamma_0(\mathfrak{n})}{\approx}|=|C_\sigma(g,\ A((a/a')t)\cap\Xi(\mathfrak{a})_+/\underset{\Gamma_0(\mathfrak{n})}{\approx}|$$

since we have $c_{\sigma,p}(f,\ A(a/a')_p,\ \Xi(\mathfrak{a})_p)=c_{\sigma,p}(f,\ A((a/a')t)_p,\ \Xi(\mathfrak{a})_p)$. We see $g_i^t=x_i^{-1}\begin{pmatrix}\bar{a} & \bar{b}t\\ 0 & \bar{a}\end{pmatrix}^\sigma x_i$, $1\leqq i\leqq 2\prod c_{\sigma,p}(f,\ A(a/a')_p,\ \Xi(\mathfrak{a})_p)$, gives a complete system of representatives of $C_\sigma(g,\ A((a/a')t))\cap\Xi(\mathfrak{a})_+/\underset{\Gamma_0(\mathfrak{n})}{\approx}$, and we may assume $\lambda(g_i^t)=\lambda(g_i)$, $|m(g_i^t)|=|m(g_i)|$, and $A(g_i^t)=tA(g_i)$. We note $|m(g_i)/A(g_i)|=a'$ and $c_{\sigma,p}(f,\ A(m),\ \Xi(\mathfrak{a})_p)=0$ if $v_p(m)<v_p(a/a')$ by the result in §3 of [8], where v_p is the valuation of $\boldsymbol{Z}$ given by $v_p(p)=1$ and $A(m)$ is the $\boldsymbol{Z}$-order of $K(f)$ such that $[A(m):\boldsymbol{Z}[\tilde{X}]]=m$. Hence we obtain

$$\begin{aligned} t_p=\lim_{s\to 0}\frac{1}{2l}a'\sum_{g_i}\mathrm{sgn}\,A(g_i)\sqrt{-1}\{&\exp(\pi/2s\ \mathrm{sgn}\,A(g_i)\sqrt{-1}) \\ &-\exp(\pi/2s\ \mathrm{sgn}(-A(g_i))\sqrt{-1})\} \\ &\sum_{t=1}\frac{1}{t^{1+ls}} \\ =-\frac{1}{2l}a'\prod c_{\sigma,p}(f,\ A(a/a')_p,\ \Xi(\mathfrak{a})_p)&. \end{aligned}$$

By the definition of Δ_F, we have $R(U_F,\ \Delta_F)\cong\bigotimes_{\mathfrak{p}\nmid\mathfrak{n}}R_{F,\mathfrak{p}}$, where $R_{F,\mathfrak{p}}=R(GL_2(\mathfrak{o}_\mathfrak{p}),$

$GL_2(F_{\mathfrak{p}})$). We put $R_F^0=\bigotimes_{\mathfrak{p}\nmid\mathfrak{n}} R_{F,\mathfrak{p}}$, and $R_{\boldsymbol{Q}}^0=\bigotimes_{p\nmid N\mathfrak{n}} R_{\boldsymbol{Q},p}$, where $R_{\boldsymbol{Q},p}=R(GL_2(\boldsymbol{Z}_p), GL_2(\boldsymbol{Q}_p))$. For a positive integer N and a character χ mod. N, we denote by $S_\kappa(\Gamma_0(N), \chi)$ the space of cusp forms on $\mathfrak{H}$ satisfying

$$g(\gamma z)=\chi(a)^{-1}(cz+d)^\kappa g(z) \qquad \text{for } \gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_0(N).$$

The we can define an action $\mathfrak{T}$ of $R_{\boldsymbol{Q}}^0$ on $S_\kappa(\Gamma_0(N), \chi)$ as in [11] if N divides some power of $N\mathfrak{n}$. Let $\lambda_{\mathfrak{p}}$ be a homomorphism from $R_{F,\mathfrak{p}}$ to $R_{\boldsymbol{Q},p}$ for $\mathfrak{p}|p$ defined in §2 of [7] and §5.1 of [8]. Then these homomorphisms $\lambda_{\mathfrak{p}}$ define a homomorphism λ from R_F^0 to $R_{\boldsymbol{Q}}^0$. Thus we obtain a representation of R_F^0 in the space $S_\kappa(\Gamma_0(N), \chi)$. Comparing these representation with the representation $\mathfrak{T}_S$, we obtain

