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Reproducing differentials and certain theta functions on open Riemann surfaces

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Introduction

The purpose of the present note is to introduce the theta function on open Riemann surfaces using some fundamental Abelian differentials with finite norm, especially reproducing kernels for analytic differentials (cf. Ahlfors-Sario [1]) and square integrable normal differentials (cf. Kusunoki [4]). In § 1 we shall construct a factor of automorphy for the Fuchsian group acting on the universal covering surface of an open Riemann surface W using the reproducing kernel for $\Gamma_a(W)$, the space of all square integrable analytic differentials on W. If we replace $\Gamma_a(W)$ by $\Gamma_{ase}(W)$, then we have a factor of automorphy associated with $\Gamma_{ase}(W)$, where $\Gamma_{ase}(W)$ is the subspace of $\Gamma_a(W)$ consisting of all semi-exact elements. In § 2 we shall define the theta function induced by a factor of automorphy above and some associated functions on W which correspond to the alternating Riemann form and the Hermitian Riemann form for theta functions of finitely many variables. We shall show in § 3 that an analogue of the Riemann's theta function can be obtained from a theta function defined in § 2 by multiplying an exponential of a certain analytic double integral provided that a surface W belongs to a restricted class of Riemann surfaces. In § 4 we shall give a condition that a theta function on a parabolic open Riemann surface of positive finite genus should be continued analytically to the compact prolongation of the surface. Finally we shall show also that our theta function defined in § 2 on a plane domain bounded by a finite number of analytic Jordan curves is obtained from the Riemann's theta series associated with the double of that domain.

§ 1. Factors of automorphy associated with reproducing kernels

Let W be an open Riemann surface, ζ be a local parameter of a given point of W and z be a local parameter of a general point of W. The following are known (cf. [1], V. 18, 19, 20):

There is a unique differential $\psi_0(z, \zeta)=k_0(z, \zeta)dz$ in Γ_a $(=\Gamma_a(W))$ with the following properties:

$$k_0(z, \zeta)=\overline{k_0(\bar{\zeta}, \bar{z})}, \tag{1.1}$$

$$(\alpha, \psi_0)=2\pi a(\zeta) \quad \text{for each} \quad \alpha(z)=a(z)dz \quad \text{in} \quad \Gamma_a. \tag{1.2}$$

Let C be a finite cycle and write as

$$\psi(C)=k(z, C)dz=\left(\int_C k_0(z, \zeta)d\bar{\zeta}\right)dz,$$

then

$$(\alpha, \psi(C))=2\pi\int_C \alpha. \tag{1.3}$$

If we set $\tau(C)=-\frac{1}{\pi}\operatorname{Im}\psi(C)$, then

$$(\tau(C_1), {}^*\tau(C_2))=C_1\times C_2 \tag{1.4}$$

and

$$(\omega, {}^*\tau(C))=\int_C \omega \quad \text{for each} \quad \omega, \tag{1.5}$$

where ω is a square integrable harmonic differential on W. $\overline{k_0(z, \zeta)}$ is said to be the reproducing kernel for analytic differentials (cf. [1], p. 302).

Next we recall the contents of ([1], V. 21), that is to say, the results just mentioned are slightly modified for Γ_{ase} $(=\Gamma_{ase}(W))$: There is a unique differential $\psi_0'(z, \zeta)=k_0'(z, \zeta)dz$ in Γ_{ase} with the following properties:

$$k_0'(z, \zeta)=\overline{k_0'(\zeta, z)}, \tag{1.1'}$$

$$(\alpha, \psi_0')=2\pi a(\zeta) \quad \text{for each} \quad \alpha(z)=a(z)dz \quad \text{in} \quad \Gamma_{ase}. \tag{1.2'}$$

Let C be a finite cycle and write as

$$\psi'(C)=k'(z, C)dz=\left(\int_C k_0'(z, \zeta)d\bar{\zeta}\right)dz,$$

then

$$(\alpha, \psi'(C))=2\pi\int_C \alpha. \tag{1.3'}$$

If we set $\tau'=-\frac{1}{\pi}\operatorname{Im}\psi'(C)$, then

$$(\tau'(C_1), {}^*\tau'(C_2))=C_1\times C_2 \tag{1.4'}$$

and

$$(\omega, {}^*\tau'(C))=\int_C \omega \quad \text{for each} \quad \omega, \tag{1.5'}$$

where ω is a square integrable harmonic differential on W such that both ω and $^*\omega$ are semi-exact.

From now on we assume that the universal covering surface $\hat{W}$ of W is conformally equivalent to the unit disc $D=\{z \mid |z|<1\}$. We may identify $\hat{W}$ with D. We denote by π: $\hat{W}\to W$ the projection and by $D(\pi)$ the group of covering transformations. Then $D(\pi)$ is a Fuchsian group and $D/D(\pi)=W$. We denote by $\pi_1(W, p_0)$ the fundamental group of W with the base point p_0 and by $H_1(W)$ the 1-dimensional homology group of W. There is a sequence of homomorphisms

$$D(\pi)\xrightarrow{\alpha_1}\pi_1(W, p_0)\xrightarrow{\alpha_2}H_1(W),$$

where α_1 is an isomorphism and α_2 is an epimorphism. We write as

$$\chi=\alpha_2\circ\alpha_1 : D(\pi)\longrightarrow H_1(W).$$

From now on we denote by z or ζ points (and their coordinates) of $\hat{W}$ and by $p=\pi(z)$ or $q=\pi(\zeta)$ those of W. We write as

$$\hat{k}_0(z, \zeta)dzd\bar{\zeta}=\pi^*\times\pi^*(k_0(p, q)dpd\bar{q})=k_0(\pi(z), \pi(\zeta))d\pi(z)d\overline{\pi(\zeta)},$$

