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A note on characteristic numbers of MSp_*

By

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Dedicated to Professor A. Komatu on his 70th birthday

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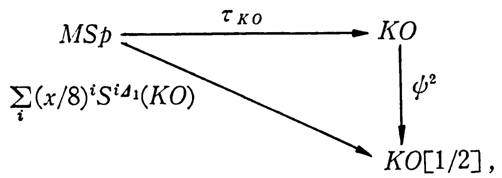
Introduction

Let MSp denote the Thom spectrum of the symplectic group, so that $MSp_n=\pi_n(MSp)$ is the symplectic cobordism ring. In this note we study some relations among KO-characteristic numbers of a class of MSp_n by considering the stable Adams operation $\psi^2 : KO^*(MSp) \to KO^*(MSp)[1/2]$.

In (2.6) *we obtain the following commutative diagram*;



where τ_{KO} *is the Thom map,* x *is the generator of* KO_4 *and* $S^{i\Delta_1}(KO)$ *is an element of* $KO^{4i}(MSp)$. *Using* (2.6), *if* $a \in MSp_{4k}$, *then we have*

$$(2.7) \qquad (\tau_{KO})_*(S^{(k-|R|)\Delta_1}(MSp)S^R(MSp))(a) \equiv 0 \mod. 8,$$

for any $R=(r_1, r_2, \cdots)$ *such that* r_i *is a non-negative integer and* $|R|=\sum_i ir_i<k$, *where* $S^R(MSp)$ *is a certain Landweber-Novikov operation in MSp-theory.*

(2.6) and (2.7) are some generalization of the result of Floyd [1].

We consider the map $\phi : KO_*(MSp) \to KO_*(MSp)[1/2]$, which is the dual of ψ^2. Let $h^{KO} : MSp_* \to KO_*(MSp)$ be the KO-Hurewicz homomorphism. For $a \in MSp_{4k}$, set $h^{KO}(a)=\sum_R \lambda^R(a)b^R(KO)$. Then we have

$$(2.12) \qquad \phi(h^{KO}(a))=\sum_R 4^{k-|R|}\lambda^R(a)^{bR}(KO),$$

$$(2.13) \qquad 4^{k-|R|}\lambda^R(a)=\sum_{|T|\geq|R|} \lambda^T(a)[\phi(b^T(KO))]_R(x/8)^{|T|-|R|},$$

where $[\phi(b^T(KO))]_R$ is the integral coefficient of $(x/8)^{|T|-|R|b^R}(KO)$ in the expansion of $\phi(b^T(KO))$.

We consider in $MSp_* \otimes Q$ the subalgebra W_*^{KO} all KO-characteristic numbers of which are integral. In (3.1) we prove that an element $\beta=\sum_R \lambda^R b^R(KO)$ of $KO_{4k}(MSp)$ satisfies the relation (2.12), i.e., $\phi(\beta)=\sum_R 4^{k-|R|}\lambda^R b^R(KO)$, if and only if β is a $h^{KO}\otimes Q$-image of an element of W_*^{KO}. In a sense this implies that W_*^{KO} is characterized by the relation (2.12). As an extension of the forgetful map $MSp_{4k}\to MO_{4k}$ we can consider a map $W_k^{KO}\to MO_{4k}$. In connection with (3.1) we have in (3.3) that Image $(W_*^{KO}\to MO_*)=P_*{}^8$, where P_* is a subalgebra of MO_* defined by E.E. Floyd [1], who proved that Image $(MSp_*\to MO_*)\subseteqq P_*{}^8$.

From (2.13) we have the following.

(2.14)
$$\sum_{|T|=k} \lambda^T(a)[\phi(b^T(KO))]_R\equiv 0 \mod 8,$$
for all R such that $|R|<k$.

Applying this relation and (2.7), we obtain the following.

(4.5) *For* $\alpha\in W_n^{KO}$,
$$\lambda^{i\Delta_1+\Delta_{n-i}}(\alpha)\equiv 0\begin{cases}\mod 8 & \text{if } n=2^m-1,\\ \mod 4 & \text{if } n=2^m \text{ or } 2m-1,\\ \mod 2 & \text{if } n=2m,\end{cases}$$
for $0\leqq i\leqq n$.

(4.5) is some generalization of the result of R. Okita [2]. Applying (2.14) we also have

(4.7) *For* $\alpha\in W_n^{KO}$,
$$2^j\lambda^{i\Delta_1+j\Delta_2+k\Delta_3}(\alpha)\equiv 0 \mod 8,$$
for $i+2j+3k=n$.

This paper is organized as follows.

In §1 we prepare some preliminary properties on cohomologies and homologies of HP^∞ and MSp and on the complex stable Adams operation ψ_c^2. In §2 we show the diagram (2.6) obtained by applying ψ^2 on the Thom class. We also define ϕ in this section and obtain some relations on characteristic numbers of MSp_*. In §3 we prove that the relation in §2 also satisfied by classes of W_*^{KO} and vice versa. In §4 by using the relations in §2 and §3, we consider some divisibility conditions on some characteristic numbers of W_*^{KO}.