Theorem 4.3. *The notation being as above, assume* $n=(N)\mathfrak{q}^{l\mu}$ *with* $N\in\boldsymbol{Z}$, $(N, q)=1$, *and a non-negative integer* μ. *If* $\kappa\geqq 4$, *and* $\mu>0$, *we have*

$$\begin{aligned}\operatorname{tr}\mathfrak{T}_S(e)=\frac{1}{l}\Big\{&\sum_{i=1}^{l}\operatorname{tr}\mathfrak{T}(\lambda(e))\,|\,S_\kappa(\Gamma_0(Nq^\mu), \chi_i)\\ &+(l-1)\sum_{i=1}^{l}\operatorname{tr}\mathfrak{T}(\lambda(e))\,|\,S_\kappa(\Gamma_0(Nq^{\mu+1}), \chi_i)\Big\}\end{aligned}$$

for $e\in R_F^0$. *If* $\mu=0$, *we have*

$$\begin{aligned}\operatorname{tr}\mathfrak{T}_S(e)=&\operatorname{tr}\mathfrak{T}(\lambda(e))\,|\,S_\kappa(\Gamma_0(N))\\ &+\frac{1}{2}\sum_{i=2}^{l}\operatorname{tr}\mathfrak{T}(\lambda(e))\,|\,S_\kappa(\Gamma_0(Nq), \chi_i)\end{aligned}$$

for $e\in R_F^0$. *Here* χ_i, $2\leqq i\leqq l$, *are all characters of order* l mod. q, *and* χ_1 *is the trivial character, and* $\operatorname{tr}\mathfrak{T}(\lambda(e))|S$ *is the trace of* $\mathfrak{T}(\lambda(e))$ *on* S.

This can be proved in the same way as Theorem 5.6 of [8] by using Proposition 3.1 and 3.5 of this paper instead of Propositions in §3 of [8] for primes which divides $N\mathfrak{n}$, and we omit the details. By means of Proposition 3.4 and 3.7, we can prove the following in the same way.

Theorem 4.4. *The notation being as above, assume* $\mathfrak{n}=(N)\mathfrak{q}^{l\mu+1}$ *with* $N\in\boldsymbol{Z}$, $(N, q)=1$, *and a non-negative integer* μ. *If* $\kappa\geqq 4$, *we have*

$$\operatorname{tr}\mathfrak{T}_S(e)=\sum_{i=1}^{l}\operatorname{tr}\mathfrak{T}(\lambda(e))\,|\,S_\kappa(\Gamma_0(Nq^{\mu+1}), \chi_i)$$

for $e\in R_F^0$.

Applying these theorems, we can give a more detailed result. We choose a totally positive element δ of $\mathfrak{o}$ such that $(\delta)=\mathfrak{q}$, and put $\mathfrak{n}=(N\delta^\nu)$ with $N\in\boldsymbol{Z}$, $(N, q)=1$, and a non-negative integer ν. Let $S_\kappa(\Gamma_0(\mathfrak{n}))$ and $S_\kappa^0(\Gamma_0(\mathfrak{n}))$ be as in §1, and for a positive integer N and a character χ mod. N, let $S_\kappa^0(\Gamma_0(N), \chi)$ denote the space of new forms in $S_\kappa(\Gamma_0(N))$. Then we can prove the following.