where $\overline{k_0(p, q)}$ is the reproducing kernel for analytic differentials on W. Let z_0 be a fixed point on $\hat{W}$ such that $\pi(z_0)=p_0$ and associate with each $\hat{C}\in D(\pi)$ a path $\langle\hat{C}\rangle$ from z_0 to $\hat{C}z_0$. Then

$$\hat{k}(z, \hat{C})dz=\left(\int_{\langle\hat{C}\rangle}\hat{k}_0(z, \zeta)d\bar{\zeta}\right)dz=\pi^*\left(\left(\int_{\pi(\langle\hat{C}\rangle)}k_0(p, q)d\bar{q}\right)dp\right)$$

does not depend on the choice of $\langle\hat{C}\rangle$. It is clear that

$$\hat{k}(z, \hat{C}_1\hat{C}_2)dz=\hat{k}(z, \hat{C}_1)dz+\hat{k}(z, \hat{C}_2)dz. \tag{1.6}$$

For each $\hat{C}\in D(\pi)$ we write as

$$\sigma(z, \hat{C})=\int_{z_0}^{z}\hat{k}(\eta, \hat{C})d\eta \quad\text{and}\quad \sigma(\hat{C})=\frac{1}{4\pi}\|\psi(C)\|^2=\frac{2}{\pi}\|\tau(C)\|^2,$$

where C is $\chi(\hat{C})$ and represented by the path $\pi(\langle\hat{C}\rangle)$.[1]

Lemma. *There is a function* ρ: $D(\pi)\to\{-1, 1\}$ *such that*

$$\rho(\hat{C}_1\hat{C}_2)=\rho(\hat{C}_1)\rho(\hat{C}_2)\exp[\pi i(\chi(\hat{C}_1)\times\chi(\hat{C}_2))].$$

Proof.[2] Let $E=\{e_1, e_2,\dots\}$ be a basis of the free Abelian group $H_1(W)$ and ρ': $E\to\{-1, 1\}$ be any mapping. Then ρ': $H_1(W)\to\{-1, 1\}$ defined as

$$\rho'\left(\sum_{j=1}^{p}n_je_j\right)=\prod_{j=1}^{p}\rho'(e_j)^{n_j}\prod_{1\le j<k\le p}\exp[\pi i(e_j\times e_k)]^{n_jn_k}$$

satisfies $\rho'(C_1+C_2)=\rho'(C_1)\rho'(C_2)\exp[\pi i(C_1\times C_2)]$ for any C_1 and C_2 in $H_1(W)$. So

1) $\chi(\hat{C})$ is often abbreviated to C for $\hat{C}\in D(\pi)$.

2) The author thanks Mr. Ishitoya who told him the simple proof.

$$(1.7) \qquad \rho(\hat{C})=\rho'(\chi(\hat{C}))$$

is a required function.

Theorem 1. *If we set*

$$(1.8) \qquad \xi(\hat{C}, z)=\rho(\hat{C})\exp(\sigma(z, \hat{C})+\sigma(\hat{C})),$$

then ξ *satisfies*

$$\xi(\hat{C}_1\hat{C}_2, z)=\xi(\hat{C}_1, \hat{C}_2 z)\xi(\hat{C}_2, z)=\xi(\hat{C}_2, \hat{C}_1 z)\xi(\hat{C}_1, z).$$

In other words ξ *is an analytic factor of automorphy for the action of the group* $D(\pi)$.

Proof. We find

$$\sigma(z, \hat{C}_1\hat{C}_2)=\sigma(z, \hat{C}_1)+\sigma(z, \hat{C}_2)=\sigma(z, \hat{C}_2\hat{C}_1)$$

from (1.6). Now

$$\begin{aligned}\sigma(z, \hat{C}_1)&=\int_{z_0}^{z}\hat{k}(\eta, \hat{C}_1)d\eta=\int_{\hat{C}_2 z_0}^{\hat{C}_2 z}\hat{k}(\eta, \hat{C}_1)d\eta\\ &=\left\{-\int_{z_0}^{\hat{C}_2 z_0}+\int_{z_0}^{\hat{C}_2 z}\right\}\hat{k}(\eta, \hat{C}_1)d\eta\\ &=-\int_{C_2}\int_{C_1}k_0(p, q)d\bar{q}dp+\sigma(\hat{C}_2 z, \hat{C}_1)\\ &=-\frac{1}{2\pi}(\psi(C_1), \psi(C_2))+\sigma(\hat{C}_2 z, \hat{C}_1)\\ &=\sigma(\hat{C}_2 z, \hat{C}_1)-\frac{1}{2\pi}\operatorname{Re}(\psi(C_1), \psi(C_2))+\pi i(C_1\times C_2).\end{aligned}$$

Since

$$\sigma(\hat{C}_1\hat{C}_2)=\sigma(\hat{C}_1)+\sigma(\hat{C}_2)+\frac{1}{2\pi}\operatorname{Re}(\psi(C_1), \psi(C_2)),$$

we see

$$(1.9) \qquad \sigma(z, \hat{C}_1\hat{C}_2)=\sigma(\hat{C}_2 z, \hat{C}_1)+\sigma(z, \hat{C}_2)+\sigma(\hat{C}_1)+\sigma(\hat{C}_2)-\sigma(\hat{C}_1\hat{C}_2)+\pi i(C_1\times C_2).$$

Substituting (1.9) in ξ, we have the first equality of (1.8). The second one is obtained similarly. q.e.d.

