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Remarks on generalized Cohen-Macaulay rings and singularities

By

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1. All rings here are commutative with identity and noetherian. When referring to a local ring A we will mostly specify the maximal ideal, i.e. $(A, \mathfrak{m})$ means a ring with one maximal ideal $\mathfrak{m}$. Recall that for any $\mathfrak{m}$-primary ideal $\mathfrak{q}$ the Hilbert-Samuel function $l(A/\mathfrak{q}^{n+1})$ is a polynomial with rational coefficients, if n is sufficiently large. By $e(\mathfrak{q}, A)$ we denote the leading coefficient of that polynomial.

In [StV-1] the notion of "I-rings" (or in [StV-2], since these rings play a role in clearing up a conjecture of Buchsbaum, "*Buchsbaum-rings*") was introduced for characterizing local rings $(A, \mathfrak{m})$ with the property: "For any system $\underline{x}=(x_1, \cdots, x_d)$ of parameters of A

$$T(\underline{x}, A)\overset{def}{=}l(A/\underline{x}A)-e(\underline{x}, A)$$

is an invariant of A (i.e. independent of the choice of $\underline{x}$)." Clearly, all Cohen-Macaulay rings $(A, \mathfrak{m})$ satisfy this property. By [StV-1] $(A, \mathfrak{m})$ is a Buchsbaum ring iff, for all $i=1, \cdots, d$,

$$(*) \qquad \mathfrak{m}[(x_1, \cdots, x_{i-1})A : x_i] \subseteqq (x_1, \cdots, x_{i-1})A.^{1)}$$

It was suggested by the authors of [StV-1], [SchCT] that one should also characterize local rings $(A, \mathfrak{m})$, for which $T(\underline{x}, A)$ is not necessarily an invariant, but $T(\underline{x}, A)\leqq$constant. This condition means [SchCT] that for all systems $\underline{x}$ of parameters of A and for all $i=1, \cdots, d-1$

$$(**) \qquad \mathfrak{m}^{\rho}[(x_1, \cdots, x_{i-1})A : x_i] \subseteqq (x_1, \cdots, x_{i-1})A, \quad \rho \text{ fixed.}$$

So it makes sense to call these rings (B, ρ)-rings. (B, ρ)-modules over local rings can be defined correspondingly, [SchCT].

In the following the notation B-ring stands for $(B, 1)$-ring. We abbreviate Cohen-Macaulay rings or Cohen-Macaulay modules to CM-rings or CM-modules respectively.

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1) Note, as a consequence, that if $(A, \mathfrak{m})$ is a Buchsbaum ring then $A_{\mathfrak{p}}$ is a Cohen-Macaulay ring for all prime ideals $\mathfrak{p}\neq\mathfrak{m}$.

The present note outlines an elementary approach (i.e. without using cohomological methods) to an understanding of (B, ρ)-singularities (especially of $(B, 1)$-singularities) on varieties.[1] We show that it is easy to construct k-varieties Y of any dimension which are Cohen-Macaulay varieties at all but a finite set of (B, ρ)-singularities. Now finding the invariant $T(\underline{x}, A)$ or, on occasion, proving the property (*) or (**) for all systems of parameters of $A=\mathcal{O}_{Y,y}$ is generally a difficult task, even in case of varieties with only one isolated singularity. But we will see that for the case just mentioned — if R is a finitely generated k-subalgebra of a CM-ring S such that S/R, as a vector space over k, is finite dimensional $-Y=\mathrm{Spec}(R)$ is obtained from $X=\mathrm{Spec}(S)$ by a finite morphism φ, in which case there is a finite set of (B, ρ_i)-singularities $y_1, \cdots, y_r$ such that

$\mathrm{res}\,\varphi\colon\ X-\varphi^{-1}(\{y_1, \cdots, y_r\})\to Y-\{y_1, \cdots, y_r\}$ is an isomorphism (Prop. 2). In particular we give sufficient conditions for the existence of $(B, 1)$-singularities. The results are partially based on the following lemma which establishes an estimation of $T(\underline{x}, E)$ for a finitely generated module E contained in a CM-module M with $l(M/E)<\infty$. This lemma can also be obtained from the results in [RStV] and [SchCT]. We sketch here a short and very simple proof using only well-known facts on commutative algebra, and we don't use—diverging from [RStV] and [SchCT]—the machinery of local cohomology.