Theorem 4.5. *The notation being as above, assume* $\mathfrak{n}=(N)\mathfrak{q}^{\nu}$ *with* $N\in \boldsymbol{Z}$, $(N, q)=1$, *and a non-negative integer* ν. *If* $\kappa\geqq 4$, *we have*

i) *If* $l\neq 2$, *we have as* R_{F}^{0}*-modules,*

a) $\mathcal{S}_{\kappa}^{0}(\Gamma_{0}((N)))\cong S_{\kappa}^{0}(\Gamma_{0}(N))\oplus S$

$$\bigoplus_{i=2}^{l} S_{\kappa}^{0}(\Gamma_{0}(Nq), \chi_{i})\cong S\oplus S .$$

b) $\mathcal{S}_{\kappa}^{0}(\Gamma_{0}((N)\mathfrak{q}))\cong S_{\kappa}^{0}(\Gamma_{0}(Nq))$.

c) $\mathcal{S}_{\kappa}^{0}(\Gamma_{0}((N)\mathfrak{q}^{2}))\cong V$, $\quad S_{\kappa}^{0}(\Gamma_{0}(Nq^{2}))\cong V\oplus S$.

d) *For* $\nu\geqq 3$,

$$\mathcal{S}_{\kappa}^{0}(\Gamma_{0}((N)\mathfrak{q}^{\nu}))\cong\begin{cases} S_{\kappa}^{0}(\Gamma_{0}(Nq^{(\nu-2)/l+2})) & \text{if } \nu\equiv 2 \text{ mod. } l \\ 0 & \text{otherwise}. \end{cases}$$

ii) *If* $l=2$, *we have as* R_{F}^{0}*-modules,*

a) $\mathcal{S}_{\kappa}^{0}(\Gamma_{0}((N)))\cong S_{\kappa}^{0}(\Gamma_{0}(N))\oplus S$

$S_{\kappa}^{0}(\Gamma_{0}(Nq), \chi_{2})\cong S\oplus S$.

b) $\mathcal{S}_{\kappa}^{0}(\Gamma_{0}((N)\mathfrak{q}))\cong S_{\kappa}^{0}(\Gamma_{0}(Nq))$.

c) $\mathcal{S}_{\kappa}^{0}(\Gamma_{0}((N(\mathfrak{q}^{2}))\cong V$

$S_{\kappa}^{0}(\Gamma_{0}(Nq^{2}))\oplus S_{\kappa}^{0}(\Gamma_{0}(Nq^{2})), \chi_{2})\cong S_{\kappa}^{0}(\Gamma_{0}(N))\oplus S_{\kappa}^{0}(\Gamma_{0}(Nq))\oplus V\oplus V$.

d) *For* $\nu\geqq 3$,

$$\mathcal{S}_{\kappa}^{0}(\Gamma_{0}((N)\mathfrak{q}^{\nu}))\cong\begin{cases} W & \text{if } \nu\equiv 0 \text{ mod. } 2 \\ 0 & \text{otherwise} \end{cases}$$

$$S_{\kappa}^{0}(\Gamma_{0}(Nq^{(\nu-2)/2+2}))\oplus S_{\kappa}^{0}(\Gamma_{0}(Nq^{(\nu-2)/2+2}), \chi_{2})\cong W\oplus W .$$

Proof. By a result of T. Miyake [5], we have

$$\mathcal{S}_{\kappa}(\Gamma_{0}((N)q^{\nu}))\cong \bigoplus_{\substack{M\mid N\\ 0\leq\lambda\leq\nu}}\ \bigoplus_{\substack{d\mid N/M\\ 0\leq\mu\leq\nu-\lambda}} \mathcal{S}_{\kappa}^{0}(\Gamma_{0}((M)q^{\lambda}))^{d\delta^{\mu}}$$

as R_{F}^{0}-modules, where $\mathcal{S}_{\kappa}^{0}(\Gamma_{0}((M)q^{\nu}))^{d\delta^{\mu}}=\{f(d\delta^{\mu}z)\mid f\in\mathcal{S}_{\kappa}^{0}(\Gamma_{0}((M)q^{\lambda}))\}$. For the Möbius function μ and a positive integer m, we define the function β by