Remark. We can show more generally that there is a function $\lambda: D(\pi)\rightarrow C^*=C-\{0\}$ such that $\lambda(\hat{C})\exp(\sigma(z, \hat{C}))$ is a factor of automorphy. Indeed since $f(\hat{C}_1, \hat{C}_2)=\exp[(\psi(C_1), \psi(C_2))/2\pi]$ satisfies $f(\hat{C}_1\hat{C}_2, \hat{C}_3)f(\hat{C}_1, \hat{C}_2)=f(\hat{C}_1, \hat{C}_2\hat{C}_3)\cdot f(\hat{C}_2, \hat{C}_3)$, f is an element of $Z^2(D(\pi), C^*)$, the group of 2-cocycles of $D(\pi)$ with coefficients in C^* on which $D(\pi)$ acts trivially. Since, W being an open Riemann surface, $D(\pi)$ is a free group (cf. [1], I. 44A), the second cohomology group $H^2(D(\pi), C^*)$ vanishes (cf. [6], p. 245, Theorem 15). Hence we have a function $\lambda: D(\pi)\rightarrow C^*$

such that $\lambda(\hat{C}_1\hat{C}_2)=\lambda(\hat{C}_1)\lambda(\hat{C}_2)f(\hat{C}_1, \hat{C}_2)$. Then $\lambda(\hat{C})\exp(\sigma(z, \hat{C}))$ is shown to be a factor of automorphy for the action of $D(\pi)$ in the same way as the proof of Theorem 1. Theorem 1 asserts that we can take $\rho(\hat{C})\exp\sigma(\hat{C})$ as $\lambda(\hat{C})$.

We note that if we use $k_0'(p, q)dpd\bar{q}$ instead of $k_0(p, q)dpd\bar{q}$ we have another factor of automorphy:

Theorem 1′. *If we set*

$$\hat{k}'(z, \hat{C})dz=\pi^*\Big(\Big(\int_{\pi(\langle C\rangle)} k_0'(p, q)d\bar{q}\Big)dp\Big),\ \sigma'(z, \hat{C})=\int_{z_0}^{z} \hat{k}'(\eta, \hat{C})d\eta$$

and

$$\sigma'(\hat{C})=\frac{1}{4\pi}\|\psi'(C)\|^2,$$

then $\xi'(\hat{C}, z)=\rho(\hat{C})\exp(\sigma'(z, \hat{C})+\sigma'(\hat{C}))$ *is an analytic factor of automorphy for the action of the group* $D(\pi)$.

We call $\xi(\hat{C}, z)$ a factor of automorphy associated with Γ_a and $\xi'(\hat{C}, z)$ that associated with Γ_{ase}.

§ 2. Theta functions associated with the reproducing kernels

Let W be an open Riemann surface whose universal covering surface $\hat{W}$ is conformally equivalent to the unit disc. We shall show that a relatively automorphic function for ξ or ξ' is capable of being called a theta function on a Riemann surface.

Definition. An analytic relatively automorphic function for a factor of automorphy ξ (resp. ξ') is called a theta function associated with the reproducing kernel for Γ_a (resp. Γ_{ase}).

It should be noted that a theta function defined above depends on the choice of the function ρ: $D(\pi)\to\{-1, 1\}$.

In the rest of the present section we deal with theta functions for Γ_a, but the similar results are also true for those for Γ_{ase}.

At first we note that there exists certainly a theta function which does not vanish identically. In fact ξ induces an analytic line bundle $[\xi]$ over W (cf. [2], § 3) and it is trivial since W is an open Riemann surface (cf. [8], § 1, Theorem 1). Therefore $[\xi]$ has a non-trivial analytic cross section which induces a non-trivial analytic relatively automorphic function for ξ. We set

$$L(z, \hat{C})=\frac{1}{2\pi i}\sigma(z, \hat{C}),\quad J(\hat{C})=\frac{1}{2\pi i}\sigma(\hat{C}).$$

Furthermore we set

$$L(z, w)=\frac{1}{2\pi i}\int_{z_0}^{z} d\eta\int_{z_0}^{w} \hat{k}_0(\eta, \zeta)d\bar{\zeta},$$

$$E(z, w) = L(z, w) - L(w, z) = \frac{1}{\pi} \operatorname{Im}\left(\int_{z_0}^{z} d\eta \int_{z_0}^{w} \hat{k}_0(\eta, \zeta) d\bar{\zeta}\right),$$

$$E(z, \hat{A}) = \frac{1}{\pi} \operatorname{Im}\left(\int_{z_0}^{z} d\eta \int_{z_0}^{\hat{A}z_0} \hat{k}_0(\eta, \zeta) d\bar{\zeta}\right)$$

and

$$E(\hat{A}, \hat{B}) = \frac{1}{\pi} \operatorname{Im}\left(\int_{z_0}^{\hat{A}z_0} d\eta \int_{z_0}^{\hat{B}z_0} \hat{k}_0(\eta, \zeta) d\bar{\zeta}\right).$$

Finally we set

$$H(z, w) = \frac{1}{\pi} \int_{z_0}^{z} d\eta \int_{z_0}^{w} \hat{k}_0(\eta, \zeta) d\bar{\zeta} = 2iL(z, w).$$

Proposition. (i) $E(z, w)$ *is alternating and* $H(z, w)$ *is hermitian in* (z, w). (ii) $E(\hat{A}, \hat{B})$ *is an integer.* (iii) *For* $\hat{A} \in D(\pi)$ *we have*

$$H(\hat{A}z, \hat{A}w) = H(z, w) + \frac{1}{\pi}\sigma(z, \hat{A}) + \frac{1}{\pi}\overline{\sigma(w, \hat{A})} + \frac{2}{\pi}\sigma(\hat{A}).$$

In particular,

$$H(\hat{A}z, \hat{A}z) = H(z, z) + \frac{2}{\pi} \operatorname{Re} \sigma(z, \hat{A}) + \frac{2}{\pi}\sigma(\hat{A}). \tag{2.1}$$