§1. Preliminaries

Let $E=MSp$, KO, K or HZ which is the representative spectrum of the symplectic cobordism theory, real K-theory, complex K-theory or ordinary cohomology theory with integral coefficients. E_* denote its coefficient ring. Then symplectic vector bundles are $E^*(\ )$-orientable. We denote the Thom map by $\tau_E: MSp \to E$. Notice that $\tau_K = c\tau_{KO}$, where $c: KO \to K$ is a complexification. The following proposition is well known. Our notations are usual ones.

(1.1) Proposition.

(1) *$E^*(HP^\infty) = E_*[[e(E)]]$, where $e(E)$ is the Euler class of the canonical symplectic line bundle over HP^∞,* i.e., *the first Pontrjagin class.*

(2) *$E_*(HP^\infty) = E_*\{\beta_1(E), \beta_2(E), \cdots\}$, where $\beta_i(E)$ is the dual of $e^i(E)$.*
Let $i: HP^\infty = MSp(1) \to MSp$ be the inclusion and set $i_*(\beta_{i+1}(E)) \doteq b_i(E)$.

(3) *$E_*(MSp) = E_*[b_1(E), b_2(E), \cdots]$, where* dim. $b_i(E) = 4i$.

(4) *$E^*(MSp)$ is the dual of $E_*(MSp)$ over E_*. We denote the dual of $b^R(E) = b_1(E)^{r_1} b_2(E)^{r_2} \cdots$ by $S^R(E)$, where $R = (r_1, r_2, \cdots)$ is an exponent sequence of non-negative and almost zero integers.*

(5) *The coproduct $\Delta_E: E^*(MSp) \to E^*(MSp) \otimes_{E_*} E^*(MSp)$ is given by the following formula;*

$$\Delta_E(S^R(E)) = \sum_{R_1+R_2=R} S^{R_1}(E) \otimes S^{R_2}(E).$$

(6) *$MSp^*(MSp)$ and $MSp_*(MSp)$ are Hopf algebras over MSp_*. In $MSp_*(MSp)$, the coproduct μ^* is given by*

$$\mu^*(b_n(MSp)) = \sum_{j\geq 0} (\underline{b}(MSp))^{j+1}_{n-j} \otimes b_j(MSp),$$

where $\underline{b}(MSp) = 1 + b_1(MSp) + b_2(MSp) + \cdots$ and $(\underline{b}(MSp))^{j+1}_{n-j}$ is the $4(n-j)$-dimensional homogeneous part of $(\underline{b}(MSp))^{j+1}$.

(7) $(\tau_E)_*(e(MSp)) = e(E)$,

$(\tau_E)_*(\beta_i(MSp)) = \beta_i(E)$,

$(\tau_E)_*(b_i(MSp)) = b_i(E)$,

$(\tau_E)_*(S^R(MSp)) = S^R(E)$.

Let $\psi_c^2: K^*(\ ) \to K^*(\ )[1/2]$ be the stable Adams operation.

(1.2) Lemma.
In $K^*(HP^\infty)$,

$$\psi_c^2(e(K)) = e(K) + (t^2/4)(e(K))^2,$$

where $t \in K_2$ is the Bott-periodicity element.

Proof. It is known that $e(K) = t^{-2}(c'(\xi) - 2)$, where ξ is the canonical symplectic line bundle over HP^∞ and $c'(\xi)$ is the complexification of ξ. Let η be a canonical complex line bundle over CP^∞, and $\pi: CP^\infty \to HP^\infty$ be a canonical projection. Then $\pi^*(c'(\xi)) = \eta + \bar{\eta}$, where $\bar{\eta}$ is a complex conjugate of η. From the properties of ψ_c^2, we have

$$\begin{aligned}\psi_c^2\pi^*(t^{-2}(c'(\xi)-2)) &= (t^{-2}/4)\psi_c^2(\eta+\bar{\eta}-2)=(t^{-2}/4)(\eta^2+\bar{\eta}^2-2)\\ &=(t^{-2}/4)(\eta+\bar{\eta}-2)^2+t^{-2}(\eta+\bar{\eta}-2)\\ &=(t^2/4)\pi^*(e(K))^2+\pi^*(e(K)).\end{aligned}$$

Since $\pi^*: K^*(HP^\infty)\to K^*(CP^\infty)$ is monomorphic, we have the required result.

Q. E. D.

§2. Relations on K and KO-characteristic numbers of MSp_*.

In this section we first consider the complex stable Adams operation ψ_c^2 on $K^*(MSp)$. In order to compute ψ_c^2 on $K^*(MSp)$, consider the following operation $\phi_c: K_*(MSp)\to K_*(MSp)[1/2]$, which is the dual of the stable Adams operation $\psi_c^2: K^*(MSp)\to K^*(MSp)[1/2]$.

(2.1) **Definition.** *For* $\alpha\in K_*(MSp)$, *put*

$$\phi_c(\alpha)=\sum_R\langle\alpha,\ \psi_c^2(S^R(K))\rangle b^R(K)\in K_*(MSp)[1/2],$$

where $\langle\ ,\ \rangle$ *denote the Kronecker pairing.*

(2.2) **Lemma.**
ϕ_c is a morphism of K_*-algebra.