$$\beta(m)=\sum_{d\mid m}\mu(d)\mu(m/d) .$$

For $\nu=0$, we see by Theorem 4.3

$$\begin{aligned}\operatorname{tr}\mathfrak{T}(e)\mid\mathcal{S}_{\kappa}^{0}(\Gamma_{0}((N)))&=\sum_{M\mid N}\beta(N/M)\operatorname{tr}\mathfrak{T}(e)\mid\mathcal{S}_{\kappa}(\Gamma_{0}((M))\\ &=\sum_{M\mid N}\beta(M/N)\operatorname{tr}\mathfrak{T}(\lambda(e))\mid S_{\kappa}(\Gamma_{0}(M))\\ &\quad+\frac{1}{2}\sum_{M\mid N}\beta(N/M)\sum_{i=2}^{l}\operatorname{tr}\mathfrak{T}(\lambda(e))\mid S_{\kappa}(\Gamma_{0}(Mq), \chi_{i})\\ &=\operatorname{tr}\mathfrak{T}(\lambda(e))\mid S_{\kappa}^{0}(\Gamma_{0}(N))\\ &\quad+\frac{1}{2}\sum_{i=2}^{l}\operatorname{tr}\mathfrak{T}(\lambda(e))\mid S_{\kappa}^{0}(\Gamma_{0}(Nq), \chi_{i}) .\end{aligned}$$

for $e\in R_F^0$. From this we obtain the assertions a) of i) and ii). For $\nu=1$, we can prove the following by Theorem 4.4 in the similar way as above.

$$\operatorname{tr}\mathfrak{T}(e)\,|\,\mathcal{S}_\kappa^0(\Gamma_0((N)\mathfrak{q}))=\operatorname{tr}\mathfrak{T}(\lambda(e))\,|\,S_\kappa^0(\Gamma_0(Nq))\,.$$

The assertions b) of i) and ii) follows from this equality. Next we consider the case $\nu\geqq 2$. First assume $l\neq 2$, then for χ_i, $2\leqq i\leqq l$, there exists j, $2\leqq j\leqq l$, such that $\chi_i\chi_j^2=$identity character. For $\nu\geqq 2$, the twisting operator $f\to f_{\chi_j}$, $f_{\chi_j}=\sum_{n=1}^{\infty}a_n\chi_j(n)e^{2\pi i n\tau}$ for $f=\sum_{n=1}^{\infty}a_n e^{2\pi i n\tau}$, induces a map from $S_\kappa(\Gamma_0(Nq^\nu)),\chi_i)$ to $S_\kappa(\Gamma_0(Nq^\nu))$ which commutes with the action of R_F^0. We see easily the twisting operator gives the following identity between traces of $T(\lambda(e))$.

$$\operatorname{tr}\mathfrak{T}(\lambda(e))\,|\bigoplus_{0\leq\mu\leq\nu}S_\kappa^0(\Gamma_0(Nq^\mu),\chi_i)=\operatorname{tr}\mathfrak{T}(\lambda(e))\,|\bigoplus_{0\leq\mu\leq\nu}S_\kappa^0(\Gamma_0(Nq^\mu))\,.$$