Moreover a good deal of effort was devoted to compile the lists of types of (B, ρ)-singularities on varieties. As for a general theory of (B, ρ)-singularities (especially of $(B, 1)$-singularities), it seems to be far from definitive however. So our examples are just designed to analyse some isolated (B, ρ)-singularities. Examples 1-3 deal with affine surfaces Y in $\boldsymbol{A}^4$ having just one $(B, 1)$-point or $(B, 2)$-point respectively. Especially in Example 1 we describe explicitly the morphism $\varphi: X\to Y$. Example 4 describes a more general situation.

2. Lemma. *Let M be a CM-module*[2] *over $(A, \mathfrak{m})$ and E a submodule of M such that $l(M/E)<\infty$.*

Then:

(i) *All systems $\underline{x}=(x_1, \cdots, x_d)$ of parameters of E (with $d:=\dim E\geqq 1$) satisfy the inequality*

$$T(\underline{x}, E)=l(E/\underline{x}E)-e(\underline{x}, E)\leqq(d-1)\,l(M/E),$$

where $e(\underline{x}, E)$ is the multiplicity symbol for E and the multiplicity system $\underline{x}$ on E [No].[3]

(ii) *Equality holds in* (i) *if $\underline{x}M\subseteqq E$.*

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1) As to singularities, only irreducible algebraic sets Y are really interesting. A point $y\in Y$ is called a (B,ρ)-singularity if the local ring $\mathcal{O}_{Y,y}$ is a (B,ρ)-ring. $(B,1)$-singularities are sometimes called B-singularities.

2) Note that M is finitely generated by definition.

3) In this case $e(\underline{x},E)$ coincides with the leading coefficient of the polynomial $l(E/\underline{x}^{n+1}E)$, $n\gg 0$, see [HVS].

Proof. Since $M \supseteqq E \supseteqq \underline{x}E$, any system $\underline{x}$ of parameters of E satisfies

$$l(M/\underline{x}E)=l(M/E)+l(E/\underline{x}E)<\infty. \tag{1}$$

Furthermore, since $M \supseteqq \underline{x}M \supseteqq \underline{x}E$,

$$l(M/\underline{x}M) \leqq l(M/\underline{x}E)<\infty. \tag{2}$$

Hence $x_1, \cdots, x_d$ form a multiplicity system on M. Observe that every submodule of a module of finite length over a noetherian ring is also of finite length. Therefore $\underline{x}$ is a multiplicity system for M/E too. Hence, by [No], 7.4, Thm. 5 and 7.8, Prop. 8, one has

$$0 \neq e(\underline{x}, E)=e(\underline{x}, M). \tag{3}$$

But $e(\underline{x}, M)=l(M/\underline{x}M)$ because of the Cohen-Macaulay property of M, see [No], 7.4. (Note that $\dim M=\dim E=d$). Hence

$$e(\underline{x}, E)=l(M/\underline{x}M)=l(M/\underline{x}E)-l(\underline{x}M/\underline{x}E). \tag{4}$$

By (1) and (4) we obtain

$$T(\underline{x}, E)=l(\underline{x}M/\underline{x}E)-l(M/E). \tag{5}$$

Take now the surjective map

$$\sigma : \bigoplus_{i=1}^{d} (M/E)T_i \longrightarrow \underline{x}M/\underline{x}E, \tag{6}$$

sending $\gamma=\sum(m_i+E)T_i \mapsto (\sum x_i m_i)+\underline{x}E$ (T_i are indeterminates). Then we obtain $l(\underline{x}M/\underline{x}E) \leqq d \cdot l(M/E)$, proving (i).

For the proof of (ii), we may assume that $\underline{x}M \subseteqq E$.

Claim: σ is an isomorphism. To see this, take $\gamma \in \ker \sigma$ in (6).

Then $\sum x_i m_i=\sum x_i e_i \in \underline{x}E$ for suitable elements $e_i \in E$. Hence $\sum x_i(m_i-e_i)=0$ and therefore, since $\underline{x}$ is a regular sequence of the CM-module M, $m_i-e_i \in \underline{x}M$ ([D]). It follows that the kernel of σ is contained in $\bigoplus (\underline{x}M+E/E)T_i=0$.

q. e. d.