Proof. The linearity is clear. Let α, $\beta\in K_*(MSp)$. Then

$$\begin{aligned}\phi_c(\alpha\beta) &= \sum_R\langle\alpha\beta,\ \psi_c^2(S^R(K))\rangle b^R(K)\\ &=\sum_R\langle\alpha\otimes\beta,\ \Delta_K\psi_c^2(S^R(K))\rangle b^R(K)\\ &=\sum_R\Big(\sum_{R_1+R_2=R}\langle\alpha,\ \psi_c^2(S^{R_1}(K))\rangle\langle\beta,\ \psi_c^2(S^{R_2}(K))\rangle\Big)b^R(K)\\ &=\phi_c(\alpha)\phi_c(\beta),\end{aligned}$$

where $\Delta_K(S^R(K))=\sum_{R_1+R_2=R}S^{R_1}(K)\otimes S^{R_2}(K)$. Q. E. D.

(2.3) **Proposition.**

$$\phi_c(b_n(K))=\sum_j\binom{j+1}{n-j}(t^2/4)^{n-j}b_j(K).$$

Proof.

$$\begin{aligned}\phi_c(b_n(K)) &= \sum_R\langle i_*(\beta_{n+1}(K)),\ \psi_c^2(S^R(K))\rangle b^R(K)\\ &=\sum_R\langle\beta_{n+1}(K),\ \psi_c^2 i^*(S^R(K))\rangle b^R(K)).\end{aligned}$$

Recall that

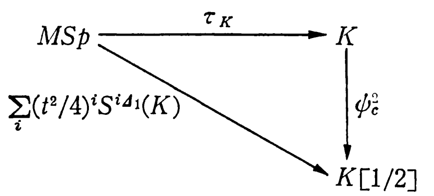
$$i^*(S^R(K))=\begin{cases} e^{j+1}(K) & \text{if } R=\Delta_j, \\ 0 & \text{otherwise,}\end{cases}$$

where Δ_j is an exponent sequence of which j-th part is 1 and others are zero, i.e., $\Delta_j=(0, 0, \cdots, 1, 0, \cdots)$. Using (1.2), we have

$$\phi_c(b_n(K))=\sum_j \langle \beta_{n+1}(K), \psi_c^2(e^{j+1}(K))\rangle b_j(K)$$

$$=\sum_j \binom{j+1}{n-j}(t^2/4)^{n-j} b_j(K). \qquad \text{Q. E. D.}$$

(2.4) **Theorem.** *The following diagram commutes.*



Proof. Let $\psi_c^2(\tau_K)=\sum_R \lambda^R S^R(K)$. Then

$$\lambda^R=\langle b^R(K), \psi_c^2(\tau_K)\rangle=\langle \phi_c(b^R(K)), \tau_K\rangle.$$

From (2.2) and (2.3), we have

$$\lambda^R=\begin{cases} \langle \phi_c(b_1(K))^j, \tau_K\rangle=(t^2/4)^j & \text{if } R=j\Delta_1, \\ 0 & \text{otherwise.}\end{cases}$$

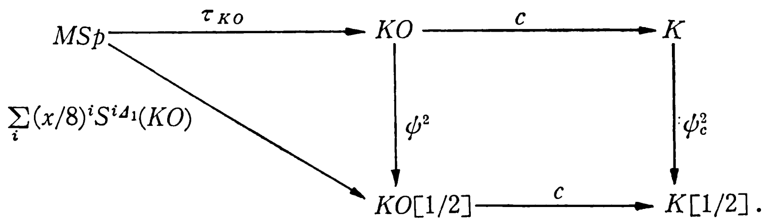
Q. E. D.

(2.5) **Corollary** (*E. E. Floyd* [1]).

$$\psi_c^2(S^R(K))=\sum_{j\geq 0}(t^2/4)^j(\tau_K)_*(S^{j\Delta_1}(MSp)S^R(MSp)).$$

Now, we consider KO-characteristic numbers. Recall that $KO_{4*}=Z[x, y]/x^2=4y$, where $x\in KO_4$ and $y\in KO_8$, and that the complexification homomorphism $c: KO_*\to K_*$ carries x, y to $2t^2$, t^4, respectively. Let ψ^2 be the stable KO-Adams operation. It is well-known that $c\psi^2=\psi_c^2 c$. So we have

(2.6) **Theorem.** *The following diagram commutes;*



Let $|R|=\sum_i ir_i$ for $R=(r_1, r_2, \cdots)$.