Now by Theorem 4.3 and the above equality, we see

$$\begin{aligned}
&\operatorname{tr}\mathfrak{T}(e)\,|\,\mathcal{S}_\kappa^0(\Gamma_0((N)\mathfrak{q}^l))\oplus 2\mathcal{S}_\kappa^0(\Gamma_0((N)\mathfrak{q}^{l-1}))\oplus\cdots\oplus(l-1)\mathcal{S}_\kappa^0(\Gamma_0((N)\mathfrak{q}^2)\\
&\qquad\oplus(l-1)\mathcal{S}_\kappa^0(\Gamma_0((N)\mathfrak{q})))\oplus(l-1)\mathcal{S}_\kappa^0(\Gamma_0((N)))\\
&=\sum_{M\mid N}\beta(N/M)\operatorname{tr}\mathfrak{T}(e)\,|\,\mathcal{S}_\kappa(\Gamma_0((M)\mathfrak{q}^l))\\
&\quad-\sum_{M\mid N}\beta(N/M)\operatorname{tr}\mathfrak{T}(e)\,|\,\mathcal{S}_\kappa(\Gamma_0((M)\mathfrak{q}))\\
&=\frac{1}{l}\Big\{\sum_{M\mid N}\beta(N/M)\sum_{i=1}^{l}\operatorname{tr}\mathfrak{T}(\lambda(e))\,|\,S_\kappa(\Gamma_0(Mq),\chi_i)\\
&\quad+(l-1)\sum_{i=1}^{l}\operatorname{tr}\mathfrak{T}(\lambda(e))\,|\,S_\kappa(\Gamma_0(Mq^2),\chi_i)\Big\}\\
&\quad-\sum_{M\mid N}\beta(N/M)\sum_{i=1}^{l}\operatorname{tr}\mathfrak{T}(\lambda(e))\,|\,S_\kappa(\Gamma_0(Mq),\chi_i)\\
&=(l-1)\operatorname{tr}\mathfrak{T}(\lambda(e))\,|\,S_\kappa^0(\Gamma_0(Nq^2))\oplus S_\kappa^0(\Gamma_0(Nq))\oplus S_\kappa^0(\Gamma_0(N))\,.
\end{aligned}$$

Since each irreducible representation of R_F^0 appears in $\mathcal{S}_\kappa^0(\Gamma_0((N)\mathfrak{q}^l))\oplus 2\mathcal{S}_\kappa^0(\Gamma_0((N))\mathfrak{q}^{l-1}))\oplus\cdots\oplus(l-1)\mathcal{S}_\kappa^0(\Gamma_0((N)\mathfrak{q}))\oplus(l-1)\mathcal{S}_\kappa^0(\Gamma_0((N))$ with multiplicity at most $l-1$, we obtain

$$\begin{aligned}
&\mathcal{S}_\kappa^0(\Gamma_0((N)\mathfrak{q}^\nu))\cong 0\,,\quad 3\leqq\nu\leqq l\\
&\mathcal{S}_\kappa^0(\Gamma_0((N)\mathfrak{q}^2))\oplus\mathcal{S}_\kappa^0(\Gamma_0((N)\mathfrak{q})\oplus\mathcal{S}_\kappa^0(\Gamma_0((N))\\
&\quad\cong S_\kappa^0(\Gamma_0(Nq^2))\oplus S_\kappa^0(\Gamma_0(Nq))\oplus S_\kappa^0(\Gamma_0(N))\,.
\end{aligned}$$

If we use a), b), we obtain

$$S_\kappa^0(\Gamma_0(\mathrm{N}q^2))\cong\mathcal{S}_\kappa^0(\Gamma_0((N)\mathfrak{q}^2))\oplus S\,.$$

Hence we proved our assertions c) and d) for ν, $2\leqq\nu\leqq l$. In the similar way as above, for $\mu\geqq 2$, we can show the following isomorphism as R_F^0-modules by considering

$$\sum_{M\mid N}\beta(N/M)\operatorname{tr}\mathfrak{T}(e)\mid\mathcal{S}_{\kappa}(\Gamma_0((N)\mathfrak{q}^{\mu l}))-\sum_{M\mid N}\beta(N/M)\operatorname{tr}\mathfrak{T}(e)\mid\mathcal{S}_{\kappa}(\Gamma_0((N)\mathfrak{q}^{(\mu-1)l}))\,;$$

$$\bigoplus_{i=1}^{l}(i+1)\mathcal{S}_{\kappa}^{0}(\Gamma_0((N)\mathfrak{q}^{\mu l-i}))\oplus l(\bigoplus_{0\leq\lambda\leq(\mu-1)l}\mathcal{S}_{\kappa}^{0}(\Gamma_0((N)\mathfrak{q}^{\lambda})) \tag{8}$$

$$\cong l(\bigoplus_{0\leq\lambda\leq\mu}S_{\kappa}^{0}(\Gamma_0(N\mathfrak{q}^{\lambda})))\oplus(l-1)S_{\kappa}^{0}(\Gamma_0(Nq^{\mu+1}))\,.$$