Proof. Since $E(\hat{A}, \hat{B}) = \chi(\hat{A}) \times \chi(\hat{B})$, it is an integer. Next

$$\begin{aligned} H(\hat{A}z, \hat{A}w) &= \frac{1}{\pi} \int_{z_0}^{\hat{A}z} d\eta \int_{z_0}^{\hat{A}w} \hat{k}_0(\eta, \zeta) d\bar{\zeta} \\ &= \frac{1}{\pi} \left\{\int_{z_0}^{z} + \int_{z}^{\hat{A}z}\right\} \left\{\int_{z_0}^{w} + \int_{w}^{\hat{A}w}\right\} \hat{k}_0(\eta, \zeta) d\eta d\bar{\zeta} \\ &= \frac{1}{\pi} \int_{z_0}^{z} d\eta \int_{z_0}^{w} \hat{k}_0(\eta, \zeta) d\bar{\zeta} + \frac{1}{\pi} \int_{z_0}^{w} d\bar{\zeta} \int_{z}^{\hat{A}z} \hat{k}_0(\eta, \zeta) d\eta \\ &\quad + \frac{1}{\pi} \int_{z_0}^{z} d\eta \int_{w}^{\hat{A}w} \hat{k}_0(\eta, \zeta) d\bar{\zeta} + \frac{1}{\pi} \int_{z}^{\hat{A}z} d\eta \int_{w}^{\hat{A}w} \hat{k}_0(\eta, \zeta) d\bar{\zeta} \\ &= H(z, w) + \frac{1}{\pi}\overline{\sigma(w, \hat{A})} + \frac{1}{\pi}\sigma(z, \hat{A}) + \frac{2}{\pi}\sigma(\hat{A}). \end{aligned}$$

Other assertions are evident. q.e.d.

$E(z, w)$ (resp. $H(z, w)$) corresponds to the alternating Riemann form (resp. the Hermitian Riemann form) in the linear theory of theta functions on finite dimensional complex vector spaces (cf. Lang [5]).

Corollary. *For a theta function* $\theta(z)$ *associated with the reproducing kernel for* Γ_a *we set*

$$\phi(z) = \theta(z) \exp\left[-\frac{\pi}{2} H(z, z)\right].$$

Then

(2.2) $$\phi(\hat{A}z)=\phi(z)\rho(\hat{A})\exp[i\operatorname{Im}\sigma(z,\hat{A})]=\phi(z)\rho(\hat{A})\exp[\pi iE(z,\hat{A})].$$

In particular, $|\phi(z)|$ *is a single valued continuous function on* W. *If* $|\phi(z)|$ *is bounded, then there is a positive constant* M *such that*

(2.3) $$|\theta(z)|\leqq \mathrm{M}\exp\left[\frac{\pi}{2}H(z,z)\right].$$

§3. Riemann's theta functions on open Riemann surfaces

Let W be an open Riemann surface of genus g $(0<g\leqq\infty)$ and $\Xi=\{A_j, B_j\}_{j=1}^{g}$ be a canonical homology basis (mod. dividing cycles) associated with some canonical exhaustion of W. We denote by $\{\phi_{A_j}, \phi_{B_j}\}_{j=1}^{g}$ the normalized semi-exact canonical differentials of the first kind, i.e. ϕ_{A_j} and ϕ_{B_j} are holomorphic semi-exact canonical differentials on W such that

$$\operatorname{Re}\int_{B_i}\phi_{A_j}=-\operatorname{Re}\int_{A_i}\phi_{B_j}=\delta_{ij},\quad \operatorname{Re}\int_{A_i}\phi_{A_j}=\operatorname{Re}\int_{B_i}\phi_{B_j}=0.$$

As for these differentials, see [3] and [4]. Then $-\pi i\phi_{A_j}$ (resp. $-\pi i\phi_{B_j}$) is the period reproducing differential for the cycle A_j (resp. B_j) for Γ_{ase} (cf. [4]):[3]

$$(\alpha, -\pi i\phi_{A_j})=2\pi\int_{A_j}\alpha,\quad (\alpha, -\pi i\phi_{B_j})=2\pi\int_{B_j}\alpha$$

for each $\alpha\in\Gamma_{ase}$. Thus by the uniqueness of the period reproducer

(3.1) $$\psi'(A_j)=-\pi i\phi_{A_j},\quad \psi'(B_j)=-\pi i\phi_{B_j}.$$

$\{\phi_{A_j}\}_{j=1}^{g}$ are linearly independent over the complex number field. If we denote by $\Gamma_{ase}(A)$ the closed subspace of Γ_{ase} spanned by $\{\phi_{A_j}\}_{j=1}^{g}$, then there exist uniquely normal differentials $\{\omega_j\}_{j=1}^{g}$ in $\Gamma_{ase}(A)$ such that

(3.2) $$\int_{A_k}\omega_j=\delta_{kj},\qquad k, j=1, 2,\dots, g,\quad (g\leqq\infty),$$

(3.3) $$\omega_j=\sum_{k=1}^{g}\alpha_{kj}\phi_{A_k},\qquad j=1, 2,\dots, g,\qquad (g<\infty)$$

with pure imaginary numbers α_{kj}.

We write some differentials on W and their lifts to $\hat{W}$ by the same letters replacing variable p of W by z of $\hat{W}$. For example $\omega_j(p)$ is a differentials on W and $\omega_j(z)$ is its lift to $\hat{W}$. We take a path from p_0 to a point of A_j and make A_j an element of $\pi_1(W, p_0)$. The corresponding element in $D(\pi)$ is denoted by $\hat{A}_j$. Similarly we have $\hat{B}_j$ in $D(\pi)$ for B_j.

Definition. An analytic factor of automorphy ξ for the action of $D(\pi)$ is called a Riemann's theta factor of automorphy if, for $j=1, 2,\dots,$

3) Since we use the definitions of the intersection number and the inner product of [1], our formulas are modified ones of those of [4].

