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Studies on the real primitive infinite Lie algebras

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Introduction

By the classification theorem of Morimoto-Tanaka [1], we know that there are fourteen classes of the real primitive infinite Lie algebras. They contain all *the classical infinite Lie algebras* (see § 1). In the previous paper [2], we determined the structure of Lie subalgebras of finite codimension of *the classical infinite Lie algebras*.

One of the purposes of the present paper is to extend our result to the case of Lie subalgebras of finite codimension of the real primitive infinite Lie algebras. The precise statement can be seen in Theorem A.

By using Theorem A, we prove that fourteen classes of the real primitive infinite Lie algebras are not isomorphic to one another (see Theorem B).

§ 1. The real primitive infinite Lie algebras

Let V be an n-dimensional vector space over the field F, where F is $\mathbf{R}$ or $\mathbf{C}$. We denote by $L_{gl}(n, F)$ the Lie algebra of all formal vector fields over V. If $F=\mathbf{C}$, Lie subalgebras of $L_{gl}(n, \mathbf{C})$ are regarded as those of $L_{gl}(2n, \mathbf{R})$, and these "real" Lie algebras are denoted by the same notations.

The complete list of the real primitive infinite Lie algebras is following.

(1) $L_{gl}(n, \mathbf{R})$.

(2) $L_{sl}(n, \mathbf{R})$: the Lie algebra of real vector fields of divergence zero.

(3) $L_{csl}(n, \mathbf{R})$: the Lie algebra of real vector fields of constant divergence.

(4) $L_{sp}(2n, \mathbf{R})$: the Lie algebra of real Hamiltonian vector fields, $(n\geqq 2)$.

(5) $L_{csp}(2n, \mathbf{R})$: the Lie algebra of real vector fields preserving a Hamiltonian form up to a constant multiple, $(n\geqq 2)$.

(6) $L_{ct}(2n+1, \mathbf{R})$: the real contact algebra.

(7) $L_{gl}(n, \mathbf{C})$.

(8) $L_{sl}(n, \mathbf{C})$: the Lie algebra of complex vector fields of divergence zero.

(9) $L_{rsl}(n, \mathbf{C})$: the Lie algebra of complex vector fields with divergence on some real line in $\mathbf{C}$.

(10) $L_{csl}(n, \mathbf{C})$: the Lie algebra of vector fields of complex constant divergence.

(11) $L_{sp}(2n, \mathbf{C})$: the Lie algebra of complex Hamiltonian vector fields, $(n \geqq 2)$.

(12) $L_{rsp}(2n, \mathbf{C})$: the Lie algebra of complex vector fields preserving a Hamiltonian form up to constant lying on a real line in $\mathbf{C}$, $(n \geqq 2)$.

(13) $L_{csp}(2n, \mathbf{C})$: the Lie algebra of complex vector fields preserving a Hamiltonian form up to a complex constant, $(n \geqq 2)$.

(14) $L_{ct}(2n+1, \mathbf{C})$: the complex contact algebra.

We will divide the above fourteen Lie algebras into four cases according to their properties.

Case 1. Lie algebras $L_{gl}(n, \mathbf{R})$, $L_{sl}(n, \mathbf{R})$, $L_{sp}(2n, \mathbf{R})$ and $L_{ct}(2n+1, \mathbf{R})$.

These Lie algebras are known as *the real classical infinite Lie algebras*, and the structure of Lie subalgebras of finite codimension of them was determined in [2].

Case 2. Lie algebras $L_{gl}(n, \mathbf{C})$, $L_{sl}(n, \mathbf{C})$, $L_{sp}(2n, \mathbf{C})$ and $L_{ct}(2n+1, \mathbf{C})$.

These "real" Lie algebras are real representations of *the complex classical infinite Lie algebras.*

Case 3. Lie algebras $L_{csl}(n, \mathbf{R})$, $L_{csp}(2n, \mathbf{R})$, $L_{rsl}(n, \mathbf{C})$ and $L_{rsp}(2n, \mathbf{C})$.

Each Lie algebra L of this case contains a unique ideal of codimension 1, which is denoted by L'.

Case 4. Lie algebras $L_{csl}(n, \mathbf{C})$ and $L_{csp}(2n, \mathbf{C})$.

Each Lie algebra L of this case contains two ideals L' and L'': L' is of codimension 1 and L'' is of codimension 2.

For a Lie algebra L of Case 3 and Case 4, L' and L'' are called "trivial" subalgebras (or ideals) of L.

§2. Summary of known results

A real primitive infinite Lie algebra (or briefly a real PLA) L has a canonical filtration $\{L_p\}_{p \in Z}$. For a contact algebra, integers p satisfy $p \geqq -2$, and for other *PLA*, $p \geqq -1$. Recall that L_0 is, by definition, a maximal subalgebra of L. Except a contact algebra, the natural representation of L_0 on L/L_0 is irreducible.

Next we topologize a real *PLA* L assigning $\{L_p\}$ as a system of fundamental neighborhood of L. Then L is a topological Lie algebra and it is separated and complete with respect to the filtration topology. We should remark that a Lie subalgebra B of L is closed if and only if $B = \bigcap_{p \in Z} (B + L_p)$.

The following lemma was essential in [2].