(2.7) **Theorem.** *Let $a\in MSp_{4k}$. Then for each R such that $|R|<k$,*

$$(\tau_{KO})_*(S^{(k-|R|)\Delta_1}(MSp)S^R(MSp))(a)\equiv 0 \mod 8,$$
$$(\tau_{HZ})_*(S^{(k-|R|)\Delta_1}(MSp)S^R(MSp))(a)\equiv 0 \mod 8.$$

Proof. From (2.6), we get the following commutative diagram;

$$\begin{array}{ccc} MSp_{4k} & \xrightarrow{(\tau_{KO})_*} & KO_{4k} \\ & \searrow^{\sum_i (x/8)^i(\tau_{KO})_*(S^{i\Delta_1}(MSp))_*} & \downarrow \psi^2 \\ & & KO_{4k}[1/2]. \end{array}$$

Recall $\psi^2 : KO_{4k}\rightarrow KO_{4k}[1/2]$ is the multiplication by 4^k. So we have the equation;

$$4^k(\tau_{KO})_*(a)=\sum_i (x/8)^i(\tau_{KO})_*(S^{i\Delta_1}(MSp)(a)).$$

Hence we have $(\tau_{KO})_*S^{k\Delta_1}(MSp)(a)\equiv 0 \mod 8$. It is obvious that $(\tau_{KO})_*S^{k\Delta_1}(MSp)(a)=(\tau_{HZ})_*S^{k\Delta_1}(MSp)(a)$. So we get the results for $R=(0, 0, \cdots)$. The general case is easily obtained replacing a by $S^R(MSp)(a)$ for R such that $|R|<k$. Q. E. D.

(2.8) **Remark.** *E. E. Floyd proved the following in* [1];

$$(\tau_{HZ})_*S^{(k-|R|)\Delta_1}(MSp)S^R(MSp)(a)\equiv\begin{cases} 0 & \mod 4, \\ 0 & \mod 8 \quad \text{if } k \text{ is even,}\end{cases}$$

for each R such that $|R|<k$.

In order to obtain more explicite relations of characteristic numbers of sympectic manifolds, we consider the KO-analogy of (2.1).

(2.9) **Definition.** *Let $\alpha\in KO_*(MSp)$, set*

$$\phi(\alpha)=\sum_R \langle\alpha, \psi^2(S^R(KO))\rangle b^R(KO).$$

Then we have the analogous properties with (2.2) and (2.3).

(2.10) **Lemma.** *ϕ is a morphism of KO_*-algebra.*

(2.11) **Proposition.**

$$\phi(b_n(KO))=\sum_j \binom{j+1}{n-j}(x/8)^{n-j}b_j(KO).$$

Let $h^{KO} : MSp_*\rightarrow KO_*(MSp)$ be the KO-Hurewicz homomorphism.

(2. 12) **Theorem.** *Let* $a \in MSp_{4k}$. *Let* $h^{KO}(a)=\sum_R \lambda^R(a) b^R(KO)$. *Then we have the following relation;*

$$\phi(h^{KO}(a))=\sum_R 4^{k-|R|} \lambda^R(a) b^R(KO).$$

Proof. If we put $\phi(h^{KO}(a))=\sum_R m^R b^R(KO)$, it holds

$$\begin{aligned} m^R &= \langle \phi(h^{KO}(a)), S^R(KO)\rangle = \langle h^{KO}(a), \psi^2(S^R(KO))\rangle \\ &= 4^{k-|R|}\langle h^{KO}(a), S^R(KO)\rangle = 4^{k-|R|}\lambda^R(a), \end{aligned}$$

by using the fact that $\psi^2 : KO_{4i} \rightarrow KO_{4i}[1/2]$ is the multiplication by 4^i.

Let $[\phi(b^T(KO)]_R$ be the integral coefficient of $(x/8)^{|T|-|R|} b^R(KO)$ in the expansion of $\phi(b^T(KO))$. We can restate (2. 12) as follows.

(2. 13) **Corollary.** *Uuder the notations of* (2. 12), *we have*

$$4^{k-|R|} \lambda^R(a)= \sum_{|T| \geq |R|} \lambda^T(a) [\phi(b^T(KO))]_R (x/8)^{|T|-|R|}.$$

The proof of (2. 13) is clear from (2. 10) and (2. 11). From (2. 13) and (2. 11), we also have

(2. 14) **Corollary.** *Under the above notations,*

$$\sum_{|T|=k} \lambda^T(a) [\phi(b^T(KO)]_R \equiv 0 \quad \text{mod. } 8 \qquad \textit{for any } R \textit{ such that } |R|<k.$$

(2. 15) **Remark.** *Compairing* (6) *in* (1. 1) *with* (2. 11), *it is easily obtained that*

$$S^{i\Delta_1}(MSp) S^R(MSp)= \sum_{|T|=i+|R|} [\phi(b^T(KO)]_R S^T(MSp).$$

So (2. 14) *is only the restatement of* (2. 7).

§ 3. A subalgebra W_*^{KO} of $MSp_* \otimes Q$.

In $KO_*(MSp) \otimes Q$, we consider all elements that satisfy the relation (2. 12) and (2. 13). Set

$$V_k = \{\alpha = \sum_{|R| \leq k} \lambda^R(\alpha) b^R(KO) \in KO_{4k}(MSp) | \phi(\alpha) = \sum_R 4^{k-|R|} \lambda^R(\alpha) b^R(KO)\}.$$

From (2. 12) $V_k \supset h^{KO}(MSp_{4k})$ holds. Now consider $h^{KO} \otimes Q : MSp_{4k} \otimes Q \rightarrow KO_{4k}(MSp) \otimes Q$, and define

$$W_k^{KO} = (h^{KO} \otimes Q)^{-1}(KO_{4k}(MSp)).$$

W_k^{KO} consists of elements all KO-characteristic numbers of which are integral. It holds $MSp_{4k}/Tor \subset W_k^{KO}$. We have the following, which implies that the KO-Hurewicz image of W_k^{KO} is characterized by the relation (2. 12).