For $\mu=2$, we have $\bigoplus_{0\leq\lambda\leq l}\mathcal{S}_{\kappa}^{0}(\Gamma_0((N)q^{\lambda}))\cong\bigoplus_{0\leq\lambda\leq 2}S_{\kappa}^{0}(\Gamma_0(Nq^{\lambda}))$ by the result proved above. If follows from this

$$\bigoplus_{i=1}^{l}(i+1)\mathcal{S}_{\kappa}^{0}(\Gamma_0((N)\mathfrak{q}^{2l-i}))\cong(l-1)S_{\kappa}^{0}(\Gamma_0(Nq^{2+1}))\,.$$

By observing the multiplicities, we see

$$\mathcal{S}_{\kappa}^{0}(\Gamma_0((N)\mathfrak{q}^{2l-i}))\cong\begin{cases} S_{\kappa}^{0}(\Gamma_0(Nq^{3}))\,, & i=l-2\\ 0 & ,\ \text{otherwise.}\end{cases}$$

In the similar way, by using (8) and induction on μ, we can show for $\mu\geqq 2$,

$$\mathcal{S}_{\kappa}^{0}(\Gamma_0((N)\mathfrak{q}^{\mu l-i}))\cong\begin{cases} \mathcal{S}_{\kappa}^{0}(\Gamma_0(Nq^{\mu+1}))\,, & i=l-2\\ 0 & ,\ \text{otherwise.}\end{cases}$$

This is nothing but the assertion d) of i). We can proceed similarly in the case of $l=2$ by using Theorem 4.3, 4.4, and we omit the details.

Example. We give a numerical example for Theorem 4.5. We take $F=\boldsymbol{Q}(\sqrt{5})$, $\mathfrak{n}=(\sqrt{5})$ and $\kappa=4$. In this case we have $\dim S_4(\Gamma_0((\sqrt{5})))=1$ and $\dim S_4(\Gamma_0(5))=1$. Hence $S_4(\Gamma_0((\sqrt{5})))\cong\mathcal{S}_4(\Gamma_0((\sqrt{5})))$. Let $f(z)$ and $g(\tau)$ be a non-zero element of $S_4(\Gamma_0((\sqrt{5})))$ and $S_4(\Gamma_0(5))$ respectively. We denote by a_p (resp. $\lambda(\mathfrak{p})$) the eigen-value of $g(\tau)$ (resp. $f(z)$) for T_p (resp. $T(\mathfrak{p})$), then by Theorem 4.5 $S_4(\Gamma_0((\sqrt{5})))\cong S_4(\Gamma_0(5))$ as $R_{\mathfrak{p}}^{0}$-modules, hence it should hold

$$\lambda(\mathfrak{p})=\begin{cases} a_p, & p=\mathfrak{p}\mathfrak{p}',\quad \mathfrak{p}\neq\mathfrak{p}'\\ a_p^2-2p^{\kappa-1}, & (p)=\mathfrak{p}\quad(\kappa=4)\,.\end{cases}$$

By Shimizu's trace formula [9], we can calculate $\lambda(\mathfrak{p})$ for several $\mathfrak{p}$ using the class numbers of totally imaginary quadratic extensions of F, and we can check the above relation. In fact we have

$a_2=-4$ $\quad\lambda((2))=0$

$a_3=2$ $\quad\lambda((3))=-50$

$a_{11}=32$ $\quad\lambda((4+\sqrt{5}))=32$ $\quad N((4+\sqrt{5}))=11$

$a_{19}=100$ $\quad\lambda\left(\left(\frac{9+\sqrt{5}}{2}\right)\right)=100$ $\quad N\left(\left(\frac{9+\sqrt{5}}{2}\right)\right)=19\,.$

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