Remark 1. The arguments just made apply again if $M \supset E$ is any couple of finite modules over $(A, \mathfrak{m})$ such that $l(M/E)<\infty$ and $\dim E=d \geqq 1$. We then obtain:

$$T(\underline{x}, E)=T(\underline{x}, M)+l(\underline{x}M/\underline{x}E)-l(M/E) \leqq T(\underline{x}, M)+(d-1) \cdot l(M/E).$$

Hence, if M satisfies the condition $T(\underline{x}, M) \leqq c<\infty$[1] for all systems $\underline{x}$ of parameters of M, E satisfies the condition:

$T(\underline{x}, E) \leqq c+(d-1) \cdot l(M/E)$, for all systems $\underline{x}$ of parameters of E.

Moreover $l(M/E)<\infty$ implies that $\mathrm{Ass}(M/E)=\mathfrak{m}$ ([Se]), i.e. E is $\mathfrak{m}$-primary in M. But then $\mathfrak{m}^{\rho}M$ must be in E for suitable ρ, hence $\underline{x}M \subseteqq E$ for all parameter-systems $\underline{x} \subseteqq \mathfrak{m}^{\rho}$.

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1) c=constant.

3. Recall that a noetherian ring B in which the unmixedness theorem[1] holds, is called a Cohen-Macaulay ring.

Proposition 1. *Let A be a subring of a CM-ring B such that* $\dim A = \dim B \geqq 2$ *and* $l_A(B/A) < \infty$. *Assume that one of the following tow conditions is fulfilled*:

(E) *B is equicodimensional*[2]

(UI) *B is an integral domain and A is universally catenarian.*

Then:

(i) Spec A *has Cohen-Macaulay property at all points except finitely many (B, ρ_i)-singularities y_i.*

(ii) *All points y_i are B-singularities if* $\mathfrak{a} = \operatorname{ann}(B/A)$ *is a radical ideal* (i. e. $\mathfrak{a} = \sqrt{\mathfrak{a}}$).

Remark 2. The conditions (E) and (UI) are technical ones. In the following applications to k-varieties they are always fulfilled. Observe that no condition of such a kind is used in Lemma.

Question: Is Proposition 1 true without these conditions (E) or (UI)?

Proof of Proposition 1.

(a) By assumption, B/A is a finitely generated A-module, hence the extension ring B of A is finitely generated as an A-module. Therefore B is integral over A.

Let $\mathfrak{a}$ be the annihilator of B/A. (Note that $\mathfrak{a} \neq 0$). Since B/A is an A-module of finite length, it turns out that $\operatorname{Ass}(B/A) = \operatorname{Supp}(B/A) =: V(\mathfrak{a})$ is a finite set of maximal ideals, see [B], IV, §1, no 4 and §2, no 5, prop. 7. We have $(B/A)_{\mathfrak{p}} = 0$ for all primes $\mathfrak{p} \notin V(\mathfrak{a})$. Therefore, taking $T := A \backslash \mathfrak{p} \subset B$, we obtain $A_{\mathfrak{p}} = T^{-1}B = B_{\mathfrak{P}}$ with $\mathfrak{P} = (\mathfrak{p}T^{-1}B) \cap B$. This means that all points of Spec A, not contained in $V(\mathfrak{a})$, are CM-points.

(b) Claim: All points $\mathfrak{m}$ of $V(\mathfrak{a})$ are (B, ρ_i)-singularities. To see this, since $l(T^{-1}B/T^{-1}A) < \infty$,[3] it suffices to prove that $T^{-1}B$ is a CM-module over $T^{-1}A = A_{\mathfrak{m}}$ for all $\mathfrak{m} \in V(\mathfrak{a})$:

First of all $T^{-1}B$ is integral over $A_{\mathfrak{m}}$, hence $T^{-1}B$ is a semilocal ring (see [Ma], (5. E) and [No], 4. 9, proof of Prop. 18), with $\sqrt{\mathfrak{m}T^{-1}B} = \operatorname{rad}(T^{-1}B) =: \mathfrak{M}$. Furthermore we have

$$\dim(T^{-1}B) = \dim(A_{\mathfrak{m}}) =: d, \quad \text{by [Ma], (13. C)}.$$

1) That means: each ideal of the principal class is unmixed with respect to the height, see [Ma], (16. C).
2) All maximal ideals have the same height; see also Nagata's notion of "Macaulay rings" in [Na], III, 25.
3) $T := A \backslash \mathfrak{m}$ for any $\mathfrak{m} \in V(\mathfrak{a})$

Remark 1 shows that $\mathfrak{m}^{\rho}T^{-1}B\subseteqq A_{\mathfrak{m}}$ for suitable ρ. Therefore, since $\sqrt{\mathfrak{m}^{\rho}T^{-1}B}=\sqrt{\mathfrak{m}T^{-1}B}=\mathfrak{M}$, it turns out that $\mathfrak{M}^{\rho_1}\subseteqq A_{\mathfrak{m}}$ for suitable ρ_1.