(i) $\tilde{\xi}(\hat{A}_j, z)=1$,

(ii) $\tilde{\xi}(\hat{B}_j, z)=\exp\left[\pi i\left(2\int_{z_0}^{z}\omega_j+\tau_{jj}\right)\right]$,

where $\tau_{jk}=\int_{B_k}\omega_j$. An analytic relatively automorphic function $\vartheta(z)$ for a Riemann's theta factor of automorphy is called a Riemann's theta function.

Since W is an open Riemann surface, ξ' and $\tilde{\xi}$ define equivalent line bundles over W. Hence these are analytically equivalent factors of automorphy (cf. [2], § 3). Thus Riemann's theta functions are derived from theta functions defined in § 2. We shall show that, under a suitable condition on W (and Ξ), this is done using Abelian integral theory on open Riemann surfaces.

We denote by $\{\alpha_j(p)\}_{j=1}^{g}$ a complete orthonormal system of $\Gamma_{ase}(A)$ obtained from $\{i\phi_{A_j}/2\}_{j=1}^{g}$ by the Schmidt's orthogonalization method. $\alpha_j(p)$ can be written as

$$\alpha_j=\sum_{k=1}^{j}s_{jk}\phi_{A_k}, \qquad j=1, 2,\dots$$

with pure imaginary numbers s_{jk} (cf. [4]).

Proposition 3.1. *$q(p, p')dpdp'=i\sum_{j=1}^{g}\alpha_j(p)\alpha_j(p')$ converges in the norm in p (resp. p') for fixed p' (resp. p), and converges in (p, p') uniformly on each compact set in $W\times W$.*

Proof. We set $\alpha'_j(p)dp=\alpha_j(p)$. Since

$$t(p, p')dp=2\pi\sum_{j=1}^{g}\overline{\alpha'_j(p')}\alpha_j(p)$$

is the reproducing kernel for $\Gamma_{ase}(A)$, $\sum_{j=1}^{g}|\alpha'_j(p')|^2$ converges to $t(p', p')/2\pi$ which is continuous in p'. Thus by a theorem of Dini $\sum_{j=1}^{g}|\alpha'_j(p')|^2$ converges uniformly on each compact coordinate neighborhood. By the Schwarz's inequality we see $\sum_{j=1}^{g}\alpha'_j(p)\alpha'_j(p')$ converges uniformly on each compact set $K\times K'\subset W\times W$, where K and K' are compact coordinate neighborhoods of W. Therefore $q(p, p')dpdp'$ converges uniformly on each compact set in $W\times W$. Since for any n and m ($\geqq n$)

$$\left\|\sum_{j=n}^{m}\alpha'_j(p')\alpha_j(p)\right\|_p^2=\sum_{j=n}^{m}|\alpha'_j(p')|^2,$$

$q(p, p')dpdp'$ converges in the norm in p for fixed p', where $\|\ ,\ \|_p$ stands for the norm of differentials in p. By the symmetry of $q(p, p')dpdp'$ with respect to p and p', it converges in the norm in p' for fixed p. q.e.d.

We set

$$\Phi_{A_j}(z)=\int_{z_0}^{z}\phi_{A_j},\ \Phi_{B_j}(z)=\int_{z_0}^{z}\phi_{B_j}\ \text{ and }\ \Omega_j=\int_{z_0}^{z}\omega_j, \qquad j=1, 2,\dots.$$

Furthermore we set

$$Q(z, w)=\int_{z_0}^{z} d\eta \int_{z_0}^{w} q(\eta, \zeta)d\zeta=\int_{z_0}^{w} d\zeta \int_{z_0}^{z} q(\eta, \zeta)d\eta.$$

We denote by Γ_{ae} the subspace of Γ_{ase} consisting of all exact elements and by Γ_A^0 the orthogonal complement of $\Gamma_{ase}(A)$ in Γ_{ase}.

Proposition 3.2. *We set*

$$J'(\hat{C})=\frac{1}{2\pi i}\left(\frac{1}{4\pi}\|\psi'(C)\|^2\right)=\frac{1}{2\pi i}\sigma'(\hat{C})$$

for $\hat{C}\in D(\pi)$. *Then*

$$Q(\hat{A}_p z, \hat{A}_p w)=Q(z, w)+\frac{1}{2}(\Phi_{A_p}(z)+\Phi_{A_p}(w))-2J'(\hat{A}_p), \tag{3.4}$$

$p=1, 2,\ldots.$ *If* $\Gamma_A^0=\Gamma_{ae}$, *then*

$$\begin{aligned} Q(\hat{B}_p z, \hat{B}_p w)&=Q(z, w)+(\Omega_p(z)+\Omega_p(w)) \\ &\quad+\frac{1}{2}(\Phi_{B_p}(z)+\Phi_{B_p}(w))+(\tau_{pp}-2J'(\hat{B}_p)), \quad p=1, 2,\ldots. \end{aligned}\tag{3.5}$$

(3.6) $\quad Q(\hat{C}z, \hat{C}w)=Q(z, w)$ for $\hat{C}$ corresponding to a dividing cycle C.