(3. 1) **Theorem.** $(h^{KO}\otimes Q)(W_k^{KO})=V_k$.

Proof. For $v\in W_k^{KO}$, we take an integer m such that $mv\in MSp_{4k}$. Since $h^{KO}(MSp_{4k})\subset V_k$, we have the equation $\phi(h^{KO}(mv))=\sum_R 4^{k-|R|}\lambda^R(h^{KO}(mv))b^R(KO)$. It holds $\phi(h^{KO}(mv))=m\phi((h^{KO}\otimes Q)(v))$ and $\lambda^R(h^{KO}(mv))=m\lambda^R((h^{KO}\otimes Q)(v))$. Since $KO_{4k}(MSp)$ is a free module, $(h^{KO}\otimes Q)(v)$ satisfies the relation $\phi((h^{KO}\otimes Q)(v))=\sum_R 4^{k-|R|}\lambda^R((h^{KO}\otimes Q)(v))b^R(KO)$, and so, belongs to V_k.

Conversely, let $a=\sum_R \lambda^R(a)b^R(KO)$ be an element of V_k. We remark that any element of V_k also satisfies the relation (2. 13). So the following relation is satisfied;

$$(1)\qquad 4^{k-|R|}\lambda^R(a)=\lambda^R(a)+\sum_{|R|<|T|}\lambda^T(a)(x/8)^{|T|-|R|}[\phi(b^T(KO))]_R.$$

If $|R|=k-1$, this relation implies that $3\lambda^R(a)=\sum_{|T|=k}\lambda^T(a)(x/8)[\phi(b^T(KO))]_R$. By induction on $|R|$, we have from (1) that $\lambda^R(a)$ for any R can be represented as a $Q[x]$-linear combination of $\lambda^T(a)$ such that $|T|=k$. Let $\tilde{a}=\sum_{|R|=k}\lambda^R(a)b^R(HZ)$ be an element of $H_{4k}(MSp)$, where we identify the coefficients $\lambda^R(a)$ with integers. Then there exists some element α of $MSp_{4k}\otimes Q$ such that $(h^H\otimes Q)(\alpha)=\tilde{a}$. If we take an integer m such that $m\alpha\in MSp_{4k}$, then $h^{KO}(m\alpha)\in V_k$. Hence in $KO_{4(k-|R|)}\otimes Q$, it holds that

$$(2)\qquad 4^{k-|R|}\lambda^R((h^{KO}\otimes Q)(\alpha))$$
$$=\lambda^R((h^{KO}\otimes Q)(\alpha))+\sum_{|R|<|T|}\lambda^T((h^{KO}\otimes Q)(\alpha))(x/8)^{|T|-|R|}[\phi(b^T(KO))]_R$$

for any R such that $|R|\leqq k$. When $|R|=k$, $\lambda^R(a)=\lambda^R((h^{KO}\otimes Q)(\alpha))$ from the definition of α. Therefore, $\lambda^R(a)=\lambda^R((h^{KO}\otimes Q)(\alpha))$ holds for any R, because from (1) and (2) both $\lambda^R(a)$ and $\lambda^R((h^{KO}\otimes Q)(\alpha))$ can be written as the same $Q[x]$-linear combination of $\lambda^T(a)$ and $\lambda^T((h^{KO}\otimes Q)(\alpha))$ such that $|T|=k$ respectively. Hence $a=\sum_R \lambda^R((h^{KO}\otimes Q)(\alpha))b^R(KO)=(h^{KO}\otimes Q)(\alpha)$, and so, a is an element of $(h^{KO}\otimes Q)(W_k^{KO})$. Q. E. D.

By (3. 1), an element of W_k^{KO} also satisfies the relation (2. 14). Especially, we have

(3. 2) **Corollary.** *Let α be an element of W_k^{KO}, and set $(h^{KO}\otimes Q)(\alpha)=\sum_R \lambda^R(\alpha)b^R(KO)$. Then it holds $\lambda^{k\Delta_1}(\alpha)\equiv 0$ mod. 8.*

Proof. From (2. 11) $\phi(b_n(KO))$ that has $b_0(KO)$ with non-zero coefficient in its expansion is only $\phi(b_1(KO))$ which equals to $(x/8)+b_1(KO)$. So by (2. 10) we obtain $[\phi(b^T(KO))]_{\underline{0}}=1$ if $T=t\Delta_1$, and 0 if otherwise, where $\underline{0}$ is the zero sequence $(0, 0, \cdots)$. Therefore (2. 14) in the case $\underline{0}$ implies $\lambda^{k\Delta_1}(\alpha)\equiv 0$ mod. 8. Q. E. D.

We remark that (2. 7) also holds for an element of W_*^{KO} by (3. 2).