We fix a system $\underline{z}$ of parameters of the *ring* $T^{-1}B$ in $\mathfrak{M}^{\rho_1}\subseteqq A_{\mathfrak{m}}$. We know that the ring $T^{-1}B$ has *CM*-property. Now, if all maximal ideals of $T^{-1}B$ have the same height, then $\underline{z}$ forms a regular sequence in $T^{-1}B$ (see [Na], 25.4 and 25.7). So, by construction, $\underline{z}$ is a system of parameters of the $A_{\mathfrak{m}}$-module $T^{-1}B$ forming a $T^{-1}B$-sequence. Hence $T^{-1}B$ is a d-dimensional *CM*-module over $A_{\mathfrak{m}}$.

Therefore, to finish the proof of (i), it remains to be shown that all maximal ideals of $T^{-1}B$ have in either case the same height:

Case (E): Since B is integral over A, the set of maximal ideals of $T^{-1}B$ is in one-to-one correspondence with the set of all maximal ideals of B which do not meet T (see [B], II and [Ma], (5.E), Thm. 5). Since B is equicodimensional, all maximal ideals of $T^{-1}B$ have the same height.

Case (UI): Let $\mathfrak{M}_i$ be any maximal ideal of $T^{-1}B$. (Note that the prime ideals of $T^{-1}B$ lying over $\mathfrak{m}A_{\mathfrak{m}}$ are precisely the maximal ideals of $T^{-1}B$.) Then, by the dimension formula in [Ma], (14.C) and by condition (UI), we have

$$\mathrm{ht}(\mathfrak{M}_i)=\mathrm{ht}(\mathfrak{m}A_{\mathfrak{m}})+\mathrm{tr.\,deg.}_{A_{\mathfrak{m}}}(T^{-1}B)-\mathrm{tr.\,deg.}_{k(\mathfrak{m}A_{\mathfrak{m}})}(k(\mathfrak{M}_i)).$$

Since B is integral over A, we obtain $\mathrm{ht}(\mathfrak{M}_i)=\mathrm{ht}(\mathfrak{m}A_{\mathfrak{m}})$, proving (i) of Proposition 1.

(c) To see (ii), we observe that now $\mathfrak{m}T^{-1}B\subseteqq A_{\mathfrak{m}}$. But then the statement (ii) of the lemma yields the argument. This completes the proof of Proposition 1.

Now, in dealing with varieties one can carry over without any difficulty the results of the preceding considerations. We change the notations of the rings (R, S instead of A, B) to accentuate the present circumstances.

Proposition 2. *Let* $X=\mathrm{Spec}(S)$ *be an affine Cohen-Macaulay variety*[1] *of dimension*$\geqq 2$ *over a field* k. *Let* R *be a subring of* S *such that* S/R, *as a vector space over* k, *is finite-dimensional. Then*:

(i) $Y=\mathrm{Spec}(R)$ *has CM-property at all points except finitely many* (B, ρ_i)-*singularities* y_i.

(ii) *The canonial map* $\varphi: X\rightarrow Y$ *defines a proper birational morphism such that*

$$X-\varphi^{-1}(\{y_1, \cdots, y_r\})\longrightarrow Y-\{y_1, \cdots, y_r\}$$

is an isomorphism.

(iii) *All points* y_i *are B-singularities if* $\mathrm{ann}(S/R)$ *is a radical ideal.*

1) Note that an affine variety is a topological space X plus a sheaf of k-valued functions on X which is isomorphic to an irreducible algebraic subset of $\boldsymbol{k}^n$ with the sheaf $\mathcal{O}_n$.

Proof. By assumption, we have $l_R(S/R) \leqq l_k(S/R) < \infty$. [Note that Spec$(S)$, as an algebraic k-scheme, is equicodimensional.]