Lemma 2.1. *Let L be a real or complex classical infinite Lie algebra. Let B be a "closed" proper Lie subalgebra of finite codimension of L. Then B is contained in L_0.*

Remark. By an easy consideration, we know that Lemma 2.1 still holds for Lie algebras of Case 2.

Using the above lemma, we proved the following, as one of the main theorem in [2].

Theorem 2.2. *Let L be a real or complex classical infinite Lie algebra. Let B be a proper Lie subalgebra of L with* $\dim L/B<\infty$. *Then there exists a positive integer k such that $L_k \subset B \subset L_0$. (This means that B is a closed Lie subalgebra of L.)*

§3. Lie subalgebras of finite codimension of real PLA

In this section, we prove the first main theorem.

Theorem A. *Let L be a real PLA with a canonical filtration $\{L_p\}_{p\in Z}$. Let B be a proper Lie subalgebra of finite codimension of L.*

a) *For a Lie algebra L of Case* 1 *and Case* 2, *there exists a positive integer k such that $L_k \subset B \subset L_0$.*

b) *For a Lie algebra L of Case* 3 *and Case* 4, *if B is not a trivial Lie subalgebra of L, then there exists a positive integer k such that $L_k \subset B \subset L_0$.*

Proof of Theorem A. The proof of Case 1 was completed in [2]. For remaining Lie algebras, the proof depends on case by case analysis.

Proof of a). In this paragraph, we denote by M a Lie algebra of Case 1. Let L be a Lie algebra of Case 2. Then there exists a natural complex structure I in L and L is written as $L=M+IM$ (a direct sum). Through this complex structure I, we also consider L as a Lie algebra over $\mathbf{C}$, and we denote it by $\tilde{L}$. Define a one-to-one mapping $f: L\to\tilde{L}$ by $f(X+IY)=X+\sqrt{-1}\,Y$ for $X, Y\in M$. Then if a Lie subalgebra K of L is I-invariant, $f(K)$ is a Lie subalgebra of $\tilde{L}$. Now for a proper Lie subalgebra B with $\dim L/B<\infty$, $B\cap IB$ is also a Lie subalgebra of finite codimension of L, and it is I-invariant. Hence $f(B\cap IB)$ becomes a Lie subalgebra of finite codimension of $\tilde{L}$. Using Theorem 2.2, we have

$$\tilde{L}_k \subset f(B\cap IB) \subset \tilde{L}_0$$

for a suitable positive integer k. This means

$$L_k \subset B\cap IB \subset L_0.$$

Thus B becomes a closed Lie subalgebra of finite codimension of L. Using Lemma 2.1, we have $B\subset L_0$. Q.E.D.

Proof of b). Let L be a Lie algebra of Case 3. If $B\neq L'$, it holds that $0<\dim L'/B\cap L'<\infty$. Note that L' is a classical infinite Lie algebra. Using Theorem 2.2, we have $L'_k\subset B\cap L'\subset L'_0$. Since $L'_k=L_k$ for $k>0$, using Lemma 2.1, we get $L_k\subset B\subset L_0$. For a Lie algebra of Case 4, it can be proved quite similarly. Q.E.D.

§ 4. Application of the main theorem

In this section, we prove

Theorem B. *Fourteen classes of the real primitive infinite Lie algebras listed in* § 1 *are not isomorphic to one another.*

Proof. First note that the complexification of each Lie algebra of Case 1 is simple. On the contrary, the complexification of each Lie algebra of Case 2 has an ideal of "infinite" codimension, and hence it is not simple. Thus each Lie algebra of Case 1 can not be isomorphic to each Lie algebra of Case 2. Now it is clear that there are no isomorphisms among Case 1, Case 2, Case 3 and Case 4. The remaining part of the proof is to show that there are no isomorphisms among Lie algebras in each case.

Proof of Case 1. In this paragraph, we prove that there are no isomorphisms among four Lie algebras of Case 1. Let L and $\bar{L}$ be arbitrary two Lie algebras of infinite irreducible Lie algebras $L_{gl}(n, \mathbf{R})$, $L_{sl}(n, \mathbf{R})$ and $L_{sp}(2n, \mathbf{R})$. Suppose that there exists an isomorphism $\varphi: L \to \bar{L}$. Then by Theorem A it holds that $\bar{L}_k \subset \varphi(L_0) \subset \bar{L}_0$. Since $\varphi(L_0)$ is a maximal subalgebra of $\bar{L}$, it satisfies $\varphi(L_0) = \bar{L}_0$. Using the transitivity of L and $\bar{L}$, we have $\varphi(L_p) = \bar{L}_p$ for $p \geqq 0$. In particular, L_0/L_1 is isomorphic to $\bar{L}_0/\bar{L}_1$. This is a contradiction because $gl(n, \mathbf{R})$, $sl(n, \mathbf{R})$ and $sp(2n, \mathbf{R})$ are not isomorphic to one another. Recalling that only the contact algebra is not irreducible, it can be concluded that there are no isomorphisms among $L_{gl}(n, \mathbf{R})$, $L_{sl}(n, \mathbf{R})$, $L_{sp}(2n, \mathbf{R})$ and the real contact algebra. Q. E. D.

For other cases, it can be similarly proved that there are no isomorphisms among Lie algebras. Q. E. D.

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References

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[2] N. Nakanishi, Lie subalgebras of finite codimension of the classical infinite Lie algebras, (to appear in J. Math. Kyoto Univ.).