The homomorphism $MSp_*\to MU_*$ induced by the inclusion $Sp\to U$ can be extended to the homomorphism $W_*^{KO}\to MU_*$ by the Hattori-Stong theorem.

We denote by $r: W_*^{KO} \to MO_*$ the composition of $W_*^{KO} \to MU_*$ and $MU_* \to MO_*$. E. E. Floyd [1] has considered a subalgebra P_* of MO_* and proved Image$(MSp_* \to MO_*)$ is contained in $P_*{}^8$. On the other hand following F. W. Roush [3], Image$(MSp_* \to MO_*)$ contains $MO_*{}^{16}$. By selecting a polynomial base x_i, $i \neq 2^k - 1$, of MO_*, P_* can be represented as the polynomial algebra $Z_2[(x_{2i})^2, (x_{2j-1})^2, x_{2j}]$ for any i and j such that $j \neq 2^k$ for any k, and so it holds $P_* \supsetneqq MO_*{}^2$. We remark the following.

(3.3) **Corollary.** Image$(r: W_*^{KO} \to MO_*) = P_*{}^8$.

Proof. Considering the method of E. E. Floyd's in [1], Image$(r: W_*^{KO} \to MO_*) \subset P_*{}^8$ holds from (3.2). Following D. M. Segal [4], there exists some Sp-manifold for each dimension $8j$, $j \neq 2^k$, and its symplectic cobordism class y_{2j} satisfies $S^{\Delta_{2j}}(MSp)(y_{2j}) \equiv 2 \mod 4$. Such a Sp-manifold was defined by R. E. Stong [5], and by [5, Th. 4] all K-characteristic numbers of y_{2j} are multiples of 2. Therefore, the K-Hurewicz image of $(1/2)y_{2j}$ is integral and so $(1/4)y_{2j}{}^2$ is an element of W_{4j}^{KO}. We can select x_{2j}, $j \neq 2^k$, as satisfying $r((1/4)y_{2j}{}^2) = x_{2j}{}^8$. Hence $\{r((1/4)y_{2j}{}^2) \mid j \neq 2^k\}$ and $MO_*{}^{16}$ generate $P_*{}^8$ and this is the required result. Q. E. D.

(3.4) **Remark.** *In fact the above* (3.3) *can be proved without using* (3.2) *by considering merely the structure of* W_2^{KO} *and* W_4^{KO}, *if we use the essential part of Floyd's method.* W_k^{KO} *is precisely studied by R. Okita* [2] *for* $1 \leqq k \leqq 7$.

§ 4. Applications.

In this section, we investigate some divisivility conditions on characteristic numbers of W_*^{KO}. We denote in this section $h^{KO} \otimes Q$ merely by h^{KO}.

(4.1) **Theorem** (*R. Okita* [2, *Prop.* 4.2]). *Let* α *be an element of* $W_{2^n-1}^{KO}$ *and let* $h^{KO}(\alpha) = \sum_R \lambda^R(\alpha) b^R(KO)$. *Then* $\lambda^{\Delta_{2^n-1}}(\alpha) \equiv 0 \mod 8$.

Proof. In the case $n=1$, it is clear from (3.2). Now inductively supposing that $\lambda^{\Delta_{2^{n-1}-1}}(\beta) \equiv 0 \mod 8$ holds for any $\beta \in W_{2^{n-1}-1}^{KO}$, we prove $\lambda^{\Delta_{2^n-1}}(\alpha) \equiv 0 \mod 8$. For any integer k such that $0 \leqq k \leqq 2^{n-1}$, it holds

$$(1) \qquad S^{k\Delta_1}(MSp) S^{\Delta_{2^n-k-1}}(MSp) = \sum_{i=0}^{k} \binom{2^n-k}{k-i} S^{i\Delta_1 + \Delta_{2^n-i-1}}(MSp).$$

By using (2.7) and (1), we have the following;

$$(2) \qquad \sum_{i=0}^{k} \binom{2^n-k}{k-i} \lambda^{i\Delta_1 + \Delta_{2^n-i-1}}(\alpha) \equiv 0 \mod 8 \qquad \text{if} \quad k \geqq 1.$$

From (2) for each $k = 1, 2, \cdots, 2^{n-1}$, we have

$$(3) \qquad \lambda^{k\Delta_1 + \Delta_{2^n-k-1}}(\alpha) \equiv m_k \lambda^{\Delta_{2^n-1}}(\alpha) \mod 8 \qquad \text{for} \quad 1 \leqq k \leqq 2^{n-1},$$

where m_k is some integer. Next we consider the equation

$$(4)\qquad S^{\Delta_{2^{n-1}-1}}(MSp)\,S^{2^{n-1}\Delta_1}(MSp) = 2\cdot S^{(2^{n-1}-1)\Delta_1+\Delta_{2^{n-1}}}(MSp)+S^{2^{n-1}\Delta_1+\Delta_{2^{n-1}-1}}(MSp).$$