Proof.

$$\begin{aligned} &Q(\hat{A}_p z, \hat{A}_p w)=Q(z, w) \\ &\quad+\int_{z_0}^{z} d\eta \int_{w}^{\hat{A}_p w} q(\eta, \zeta)d\zeta+\int_{z_0}^{w} d\zeta \int_{z}^{\hat{A}_p z} q(\eta, \zeta)d\eta+\int_{z}^{\hat{A}_p z} d\eta \int_{w}^{\hat{A}_p w} q(\eta, \zeta)d\zeta. \end{aligned}$$

Since (α_j, ϕ_{A_p}) is pure imaginary,

$$\begin{aligned} \left(\int_{w}^{\hat{A}_p w} q(\eta, \zeta)d\zeta\right)d\eta&=i\sum_{j=1}^{g}\left(\int_{A_p}\alpha_j\right)\alpha_j(\eta)=i\sum_{j=1}^{g}\frac{1}{2\pi}(\alpha_j, -\pi i\phi_{A_p})\alpha_j(\eta) \\ &=-\frac{1}{2}\sum_{j=1}^{g}(\alpha_j, \phi_{A_p})\alpha_j(\eta)=\frac{1}{2}\sum_{j=1}^{g}(\phi_{A_p}, \alpha_j)\alpha_j(\eta)=\frac{1}{2}\phi_{A_p}(\eta). \end{aligned}$$

Therefore

$$\int_{z_0}^{z} d\eta \int_{w}^{\hat{A}_p w} q(\eta, \zeta)d\zeta=\frac{1}{2}\Phi_{A_p}(z).$$

Similarly

$$\int_{z_0}^{w} d\zeta \int_{z}^{\hat{A}_p z} q(\eta, \zeta)d\eta=\frac{1}{2}\Phi_{A_p}(w)$$

and

$$\int_{z}^{\hat{A}_p z} d\eta \int_{w}^{\hat{A}_p w} q(\eta, \zeta)d\zeta=\frac{1}{2}\int_{A_p}\phi_{A_p}=-\frac{1}{\pi i}\left(\frac{1}{4\pi}\|\psi'(A_p)\|^2\right)=-2J'(\hat{A}_p).$$

Therefore we have (3.4).

$$Q(\hat{B}_p z, \hat{B}_p w)=Q(z, w)+\int_{z_0}^{z} d\eta \int_{w}^{\hat{B}_p w} q(\eta, \zeta)d\zeta+\int_{z_0}^{w} d\zeta \int_{z}^{\hat{B}_p z} q(\eta, \zeta)d\eta$$
$$+\int_{z}^{\hat{B}_p z} d\eta \int_{w}^{\hat{B}_p w} q(\eta, \zeta)d\zeta.$$

As before

$$\left(\int_{w}^{\hat{B}_p w} q(\eta, \zeta)d\zeta\right)d\eta=\frac{1}{2}\sum_{j=1}^{g} -(\alpha_j, \phi_{B_p})\alpha_j(\eta).$$

We shall show that $(2\omega_p+\phi_{B_p}, \alpha_j)=-(\alpha_j, \phi_{B_p})$. To this end we must show

$$(\omega_p, \alpha_j)=-\mathrm{Re}\,(\alpha_j, \phi_{B_p}).$$

On the one hand

$$(\omega_p, \alpha_j)=(\omega_p, \sum_{k=1}^{j} s_{jk}\phi_{A_k})=-\sum_{k=1}^{j} s_{jk}(\omega_p, \phi_{A_k})=2i\sum_{k=1}^{j} s_{jk}\delta_{kp}.$$

On the other hand

$$-\mathrm{Re}\,(\alpha_j, \phi_{B_p})=-\mathrm{Re}(\sum_{k=1}^{j} s_{jk}\phi_{A_k}, \phi_{B_p})=-\mathrm{Re}\sum_{k=1}^{j} s_{jk}\overline{(\phi_{B_p}, \phi_{A_k})}$$
$$=-\mathrm{Re}\sum_{k=1}^{j} s_{jk}\overline{\left(-2i\int_{A_k}\phi_{B_p}\right)}=-2i\sum_{k=1}^{j} s_{jk}\,\mathrm{Re}\int_{A_k}\phi_{B_p}$$
$$=2i\sum_{k=1}^{j} s_{jk}\delta_{kp}.$$

Since $\Gamma_A^0=\Gamma_{ae}$ by assumption, ϕ_{B_p} belongs to $\Gamma_{ase}(A)$ (cf. [4]) and

$$2\omega_p+\phi_{B_p}=\sum_{j=1}^{g}(2\omega_p+\phi_{B_p}, \alpha_j)\alpha_j.$$

Therefore

$$\int_{z_0}^{z} d\eta \int_{w}^{\hat{B}_p w} q(\eta, \zeta)d\zeta=\frac{1}{2}(2\Omega_p(z)+\Phi_{B_p}(z)).$$

Similarly

$$\int_{z_0}^{w} d\zeta \int_{z}^{\hat{B}_p z} q(\eta, \zeta)d\eta=\frac{1}{2}(2\Omega_p(w)+\Phi_{B_p}(w))$$

and

$$\int_{z}^{\hat{B}_p z} d\eta \int_{z}^{\hat{B}_p w} q(\eta, \zeta)d\zeta=\frac{1}{2}\int_{B_p}(2\omega_p+\phi_{B_p})=\tau_{pp}+\frac{1}{2}\int_{B_p}\phi_{B_p}=\tau_{pp}-2J'(\hat{B}_p).$$

Thus we have obtained (3.5). (3.6) is proved similarly using the fact that all the α_j are semi-exact. q.e.d.

We shall use the same notations as in §2 just adding the prime to represent corresponding notions for a theta function associated with the reproducing kernel for Γ_{ase}, for example we write as

$$L'(z, \hat{A}_k)=\frac{1}{2\pi i}\int_{z_0}^{z} d\eta\int_{z_0}^{\hat{A}_k z_0} \hat{k}_0'(\eta, \zeta)d\bar{\zeta}=\frac{1}{2\pi i}\int_{z_0}^{z} \hat{k}_0'(\eta, \hat{A}_k)d\eta.$$

Note that

$$\phi_{A_j}(z)=-2dL'(z, \hat{A}_j), \quad \phi_{B_j}=-2dL'(z, \hat{B}_j). \tag{3.7}$$