We know by the proof of Proposition 1 that, for $\mathfrak{a} = \mathrm{ann}(S/R)$, $V(\mathfrak{a})$ is a finite set of closed points $y_1, \cdots, y_r \in Y$. In case $f \in \mathfrak{a}$ we have $\mathfrak{a}_f \subseteqq \mathrm{ann}(S_f/R_f)$ and therefore $R_f \cong S_f$.

So the restriction

$$\mathrm{res}\,\varphi:\quad X - \varphi^{-1}(\{y_1, \cdots, y_r\}) \longrightarrow Y - \{y_1, \cdots, y_r\}$$

of φ is an isomorphism. Hence φ is a proper birational morphism ([Mu], Chap. II). In particular, Y has *CM*-property at all points except $y_1, \cdots, y_r$ by Proposition 1. q. e. d.

Remark 3. If we start in Proposition 2 with a *regular* variety $X = \mathrm{Spec}(S)$, then $Y = \mathrm{Spec}(R)$ is normal at all points except $y_1, \cdots, y_r$. These points y_i are definitely not Cohen-Macaulay points (because of *Serre's* lemma of normality).

Remark 4. A specific application of our methods yields the following statement:

"*Let R be an excellent integral domain,* $\dim R = 2$, *such that* Spec(R) *is non-singular in codimension* 1. *Then all* (*isolated*) *singularities are* (B, ρ)-*points.*"

Proof. Suppose that R is not a *CM*-ring (otherwise the statement is trivial). Let S be the integral closure of R in its quotient field $Q(R)$. Since R is excellent, S is of finite type over R. Furthermore S is a *CM*-ring [note that S, as the normalization of R, is 2-dimensional, so it satisfies *Serre's* condition S_2], and condition (UI) of Proposition 1 is fulfilled. Hence it suffices to show that $\dim_R(S/R) = 0$.

Suppose that $\dim(S/R) > 0$. Then there exists a prime ideal $\mathfrak{p} \in \mathrm{Supp}(S/R)$ such that $\dim(R/\mathfrak{p}) = 1$. Since Spec(R) is non-singular in codimension 1, $R_\mathfrak{p}$ must be regular (hence integrally closed). But on the other hand we have:

$$R_\mathfrak{p} \subsetneqq S_\mathfrak{p} \subseteqq Q(R) = Q(R_\mathfrak{p}),$$

contradiction!

4. In Example 1 we construct explicitly the morphism φ. Examples 2 and 3 are of the same type but with (B, ρ)-points of a different kind.

Example 1. We denote by k the field of complex numbers; x, y are indeterminates. Take: $R = \{f \in k[x, y] / f(1, 0) = f(-1, 0)\}$ and $S = k[x, y]$.

Then R is the finitely generated subring $k[1-x^2, xy, y, x-x^3]$ of S. Clearly S is integrally dependent on R and with the same quotient field.

Therefore $Y=\mathrm{Spec}(R)$ is not normal.

Let X be the normal variety[1] $\mathrm{Spec}(k[x, y])\cong \boldsymbol{k}^2$ and φ the canonical map $X\to Y$. We see immediately that $\mathfrak{a}=\mathrm{ann}(S/R)=(1-x^2, xy, y, x-x^3)=:\mathfrak{m}$ (maximal ideal in R). Therefore the corresponding point $y_0\in Y$ is a B-point by Proposition 2.

Furthermore $\varphi^{-1}\{y_0\}$ is the set of the points $(1, 0)$ and $(-1, 0)\in \boldsymbol{k}^2$, i.e. $\varphi^{-1}\{y_0\}$ is not connected. This shows anew that Y is not normal in y_0.

Setting $v_1=1-x^2$, $v_2=xy$, $v_3=y$, $v_4=x-x^3$, we obtain

$$R\cong k[v_1, v_2, v_3, v_4]/(v_4v_3-v_2v_1,\ v_2^2-v_3^2+v_1v_3^2,\ v_4^2+v_1^3-v_1^2,\ v_1v_3-v_2v_4-v_1^2v_3).$$

So Y can be regarded as an affine surface in $\boldsymbol{k}^4$, which is non-singular in codimension 1, but with a B-point in the origin.