Since it holds $\lambda^{\Delta_{2^{n-1}-1}}(S^{2^{n-1}\Delta_1}(MSp)(\alpha))\equiv 0$ mod. 8 by our inductive hypothesis, it holds from (4) that

$$(5)\qquad \lambda^{2^{n-1}\Delta_1+\Delta_{2^{n-1}-1}}(\alpha)\equiv -2\cdot\lambda^{(2^{n-1}-1)\Delta_1+\Delta_{2^{n-1}}}(\alpha) \quad \text{mod. } 8.$$

Considering the equation (2) in the case $k=2^{n-1}$ and using (3) and (5), we obtain

$$\left(1+\sum_{i=1}^{2^{n-1}-1}\binom{2^{n-1}}{i}m_i-2\cdot m_{2^{n-1}-1}\right)\lambda^{\Delta_{2^{n-1}}}(\alpha)\equiv 0 \quad \text{mod. } 8.$$

Since $\binom{2^{n-1}}{i}\equiv 0$ mod. 2 for $1\leqq i\leqq 2^{n-1}-1$, we have $\lambda^{\Delta_{2^{n-1}}}(\alpha)\equiv 0$ mod. 8.

Q. E. D.

(4. 2) Corollary (*Segal* [4] *or Okita* [2, *Prop.* 4. 1]). *Let* α_1 *and* α_2 *be classes of* W_{2n}^{KO} *and* W_{2n-1}^{KO} *respectively. Then, we have* $\lambda^{\Delta_{2n}}(\alpha_1)\equiv 0$ mod. 4 *and* $S^{\Delta_{2n-1}}(\alpha_2)\equiv 0$ mod. 4.

Proof. We consider the following equation;

$$S^{\Delta_1}(MSp)\,S^{\Delta_{2n-1}}(MSp)-S^{\Delta_{2n-1}}(MSp)\,S^{\Delta_1}(MSp)=(2^n-2)\,S^{\Delta_{2n}}(MSp)$$

for $n>1$, and

$$(S^{\Delta_1}(MSp))^2=2\cdot S^{\Delta_2}(MSp).$$

From these equations, (2. 7) and (4. 1), we have $\lambda^{\Delta_{2n}}(\alpha_1)\equiv 0$ mod. 4 for $n\geqq 1$. We also consider the following relation;

$$S^{\Delta_2}(MSp)\,S^{\Delta_{2n-1}}(MSp)-S^{\Delta_{2n-1}}(MSp)\,S^{\Delta_2}(MSp)=(2n-3)\,S^{\Delta_{2n+1}}(MSp)$$

for $n\geqq 1$. By using this equation and $\lambda^{\Delta_2}(\alpha_1)\equiv 0$ mod. 4, we have inductively $\lambda^{\Delta_{2n-1}}(\alpha_2)\equiv 0$ mod. 4. Q. E. D.

(4. 3) Proposition (*R. Okita* [2, *Prop.* 4. 1]). *For any* $\alpha\in W_n^{KO}$, $\lambda^{\Delta_n}(\alpha)\equiv 0$ mod. 2.

Proof. This is clear from Image$(r: W_*^{KO}\to MO_*)\subset MO_*^8$ by (3. 3).

Q. E. D.

We apply (2. 14) in the case $R=\Delta_r$. For this, we first consider the following lemma.

(4. 4) Lemma.

$$[\phi(b^T(KO))]_{\Delta_r}=\begin{cases}\binom{r+1}{j} & \text{if } \; T=i\Delta_1+\Delta_{r+j},\\ 0 & \text{otherwise.}\end{cases}$$

Proof. From (2.11), all $\phi(b_i(KO))$ that have $b_r(KO)$ with non-zero coefficients in their expansions are only $\phi(b_{r+j}(KO))$ for $0\leqq j\leqq r+1$, and all $\phi(b_i(KO))$ that have $b_0(KO)$ with non-zero coefficients are only $\phi(b_1(KO))$. By this and (2.10), we have that all $\phi(b^T(KO))$ that have $b_r(KO)$ with non-zero coefficients in their expansions are only $\phi(b^T(KO))$ such that $T=i\Delta_1+\Delta_{r+j}$ for $0\leqq j\leqq r+1$, and we can easily deduce the required relation from (2.10) and (2.11). Q. E. D.

(4.5) **Theorem.** *Let α be an element of W_n^{KO} and set $h^{KO}(\alpha)=\sum_R \lambda^R(\alpha)b^R(KO)$. For any i such that $0\leqq i\leqq n$, we have $\lambda^{i\Delta_1+\Delta_{n-i}}(\alpha)\equiv 0$ mod. 8, 4, 4 or 2 if $n=2^m-1$, 2^m, $2m-1$ or $2m$ for some m, respectively.*

Proof. In the case $i=0$, this is just (4.1), (4.2) and (4.3). By using (4.4), we have the following relation from (2.14) for $R=\Delta_r$;

$$(*)\qquad \sum_{j=0}^{r+1}\binom{r+1}{j}\lambda^{(n-r-j)\Delta_1+\Delta_{r+j}}(\alpha)\equiv 0 \mod. 8$$

for $0\leqq j\leqq r+1\leqq n$. Inductively supposing that the result holds in the case $0\leqq i\leqq k<n$, we can prove the result in the case $i=k+1$ by using the relation (*) for $r=n-(k+1)$. Q. E. D.