Let $\Xi'=\Xi\cup\{C_j, j=1, 2,\dots\}$ be a basis of $H_1(W)$, where all the C_j are dividing cycles. Let ρ_0': $\Xi'\to\{-1, 1\}$ be a function such that $\rho_0'(\Xi)=\{1\}$. Using (1.7) we obtain a function ρ_0: $D(\pi)\to\{-1, 1\}$ from ρ_0' (cf. the proof of Lemma in § 1). Then there corresponds a factor of automorphy ξ' for Γ_{ase}:

$$\xi'(\hat{C}, z)=\rho_0(\hat{C})\exp(\sigma'(z, \hat{C})+\sigma'(\hat{C})).$$

Theorem 2. *Let $h(z)=\exp[\pi iQ(z, z)]$ and $\tilde{\xi}(\hat{C}, z)=\xi'(\hat{C}, z)h(\hat{C}z)/h(z)$. If $\Gamma_A^0=\Gamma_{ae}$, then $\tilde{\xi}(\hat{C}, z)$ is a Riemann's theta factor of automorphy for the action of $D(\pi)$. If ρ_0' is such as $\rho_0'(\Xi')=\{1\}$, then $\tilde{\xi}(\hat{C}, z)=1$ for $\hat{C}$ corresponding to a dividing cycle C.*

Proof. The first assertion is a consequence of Proposition 3.2 and (3.7). For example

$$\begin{aligned}\tilde{\xi}(\hat{B}_p, z)&=\rho_0(\hat{B}_p)\exp[2\pi i(L'(z, \hat{B}_p)+J'(\hat{B}_p))]\\ &\qquad\qquad\exp[\pi i(2\Omega_p(z)+\Phi_{B_p}(z)+\tau_{pp}-2J'(\hat{B}_p))]\\ &=\exp[\pi i(2\Omega_p(z)+\tau_{pp})].\end{aligned}$$

For the second assertion, note that for any $\hat{A}$ and $\hat{B}\in D(\pi)$

$$\xi'(\hat{A}\hat{B}, z)=\xi'(\hat{B}\hat{A}, z), \quad h(\hat{A}\hat{B}z)=h(\hat{B}\hat{A}z).$$

The former is proved as Theorem 1 and the latter is verified by direct calculation. Therefore $\tilde{\xi}(\hat{A}\hat{B}, z)=\tilde{\xi}(\hat{B}\hat{A}, z)$, from which we see $\tilde{\xi}(X, z)=1$ for each element X in the commutator subgroup $[D(\pi), D(\pi)]$ of $D(\pi)$. Now let $\hat{C}$ be such that $\chi(\hat{C})=C$ is a dividing cycle. Then there are X in $[D(\pi), D(\pi)]$ and integers $n_1,\dots, n_p$ such that $\hat{C}=X\hat{C}_1^{n_1}\cdots\hat{C}_p^{n_p}$, where $\hat{C}_j$ is an element of $D(\pi)$ with $\chi(\hat{C}_j)=C_j$. By construction $\rho_0(\hat{C}_j^{n_j})=1$ and $\sigma'(z, \hat{C}_j^{n_j})=\sigma'(\hat{C}_j^{n_j})=0$. Hence

$$\begin{aligned}\tilde{\xi}(\hat{C}, z)&=\tilde{\xi}(X, \hat{C}_1^{n_1}\cdots\hat{C}_p^{n_p}z)\tilde{\xi}(\hat{C}_1^{n_1}\cdots\hat{C}_p^{n_p}, z)\\ &=\prod_{j=1}^{p}\tilde{\xi}(\hat{C}_j^{n_j}, \hat{C}_{j+1}^{n_{j+1}}\cdots\hat{C}_p^{n_p}z)=1.\end{aligned}$$

q. e. d.

§ 4. Theta functions on open Riemann surfaces of finite genus

Proposition. *Let W be an open Riemann surface of positive finite genus g. Then $\Gamma_A^0=\Gamma_{ae}$ holds for W if and only if W belongs to the class O_{KD}.*

Proof. Let ω be a real square integrable exact harmonic differential such that $^*\omega$ is semi-exact. $\phi=\omega+i^*\omega$ is in Γ_{ase}. If $\Gamma_A^0=\Gamma_{ae}$, then

$$\phi=i\sum_{j=1}^{g}\gamma_j\omega_j+df,\quad \gamma_j=\int_{A_j}{}^*\omega \quad\text{and}\quad df\in\Gamma_{ae}.$$

Therefore

$$i\beta_k=\int_{B_k}\phi=i\sum_{j=1}^{g}\gamma_j\tau_{jk}.$$

Hence $(\gamma_1,\dots,\gamma_g)(\operatorname{Im}\tau_{ij})=0$. Since $(\operatorname{Im}\tau_{ij})$ is nonsingular, $\gamma_j=0$ for all j. Therefore $\phi=df$ and $^*\omega$ is exact. Hence $W\in O_{KD}$ by a theorem of Rodin [7]. Conversely if $W\in O_{KD}$, then each $\phi\in\Gamma_A^0$ is continued analytically to the compact prolongation $\overline{W}$ of W. Hence $\phi=0$, i.e. $\Gamma_{ae}\subset\Gamma_A^0=\{0\}$.

Theorem 3. *Let W be a parabolic open Riemann surface of positive finite genus g. Let $\theta(z)$ be a theta function associated with the reproducing kernel for Γ_{ase} $(=\Gamma_a)$ on $\hat{W}$ such that $\theta(\hat{C}z)=\theta(z)$ for $\hat{C}$ corresponding to a dividing cycle C. Then $\theta(z)$ as a multiplicative function on W is continued analytically to the compact prolongation $\overline{W}$ of W if and only if $\theta(z)$ satisfies* (2.3).

Proof. Since $W\in O_G$, $\Gamma_{ase}=\Gamma_a$. If $\theta(z)$ is continued to $\overline{W}$, then $|\phi|$ in Corollary in §2 is continuous on $\overline{W}$ and bounded. Conversely if (2.3) is satisfied, then each branch of $\theta(z)$ on a planar neighborhood of $\overline{W}-W$ is a single valued bounded analytic function. Therefore $\theta(z)$ is continued analytically to $\overline{W}$.