The blowing-up $B_{y_0}(Y)$ of Y with center y_0 is a surface in B_4 (=blowing-up of $\boldsymbol{k}^4$, which is covered by the pieces $B_4^{(i)}=\mathrm{Spec}\left(k\left[v_i, \frac{v_1}{v_i}, \cdots, \frac{v_4}{v_i}\right]\right)$. $B_{y_0}(Y)$ yields in $B_4^{(1)}$ as exceptional divisor[2] two different lines with the generic points:

$$\left(v_1=0,\ \frac{v_2}{v_1},\ \frac{v_2}{v_1},\ \frac{v_4}{v_1}=1\right) \text{ and } \left(v_1=0,\ \frac{v_2}{v_1},\ -\frac{v_2}{v_1},\ \frac{v_4}{v_1}=-1\right).$$

[Compare this with the corresponding statement of the following example.]

Example 2. Take: $R=k[x^2, xy, y, x^3]$ and $S=k[x, y]$.

We obtain $\mathfrak{a}=\mathrm{ann}(S/R)=(x^2, xy, y, x^3)=:\mathfrak{m}$. Hence the corresponding point $y_0\in Y$ is again a B-point. But $\varphi^{-1}(y_0)$ contains only the point $(0, 0)\in \boldsymbol{k}^2$.

Setting $v_1=x^2$, $v_2=xy$, $v_3=y$, $v_4=x^3$, we obtain:

$$R\cong k[v_1, \cdots, v_4]/(v_4v_3-v_2v_1,\ v_1^3-v_4^2,\ v_2^2-v_1v_3^2,\ v_2v_4-v_1^2v_3).$$

Therefore Y can be regarded as a surface in $\boldsymbol{k}^4$ with a B-point in the origin. The blowing-up $B_{y_0}(Y)$ yields in $B_4^{(1)}$ as exceptional divisor the line with the generic point $\left(v_1=0, \frac{v_2}{v_1}=0, \frac{v_3}{v_1}, \frac{v_4}{v_1}=0\right)$, taken with a certain multiplicity.

Example 3. Take:

$$R=k[x^2, xy, y, x^5] \text{ and } S=k[x, y].$$

It is easily seen that $\mathfrak{a}=\mathrm{ann}(S/R)\subsetneqq(x^2, xy, y, x^5)=:\mathfrak{m}$ (more exactly: $\mathfrak{m}^2\subsetneqq\mathfrak{a}\subsetneqq\mathfrak{m}$). Hence the corresponding point y_0 is a $(B, 2)$-point.

Setting $v_1=x^2$, $v_2=xy$, $v_3=y$, $v_4=x^5$, we have:

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1) We identify $\mathrm{Spec}(S)$ with $\boldsymbol{k}^2$.

2) That means $E\subset B_4^{(1)}$, where E is the exceptional divisor ($=\boldsymbol{P}^3$) of B_4.

$$R\cong k[v_1, \cdots, v_4]/(v_4v_3-v_1^2v_2,\ v_1^5-v_4^2,\ v_2^2-v_1v_3^2,\ v_2v_4-v_1^3v_3).$$

In $B_4^{(1)}$ the blowing-up $B_{y_0}(Y)$ yields a surface of the type of Example 2.

Example 4. Let A be an excellent CM-domain, $d=\dim A\geqq 1$, and T an indeterminate over A. Let $\mathfrak{m}$ be any maximal ideal in A. Consider the rings

$$R:=A[\mathfrak{m}T, T^2, T^3]\subset S:=A[T]$$

R contains all powers T^n with $n\geqq 2$, hence $S=R+AT=R+RT$. Take the maximal ideal $\mathfrak{M}:=\mathfrak{m}\cdot R+\mathfrak{m}T\cdot R+T^2\cdot R+T^3\cdot R\subset R$. Since $\mathfrak{M}S\subseteqq R$, $\mathfrak{M}$ is the annihilator of S/R (regarded as R-module). Furthermore, since $S=R+AT$ and $\mathfrak{m}T\subseteqq R$, we obtain $S/R\cong A/\mathfrak{m}$ (as A-modules). Hence $l_R(S/R)\leqq l_A(S/R)=1<\infty$.

Therefore Spec(R) contains only one B-singularity, and $T(\underline{x}, R_{\mathfrak{M}})=d$ for all systems $\underline{x}$ of parameters of $R_{\mathfrak{M}}$.

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