Next we consider (2.14) in the case $R=r\Delta_1$.

(4.6) **Lemma.**

$$[\phi(b^T(KO))]_{r\Delta_1}=\begin{cases}\dbinom{i}{\|T\|-r}2^j & \text{if } T=i\Delta_1+j\Delta_2+k\Delta_3,\\ 0 & \text{otherwise,}\end{cases}$$

where $\|T\|=i+j+k$ for $T=i\Delta_1+j\Delta_2+k\Delta_3$.

Proof. From (2.11), $\phi(b_i(KO))$ that has $b_1(KO)$ with non-zero coefficient in its expansion is only $\phi(b_1(KO))$, $\phi(b_2(KO))$ or $\phi(b_3(KO))$. Therefore, by using (2.10) and (2.11), we have that all $\phi(b^T(KO))$ which have $(b_1(KO))^r$ with non-zero coefficients in their expansions are only $\phi(b^T(KO))$ such that $T=i\Delta_1+j\Delta_2+k\Delta_3$, and we can easily obtain the required relation by using (2.11).

Q. E. D.

(4.7) **Theorem.** *Let α be an element of W_n^{KO} and set $h^{KO}(\alpha)=\sum_R \lambda^R(\alpha)b^R(KO)$. It holds*

$$2^j\cdot\lambda^{i\Delta_1+j\Delta_2+k\Delta_3}(\alpha)\equiv 0 \mod. 8$$

if $i+2j+3k=n$.

Proof. We may suppose $j=0$, 1 or 2. The following relation holds from (2.14) for $R=r\Delta_1$ by using (4.6);

$$(1)\qquad \sum_{i+2j+3k=n}\binom{i}{i+j+k-r}2^j\cdot\lambda^{i\Delta_1+j\Delta_2+k\Delta_3}(\alpha)\equiv 0 \mod. 8,$$

for any $\alpha \in W_n^{KO}$. The binomial coefficients $\binom{i}{i+j+k-r}$ are zero unless the following cases;

$$n-r \geqq j+2k \quad \text{and} \quad r \geqq j+k. \tag{2}$$

For fixed n, we prove the theorem by induction on $j+k$. In the case $j+k=0$, it holds from (2.7). Now inductively we suppose that it holds $2^j \cdot \lambda^{i\Delta_1+j\Delta_2+k\Delta_3}(\alpha) \equiv 0$ mod. 8 if $0 \leqq j+k<t$ for any $\alpha \in W_n^{KO}$ and for $i+2j+3k=n$. We prove the theorem in the case $j_0+k_0=t$.

Case (i) $j_0=2$. In this case we consider the relation (1) for $r=n-2t+2$. If there exist j and k which satisfy that $j+k \geqq t$ and that the binomial coefficient $\binom{i}{i+j+k-(n-2t+2)}$ is non-zero, then it must be $j=2=j_0$ and $k=k_0$ from (2). So using the inductive hypothesis, we have from (1) for $r=n-2t+2$

$$4 \cdot \lambda^{(n-3t+2)\Delta_1+2\Delta_2+(t-2)\Delta_3}(\alpha) \equiv 0 \quad \text{mod. } 8$$

for any $\alpha \in W_n^{KO}$. Hence the required result holds in this case.

Case (ii) $j_0=1$. We consider (1) for $r=n-2t+1$. If there exist j and k which satisfy that $j+k \geqq t$ and that the binomial coefficient $\binom{i}{i+j+k-(n-2t+1)}$ is non-zero, it must be $j=1=j_0$ and $k=t-1=k_0$ or $j=2$ and $k=t-2$. So using the inductive hypothesis, we have from (1) for $r=n-2t+1$

$$4(n-3t+2) \cdot \lambda^{(n-3t+2)\Delta_1+2\Delta_2+(t-2)\Delta_3}(\alpha)+2 \cdot \lambda^{(n-3t+1)\Delta_1+\Delta_2+(t-1)\Delta_3}(\alpha) \equiv 0 \quad \text{mod. } 8$$

for any $\alpha \in W_n^{KO}$. The former term is 0 mod. 8 by the case (i), where we consider the former term is zere when $t=1$. Hence we have the required result in this case.

Case (iii) $j_0=0$. We consider (1) for $r=t$. If there exist j and k which satisfy $j+k \geqq t$ and that the binomial coefficient $\binom{i}{i+j+k-t}$ is non zero, then it must be $j+k=t$. So using the inductive hypothesis, we have from (1) for $r=t$

$$4 \cdot \lambda^{(n-3t+2)\Delta_1+2\Delta_2+(t-2)\Delta_3}(\alpha)+2 \cdot \lambda^{(n-3t+1)\Delta_1+\Delta_2+(t-1)\Delta_3}(\alpha) + \lambda^{(n-3t)\Delta_1+t\Delta_3}(\alpha) \equiv 0 \quad \text{mod. } 8$$

for any $\alpha \in W_n^{KO}$. The former two terms are 0 mod. 8 by cases (i) and (ii), where we consider the first term is zero when $t=1$. Hence we have the required result in this case and it completes the proof. Q. E. D.

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