Corollary. *Let W and $\theta(z)$ be those in Theorem 3. Then a Riemann's theta function $\vartheta(z)=\theta(z)h(z)$ is continued analytically to $\overline{W}$ if and only if $\vartheta(z)$ satisfies the following inequality*

$$|\vartheta(z)|\leq \mathrm{M}\exp\Big[-\pi\sum_{j,k=1}^{g}(\operatorname{Im}\tau)^{-1}{}_{jk}(\operatorname{Im}\Omega_j(z))(\operatorname{Im}\Omega_k(z))\Big]^{4)}$$

for some positive constant M.

Proof. We must show that

$$\text{(4.1)}\qquad \frac{\pi}{2}H(z,z)+\operatorname{Re}(\pi iQ(z,z))=-\pi\sum_{j,k=1}^{g}(\operatorname{Im}\tau)^{-1}{}_{jk}(\operatorname{Im}\Omega_j(z))(\operatorname{Im}\Omega_k(z)).$$

Since

$$\alpha_j=\sum_{k=1}^{j}s_{jk}\phi_{A_k},\qquad j=1,\dots,g,$$

$$\omega_k=\sum_{j=1}^{g}(\omega_k,\alpha_j)\alpha_j=2i\sum_{j=k}^{g}s_{jk}\alpha_j$$

and

4) $\operatorname{Im}\tau=$ the imaginary part of the period matrix $\tau=(\tau_{ij})$.

$$q(p, p')dpdp' = i\sum_{j=1}^{g}\alpha_j(p)\alpha_j(p') = \frac{1}{2}\sum_{k=1}^{g}\phi_{A_k}(p)\omega_k(p')$$

$$= \frac{1}{2}\sum_{j,k=1}^{g} l_{jk}\omega_j(p)\omega_k(p'),$$

where $l_{jk} = \int_{A_j}\phi_{A_k}$. Similarly we have

$$k_0'(p, p')dpd\bar{p}' = -\pi i\sum_{j,k=1}^{g} l_{jk}\omega_j(p)\overline{\omega_k(p')}$$

and hence

$$H(z, z) = -i\sum_{j,k=1}^{g} l_{jk}\Omega_j(z)\overline{\Omega_k(z)}.$$

Thus

$$\frac{\pi}{2}H(z, z) + \mathrm{Re}\,(\pi i Q(z, z)) = \pi\sum_{j,k=1}^{g}(\mathrm{Im}\, l_{jk})(\mathrm{Im}\,\Omega_j(z))(\mathrm{Im}\,\Omega_k(z)).$$

Since $\mathrm{Im}\, l_{jk} = -(\mathrm{Im}\,\tau)^{-1}{}_{jk}$ by (3.3), we have (4.1). q.e.d.

Finally we remark on theta functions on planar Riemann surfaces. If a surface is planar, then it is shown in the same way as the proof of Theorem 1 that

(4.2) $$\xi(\hat{C}, z) = \exp(\sigma(z, \hat{C}) + \sigma(\hat{C}))$$

is a factor of automorphy for Γ_a for the action of $D(\pi)$ since $C_1 \times C_2 = 0$ for all cycles C_1 and C_2. This means that we can take ρ to be such as $\rho(D(\pi)) = \{1\}$ for a planar surface. Let W be a bounded plane domain whose boundary consists of a finite number of analytic Jordan curves B_k, $k = 0,\dots, g$ $(\geqq 1)$. B_0 is the outer boundary and all the curves are oriented positively with respect to W. Let Ω_k be the harmonic measure of B_k and ω_k be $(\partial\Omega_k/\partial z)dz$, $k = 1,\dots, g$. If we denote by Γ' the closed subspace of $\Gamma_a(W)$ spanned by $\{\omega_k\}_{k=1}^{g}$, then there is an orthogonal decomposition

$$\Gamma_a(W) = \Gamma_{ae}(W) + \Gamma'.$$

From this we have

$$k_0(p, q)dpd\bar{q} = \kappa(p, q)dpd\bar{q} + \kappa'(p, q)dpd\bar{q},$$

where $\bar{\kappa}'$ (resp. $\bar{\kappa}$) is the reproducing kernel of Γ' (resp. Γ_{ae}). If we set

$$P_{jk} = \int_{B_j}\omega_k = \frac{i}{2}\int_{B_j}{}^*d\Omega_k = iP'_{jk},$$

then (P_{jk}) is symmetric and (P'_{jk}) is positive definite. Explicitly

$$\kappa'(p, q)dpd\bar{q} = 2\pi\sum\alpha_{jk}\omega_j(p)\omega_k(q),$$

where (α_{jk}) is the inverse matrix of (P'_{jk}). Moreover

$$\int_C k_0(p, q)d\bar{q} = \int_C \kappa'(p, q)d\bar{q}$$

for all closed curves C in W. Therefore a theta function defined as a relatively automorphic function for ξ in (4.2) satisfies

$$\theta(\hat{B}_k z) = \theta(z) \exp\left[\pi i\left(\int_{z_0}^{z} (-2\omega_k) + (-P_{kk})\right)\right]. \tag{4.3}$$

A multiplicative analytic function satisfying (4.3) is given by

$$\theta(z) = \sum_{m \in Z^g} \exp\left[-\pi i\left({}^t m(-P)m + 2{}^t m\left(-\int_{z_0}^{z} \omega\right)\right)\right],$$

where $(-P) = (-P_{jk})$ and $-\int_{z_0}^{z} \omega = {}^t\left(-\int_{z_0}^{z} \omega_1, \ldots, -\int_{z_0}^{z} \omega_g\right)$. This is the original Riemann's theta function on the double $\tilde{W}$ of W.

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